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A Kleene Theorem for Weighted ω-Pushdown Automata*

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Abstract

Weighted ω-pushdown automata were introduced as generalization of the classical pushdown automata accepting infinite words by Büchi acceptance. The main result in the proof of the Kleene Theorem is the construction of a weighted ω-pushdown automaton for the ω-algebraic closure of subsets of a continuous star-omega semiring.

1 Introduction

Weighted ω-pushdown automata were introduced by Droste, Kuich [4] as generalization of the classical pushdown automata accepting infinite words by Büchi acceptance (see Cohen, Gold [2]). To achieve the Kleene Theorem, the following result is needed.

Let S be a continuous star-omega semiring and let (s, v), $s, v \in S$, with $v = \sum_{1 \leq k \leq m} s_k t_k^\omega$ be a pair, where s, s_k, t_k, $1 \leq k \leq m$, are algebraic elements. Then an ω-pushdown automaton $\mathcal{P}$ can be constructed whose behavior $\|\mathcal{P}\|$ equals (s, v). The construction is split into three lemmas for the construction of t_k^ω, $s_k t_k^\omega$ and v.

This proves a Kleene Theorem that is in some aspects a generalization of Theorem 4.1.8 of Cohen, Gold [2].

The paper consists of this and three more sections. In Section 2 we refer the necessary preliminaries from the theories of semirings and semiring-semimodule pairs. In Section 3, we present some definitions and results from Droste, Kuich [4] that are needed in Section 4. In the last section, existing results in connection with the Kleene Theorem are quoted and the already mentioned constructions on ω-pushdown automata are performed.

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2 Preliminaries

For the convenience of the reader, we quote definitions and results of Ésik, Kuich [6, 7, 9] from Ésik, Kuich [10]. The reader should be familiar with Sections 5.1-5.6 of Ésik, Kuich [10].

A semiring S is called *complete* if it is possible to define sums for all families $(a_i \mid i \in I)$ of elements of S, where I is an arbitrary index set, such that the following conditions are satisfied (see Conway [3], Eilenberg [5], Kuich [11]):

$$\text{(i)} \quad \sum_{i\in\emptyset} a_i = 0, \qquad \sum_{i\in\{j\}} a_i = a_j, \qquad \sum_{i\in\{j,k\}} a_i = a_j + a_k \text{ for } j \neq k\,,$$

$$\text{(ii)} \quad \sum_{j\in J}\Big(\sum_{i\in I_j} a_i\Big) = \sum_{i\in I} a_i\,, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j'\,,$$

$$\text{(iii)} \quad \sum_{i\in I}(c\cdot a_i) = c\cdot\Big(\sum_{i\in I} a_i\Big), \qquad \sum_{i\in I}(a_i\cdot c) = \Big(\sum_{i\in I} a_i\Big)\cdot c\,.$$

This means that a semiring S is complete if it is possible to define "infinite sums" (i) that are an extension of the finite sums, (ii) that are associative and commutative and (iii) that satisfy the distribution laws.

A semiring S equipped with an additional unary star operation $^* : S \to S$ is called a starsemiring. In complete semirings for each element a, the *star* a^* of a is defined by

$$a^* = \sum_{j\geq 0} a^j\,.$$

Hence, each complete semiring is a starsemiring, called a *complete starsemiring.* A *Conway semiring* (see Conway [3], Bloom, Ésik [1]) is a starsemiring S satisfying the *sum star identity*

$$(a+b)^* = a^*(ba^*)^*$$

and the *product star identity*

$$(ab)^* = 1 + a(ba)^*b$$

for all $a, b \in S$. Observe that by Ésik, Kuich [10], Theorem 1.2.24, each complete starsemiring is a Conway semiring.

Suppose that S is a semiring and V is a commutative monoid written additively. We call V a (left) S-semimodule if V is equipped with a (left) action

$$\begin{aligned} S\times V &\to V \\ (s,v) &\mapsto sv \end{aligned}$$

subject to the following rules:

$$\begin{aligned} &s(s'v) = (ss')v\,, \quad (s+s')v = sv + s'v\,, \quad s(v+v') = sv + sv'\,, \\ &1v = v\,, \quad 0v = 0\,, \quad s0 = 0\,, \end{aligned}$$

for all $s, s' \in S$ and $v, v' \in V$. When V is an S-semimodule, we call (S, V) a *semiring-semimodule pair.*

Suppose that (S, V) is a semiring-semimodule pair such that S is a starsemiring and S and V are equipped with an omega operation ${}^\omega : S \to V$. Then we call (S, V) a *starsemiring-omegasemimodule pair.* Following Bloom, Ésik [1], we call a starsemiring-omegasemimodule pair (S, V) a *Conway semiring-semimodule pair* if S is a Conway semiring and if the omega operation satisfies the *sum omega identity* and the *product omega identity*:

$$(a+b)^\omega = (a^*b)^\omega + (a^*b)^*a^\omega \qquad \text{and} \qquad (ab)^\omega = a(ba)^\omega,$$

for all $a, b \in S$. It then follows that the *omega fixed-point equation* holds, i.e.

$$aa^\omega = a^\omega,$$

for all $a \in S$.

Ésik, Kuich [8] define a *complete semiring-semimodule pair* to be a semiring-semimodule pair (S, V) such that S is a complete semiring and V is a complete monoid with

$$s\Big(\sum_{i \in I} v_i\Big) = \sum_{i \in I} sv_i \qquad \text{and} \qquad \Big(\sum_{i \in I} s_i\Big)v = \sum_{i \in I} s_i v\,,$$

for all $s \in S$, $v \in V$, and for all families $(s_i)_{i \in I}$ over S and $(v_i)_{i \in I}$ over V; moreover, it is required that an *infinite product operation*

$$(s_1, s_2, \ldots) \;\mapsto\; \prod_{j \geq 1} s_j$$

is given mapping infinite sequences over S to V subject to the following three conditions:

$$\begin{aligned}
\prod_{i \geq 1} s_i \;&=\; \prod_{i \geq 1}(s_{n_{i-1}+1} \cdot \cdots \cdot s_{n_i}) \\
s_1 \cdot \prod_{i \geq 1} s_{i+1} \;&=\; \prod_{i \geq 1} s_i \\
\prod_{j \geq 1} \sum_{i_j \in I_j} s_{i_j} \;&=\; \sum_{(i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots} \prod_{j \geq 1} s_{i_j}\,,
\end{aligned}$$

where in the first equation $0 = n_0 \leq n_1 \leq n_2 \leq \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets. Suppose that (S, V) is complete. Then we define

$$s^* \;=\; \sum_{i \geq 0} s^i \qquad \text{and} \qquad s^\omega \;=\; \prod_{i \geq 1} s\,,$$

for all $s \in S$. This turns (S, V) into a starsemiring-omegasemimodule pair. By Ésik, Kuich [8], each complete semiring-semimodule pair is a Conway semiring-semimodule pair. Observe that, if (S, V) is a complete semiring-semimodule pair, then $0^\omega = 0$.

A *star-omega semiring* is a semiring S equipped with unary operations * and $^\omega : S \to S$. A star-omega semiring S is called *complete* if (S, S) is a complete semiring semimodule pair, i.e., if S is complete and is equipped with an infinite product operation that satisfies the three conditions stated above.

A commutative monoid $(V, +, 0)$ is *continuous* (cf. Section 2.2 of [10]) if it is equipped with a a partial order $\leq$ such that the supremum of any chain exists and 0 is the least element. Moreover, the sum operation $+$ is continuous:

$$x + \sup Y \ = \ \sup(x + Y)$$

for all nonempty chains, where $x + Y = \{x + y : y \in Y\}$. (Actually this also holds when the set is empty.) It follows that the sum operation is monotonic: if $x \leq y$ in V, then $x + z \leq y + z$ for all $z \in V$.

Suppose now that $S = (S, +, \cdot, 0, 1)$ is a semiring. We say that S is a *continuous semiring* (cf. Section 2.2 of [10]) if $(S, +, 0)$ is a continuous commutative monoid equipped with a partial order $\leq$ and the product operation is continuous (hence, also monotonic), i.e., it preserves the supremum of nonempty chains in either argument:

$$\begin{aligned} (\sup X)y &= \sup(Xy) \\ y(\sup X) &= \sup(yX) \ , \end{aligned}$$

for all nonempty chains $X \subseteq S$, where $Xy = \{xy \ : \ x \in X\}$ and yX is defined in the same way.

By Corollary 2.2.2 of Ésik, Kuich [10] any continuous semiring is complete.

3 Weighted ω-pushdown automata

Weighted ω-pushdown automata were introduced by Droste, Kuich [4] as generalization of the classical pushdown automata accepting infinite words by Büchi acceptance (see Cohen, Gold [2]). In this section we refer to definitions and results of Droste, Kuich [4] that are needed for this paper.

Following Kuich, Salomaa [12] and Kuich [11], we introduce pushdown transitions matrices. Let Γ be an alphabet, called *pushdown alphabet* and let $n \geq 1$. A matrix $M \in (S^{n \times n})^{\Gamma^* \times \Gamma^*}$ is termed a *pushdown transition matrix* (with *pushdown alphabet* Γ and *stateset* $\{1, \ldots, n\}$) if

(i) for each $p \in \Gamma$ there exist only finitely many blocks $M_{p,\pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$M_{\pi_1,\pi_2} = \begin{cases} M_{p,\pi} & \text{if there exist } p \in \Gamma, \pi, \pi' \in \Gamma^* \text{ with } \pi_1 = p\pi', \pi_2 = \pi\pi', \\ 0 & \text{otherwise.} \end{cases}$$

For the remaining of this paper, $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ will denote a pushdown transition matrix with pushdown alphabet Γ and stateset $\{1,\dots,n\}$.

When we say "G is the graph with adjacency matrix $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$" then it means that G is the graph with adjacency matrix $M' \in S^{(\Gamma^*\times n)\times(\Gamma^*\times n)}$, where M' corresponds to M with respect to the canonical isomorphism between $((S^{n\times n})^{\Gamma^*\times\Gamma^*}$ and $S^{(\Gamma^*\times n)(\Gamma^*\times n)}$.

Let now M be a pushdown transition matrix and $0 \le k \le n$. Then $M^{\omega,k}$ is the column vector in $(S^n)^{\Gamma^*}$ defined as follows: For $\pi \in \Gamma^*$ and $1 \le i \le n$, let $((M^{\omega,k})_\pi)_i$ be the sum of all weights of paths in the graph with adjacency matrix M that have initial vertex (π, i) and visit vertices (π', i'), $\pi' \in \Gamma^*$, $1 \le i' \le k$, infinitely often. Observe that $M^{\omega,0} = 0$ and $M^{\omega,n} = M^\omega$.

Let $P_k = \{(j_1, j_2, \dots) \in \{1,\dots,n\}^\omega \mid j_t \le k \text{ for infinitely many } t \ge 1\}$.

Then for $\pi \in \Gamma^+$, $1 \le j \le n$, we obtain

$$((M^{\omega,k})_\pi)_j = \sum_{\pi_1,\pi_2,\dots\in\Gamma^+} \sum_{(j_1,j_2,\dots)\in P_k} (M_{\pi,\pi_1})_{j,j_1}(M_{\pi_1,\pi_2})_{j_1,j_2}(M_{\pi_2,\pi_3})_{j_2,j_3}\dots\,.$$

For the definition of an S'-algebraic system over a quemiring $S \times V$ we refer the reader to [10], page 136, and for the definition of quemirings to [10], page 110. Here we note that a quemiring T is isomorphic to a quemiring $S \times V$ determined by the semiring-semimodule pair (S, V), cf. [10], page 110.

Let $S' \subseteq S$, with $0, 1 \in S'$, and let $M \in ({S'}^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown matrix. Consider the ${S'}^{n\times n}$-algebraic system over the complete semiring-semimodule pair $(S^{n\times n}, S^n)$

$$y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi\,,\ p \in \Gamma\,. \tag{1}$$

(See Section 5.6 of Ésik, Kuich [10].) The variables of this system (1) are $y_p, p \in \Gamma$, and $y_\pi, \pi \in \Gamma^*$, is defined by $y_{p\pi} = y_p y_\pi$ for $p \in \Gamma$, $\pi \in \Gamma^*$ and $y_\varepsilon = 1$. Hence, for $\pi = p_1 \dots p_k$, $y_\pi = y_{p_1} \dots y_{p_k}$. The variables y_p are variables for $(S^{n\times n}, S^n)$.

Let $x = (x_p)_{p\in\Gamma}$, where x_p, $p \in \Gamma$, are variables for $S^{n\times n}$. Then, for $p \in \Gamma$, $\pi = p_1 p_2 \dots p_k$, $(M_{p,\pi} y_\pi)_x$ is defined to be

$$\begin{aligned}
&(M_{p,\pi} y_\pi)_x \\
&= (M_{p,\pi} y_{p_1} \dots y_{p_k})_x \\
&= M_{p,\pi} z_{p_1} + M_{p,\pi} x_{p_1} z_{p_2} + \dots + M_{p,\pi} x_{p_1} \dots x_{p_{k-1}} z_{p_k}\,.
\end{aligned}$$

Here z_p, $p \in \Gamma$, are variables for S^n.

We obtain, for $p \in \Gamma$, $\pi = p_1 \dots p_k$,

$$\begin{aligned}
(M_{p,\pi} y_\pi)_x &= \sum_{p'\in\Gamma} \sum_{\substack{\pi=p_1\dots p_k\in\Gamma^+\\ p_j=p'}} M_{p,\pi} x_{p_1} \dots x_{p_{j-1}} z_{p'} \\
&= \sum_{\pi=p_1\dots p_k\in\Gamma^+} M_{p,\pi} \sum_{1\le j\le k} x_{p_1} \dots x_{p_{j-1}} z_{p_j}\,.
\end{aligned}$$

The system (1) induces the following mixed ω-algebraic system:

$$x_p = \sum_{\pi \in \Gamma^*} M_{p\pi} x_\pi \,,\; p \in \Gamma, \tag{2}$$

$$z_p = \sum_{\pi \in \Gamma^*} (M_{p,\pi} y_\pi)_{(x_p)_{p \in \Gamma}} = \sum_{p' \in \Gamma} \sum_{\substack{\pi = p_1 \dots p_k \in \Gamma^+ \\ p_j = p'}} M_{p,\pi} x_{p_1} \dots x_{p_{j-1}} z_{p'} \,. \tag{3}$$

Here (2) is an ${S'}^{n\times n}$-algebraic system over the semiring $S^{n\times n}$ (see Section 2.3 of Ésik, Kuich [10]) and (3) is an $S^{n\times n}$-linear system over the semimodule S^n (see Section 5.5 of Ésik, Kuich [10]).

By Theorem 5.6.1 of Ésik, Kuich [10], $(A, U) \in ((S^{n\times n})^\Gamma, (S^n)^\Gamma)$ is a solution of (1) iff A is a solution of (2) and (A, U) is a solution of (3).

Theorem 3.1. *Let S be a complete star-omega semiring and $M \in ({S'}^{n\times n})^{\Gamma^* \times \Gamma^*}$ be a pushdown transition matrix. Then, for all $0 \le k \le n$,*

$$(((M^*)_{p,\varepsilon})_{p\in\Gamma}, ((M^{\omega,k})_p)_{p\in\Gamma})$$

is a solution of (1).

We now introduce pushdown automata and ω-pushdown automata (see Kuich, Salomaa [12], Kuich [11], Cohen, Gold [2]).

Let S be a complete semiring and $S' \subseteq S$ with $0, 1 \in S'$. An *S'-pushdown automaton over S*

$$\mathcal{P} = (n, \Gamma, I, M, P, p_0)$$

is given by

(i) a finite set of *states* $\{1, \dots, n\}$, $n \ge 1$,

(ii) an alphabet Γ of *pushdown symbols*,

(iii) a *pushdown transition matrix* $M \in ({S'}^{n\times n})^{\Gamma^* \times \Gamma^*}$,

(iv) an *initial state vector* $I \in {S'}^{1\times n}$,

(v) a *final state vector* $P \in {S'}^{n\times 1}$,

(vi) an *initial pushdown symbol* $p_0 \in \Gamma$,

The *behavior* $\|\mathcal{P}\|$ *of* $\mathcal{P}$ is an element of S and is defined by $\|\mathcal{P}\| = I(M^*)_{p_0,\varepsilon} P$.

For a complete semiring-semimodule pair (S, V), an S'-ω-pushdown automaton (over (S, V))

$$\mathcal{P} = (n, \Gamma, I, M, P, p_0, k)$$

is given by an S'-pushdown automaton $(n, \Gamma, I, M, P, p_0)$ and an $k \in \{0, \dots, n\}$ indicating that the states $1, \dots, k$ are *repeated states.*

The *behavior* $\|\mathcal{P}\|$ of the S'-ω-pushdown automaton $\mathcal{P}$ is defined by

$$\|\mathcal{P}\| = I(M^*)_{p_0,\varepsilon}P + I(M^{\omega,k})_{p_0} .$$

Here $I(M^*)_{p_0,\varepsilon}P$ is the behavior of the S'-ω-pushdown automaton $\mathcal{P}_1 = (n, \Gamma, I, M, P, p_0, 0)$ and $I(M^{\omega,k})_{p_0}$ is the behavior of the S'-ω-pushdown automaton $\mathcal{P}_2 = (n, \Gamma, I, M, 0, p_0, k)$. Observe that $\mathcal{P}_2$ is an automaton with the Büchi acceptance condition: if G is the graph with adjacency matrix M, then only paths that visit the repeated states $1, \dots, k$ infinitely often contribute to $\|\mathcal{P}_2\|$. Furthermore, $\mathcal{P}_1$ contains no repeated states and behaves like an ordinary S'-pushdown automaton.

Theorem 3.2. *Let S be a complete star-omega semiring and let $\mathcal{P} = (n, \Gamma, I, M, P, p_0, k)$ be an S'-ω-pushdown automaton over (S, S). Then $(\|\mathcal{P}\|, (((M^*)_{p,\varepsilon})_{p\in\Gamma}, ((M^{\omega,k})_p)_{p\in\Gamma}))$, $0 \le k \le n$, is a solution of the ${S'}^{n\times n}$-algebraic system*

$$y_0 = Iy_{p_0}P, y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi,\ p \in \Gamma$$

over the complete semiring-semimodule pair $(S^{n\times n}, S^n)$.

Let now S be a continuous star-omega semiring and consider an S'-algebraic system $y = p(y)$ over (S, S). Then the least solution of the S'-algebraic system $x = p(x)$ over S, say σ, exists, and the components of σ are elements of $\mathfrak{Alg}\,(S')$. Moreover, write the $\mathfrak{Alg}\,(S')$-linear system $z = p_0(z)$ over S in the form $z = Mz$, where M is an $n\times n$-matrix. Then, by Theorem 5.6.1 of Ésik, Kuich [10], $(\sigma, M^{\omega,k})$, $0 \le k \le n$, is a solution of $y = p(y)$. Given a $k \in \{0, 1, \dots, n\}$, we call this solution the solution of order k of $y = p(y)$. By ω-$\mathfrak{Alg}\,(S')$ we denote the collection of all components of solutions of all orders k of S'-algebraic systems over (S, S). (For details see Section 5.6 of Ésik, Kuich [10].)

4 The Kleene Theorem

The main result of this section is the following Kleene Theorem.

Theorem 4.1. *Let S be a continuous star-omega semiring. Then the following statements are equivalent for $(s, v) \in S \times S$:*

(i) $(s, v) = \|\mathfrak{A}\|$*, where* $\mathfrak{A}$ *is a finite* $\mathfrak{Alg}\,(S')$*-automaton over the quemiring* (S, S)*,*

(ii) $(s, v) \in \omega\text{-}\mathfrak{Alg}\,(S')$*,*

(iii) $s \in \mathfrak{Alg}\,(S')$ *and* $v = \sum_{1\le k\le m} s_k t_k^\omega$*, where* $s_k, t_k \in \mathfrak{Alg}\,(S')\,, 1 \le k \le m$*,*

(iv) $(s, v) = \|\mathcal{P}\|$*, where* $\mathcal{P}$ *is an* S'*-*ω*-pushdown automaton.*

The proof of this Kleene Theorem is performed as follows:

1. The equivalence of (i), (ii) and (iii) is proved in [10], Theorem 5.4.9.

2. The implication *(iv)* $\Rightarrow$ *(ii)* is a simple corollary of Theorem 13 of [4].

3. The proof of the implication *(iii)* $\Rightarrow$ *(iv)* is performed by Lemmas 4.1, 4.2 and 4.3 proved in the following pages.

Lemma 4.1. *Let S be a complete star-omega semiring and $\mathcal{P}$ be an S'-pushdown automaton. Then there exists an S'-ω-pushdown automaton $\mathcal{P}'$ such that $\|\mathcal{P}'\| = \|\mathcal{P}\|^\omega$.*

Proof. Let $\mathcal{P} = (n, \Gamma, M, I, P, p_0)$. Then we construct $\mathcal{P}' = (2n, \Gamma', M', I', 0, p_0', n)$, $\Gamma' = \Gamma \cup \{p_0'\}$ as follows.

The pushdown transition matrix $M' \in \left(S'^{2n\times 2n}\right)^{\Gamma'^* \times \Gamma'^*}$ has, for $\pi \in \Gamma^*$, $1 \le j \le n$, the entries

$$\begin{aligned}(M'_{p_0',p_0'})_{n+i,j} &= (PI)_{i,j},\\ (M'_{p_0',\pi p_0'})_{i,n+j} &= (M_{p_0,\pi})_{i,j}\\ (M'_{p,\pi})_{n+i,n+j} &= (M_{p,\pi})_{i,j};\end{aligned}$$

all other entries of the matrices $M'_{p,\pi}, p \in \Gamma', \pi \in \Gamma'^*$, are 0.

The initial state vector $I' \in S'^{2n\times 1}$ has, for $1 \le i \le n$, the entries

$$I_i' = I_i, I_{n+i}' = 0.$$

We have to prove that

$$\|\mathcal{P}'\| = I'\,(M'^{\omega,n})_{p_0'} = \|\mathcal{P}\|^\omega = \Big(I\,(M^*)_{p_0,\varepsilon}\,P\Big)^\omega .$$

The proof of this claim is as follows.

By definition, for $1 \le i \le 2n$,

$$\Big((M'^{\omega,n})_{p_0'}\Big)_i = \sum_{\pi_1,\pi_2,\ldots\in\Gamma'^*}\ \sum_{\substack{i_1,i_2,\ldots\in P_n\\ 1\le i_1,i_2,\ldots\le 2n}} \Big(M'_{p_0',\pi_1}\Big)_{i,i_1} \big(M'_{\pi_1,\pi_2}\big)_{i_1,i_2}\cdots$$

Inspection shows that a repeated state in the sequence $i_1, i_2, \ldots$ appears only if in the run $p_0', \pi_1, \pi_2, \ldots$ a transition from p_0' to p_0' appears.

Hence, we obtain, with $i_0^1 = i$, $\pi_0^t = \varepsilon$ for $t \geq 1$,

$$\begin{aligned}
&\left((M'^{\omega,n})_{p'_0}\right)_i \\
&= \prod_{t\geq 1} \sum_{k_t \geq 1} \sum_{1\leq i_0^t,\dots,i_{k_t}^t \leq n} \sum_{\pi_1^t,\cdots,\pi_{k_t-1}^t \in \Gamma^*} \left(M'_{p'_0,\pi_1^t p'_0}\right)_{i_0^t, n+i_1^t} \left(M'_{\pi_1^t p'_0, \pi_2^t p'_0}\right)_{n+i_1^t, n+i_2^t} \cdots \\
&\qquad\qquad \left(M'_{\pi_{k_t-1}^t p'_0, p'_0}\right)_{n+i_{k_t-1}^t, n+i_{k_t}^t} \left(M'_{p'_0,p'_0}\right)_{n+i_{k_t}^t, i_0^{t+1}} \\
&= \prod_{t\geq 1} \sum_{k_t \geq 1} \sum_{1\leq i_0^t,\dots,i_{k_t}^t \leq n} \sum_{\pi_1^t,\dots,\pi_{k_t-1}^t \in \Gamma^*} \left(M_{p_0,\pi_1^t}\right)_{i_0^t, i_1^t} \left(M_{\pi_1^t,\pi_2^t}\right)_{i_1^t, i_2^t} \cdots \\
&\qquad\qquad \left(M_{\pi_{k_t-1}^t,\varepsilon}\right)_{i_{k_t-1}^t, i_{k_t}^t} (PI)_{i_{k_t}^t, i_0^{t+1}} \\
&= \prod_{t\geq 1} \sum_{k_t \geq 1} \sum_{1\leq i_0^t \leq n} \left((M^{k_t})_{p_0,\varepsilon} PI\right)_{i_0^t, i_0^{t+1}} \\
&= \prod_{t\geq 1} \sum_{1\leq i_0^t \leq n} \left(\sum_{k_t \geq 1} (M^{k_t})_{p_0,\varepsilon} PI\right)_{i_0^t, i_0^{t+1}} \\
&= \prod_{t\geq 1} \sum_{1\leq i_0^t \leq n} \left((M^*)_{p_0,\varepsilon} PI\right)_{i_0^t, i_0^{t+1}} \\
&= \left((M^*)_{p_0,\varepsilon} PI\right)_i^\omega .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{P}'\| &= \sum_{1\leq i\leq 2n} I_i \left((M'^{\omega,n})_{p'_0}\right)_i \\
&= \sum_{1\leq i\leq n} I_i \left((M'^{\omega,n})_{p'_0}\right)_i \\
&= I\,(M'^{\omega,n})_{p'_0} \\
&= I\left((M^*)_{p_0,\varepsilon} PI\right)^\omega \\
&= \left(I\,(M^*)_{p_0,\varepsilon} P\right)^\omega \\
&= \|\mathcal{P}\|^\omega.
\end{aligned}$$

□

Lemma 4.2. *Let S be a complete star-omega semiring, $\mathcal{P}_1$ be an S'-ω-pushdown automaton and $\mathcal{P}_2$ be an S'-pushdown automaton. Then there exists an S'-ω-pushdown automaton $\mathcal{P}$ such that $\|\mathcal{P}\| = \|\mathcal{P}_2\|\|\mathcal{P}_1\|$.*

Proof. Let $\mathcal{P}_1 = (n_1, \Gamma_1, I_1, M_1, P_1, p_1, k)$ and $\mathcal{P}_2 = (n_2, \Gamma_2, I_2, M_2, P_2, p_2)$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then we construct $\mathcal{P} = (n_1 + n_2, \Gamma_1 \cup \Gamma_2, I, M, P, p_2, k)$ as follows.

Let $Q_1 = \{1, \ldots, n_1\}$ and $Q_2 = \{n_1 + 1, \ldots, n_2\}$. The pushdown transition matrix $M \in (S'^{(n_1+n_2)\times(n_1+n_2)})^{(\Gamma_1\cup\Gamma_2)^*\times(\Gamma_1\cup\Gamma_2)^*}$ has entries

1. transitions from Q_2 to Q_2

$$\begin{aligned}
\left(M_{p_2,\pi p_1}\right)_{i,j} &= \Big(\left(M_2\right)_{p_2,\pi}\Big)_{i,j}, \quad i, j \in Q_2, \pi \in \Gamma_2^+, \\
\left(M_{p,\pi}\right)_{i,j} &= \Big(\left(M_2\right)_{p,\pi}\Big)_{i,j}, \quad i, j \in Q_2, p \in \Gamma_2, \pi \in \Gamma_2^+, \\
\left(M_{p,\varepsilon}\right)_{i,j} &= \Big(\left(M_2\right)_{p,\varepsilon}\Big)_{i,j}, \quad i, j \in Q_2, p \in \Gamma_2;
\end{aligned}$$

2. transitions from Q_2 to Q_1

$$\begin{aligned}
\left(M_{p_2,p_1}\right)_{i,j} &= \Big(\left(M_2\right)_{p_2,\varepsilon} P_2 I_1\Big)_{i,j}, \quad i \in Q_2, j \in Q_1, \\
\left(M_{p,\varepsilon}\right)_{i,j} &= \Big(\left(M_2\right)_{p,\varepsilon} P_2 I_1\Big)_{i,j}, \quad i \in Q_2, j \in Q_1, p \in \Gamma_2;
\end{aligned}$$

3. transitions from Q_1 to Q_1

$$\left(M_{p,\pi}\right)_{i,j} = \Big(\left(M_1\right)_{p,\pi}\Big)_{i,j}, \quad i, j \in Q_1, p \in \Gamma_1, \pi \in \Gamma_1^*.$$

All other entries of the matrices $M_{p,\pi}$, $p \in \Gamma_1 \cup \Gamma_2$, $\pi \in (\Gamma_1 \cup \Gamma_2)^*$, are 0.

The initial state vector $I \in S'^{1\times(n_1+n_2)}$ and the final state vector $P \in S'^{(n_1+n_2)\times 1}$ have the entries

$$\begin{aligned}
I_i &= 0, \ i \in Q_1, & I_i &= (I_2)_i, \ i \in Q_2; \\
P_i &= (P_1)_i, \ i \in Q_1, & P_i &= 0, \ i \in Q_2.
\end{aligned}$$

We have to prove that

$$\begin{aligned}
\|\mathcal{P}\| &= I \left(M^*\right)_{p_2,\varepsilon} P + I \left(M^{\omega,k}\right)_{p_2} \\
&= I_2 \left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^*\right)_{p_1,\varepsilon} P_1 + I_2 \left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^{\omega,k}\right)_{p_1} \\
&= \|\mathcal{P}_2\| \|\mathcal{P}_1\|.
\end{aligned}$$

The proof of this claim is as follows.

By definition,

$$\Big(\left(M^{\omega,k}\right)_{p_2}\Big)_{i_0} = \sum_{\pi_1,\pi_2,\ldots\in(\Gamma_1\cup\Gamma_2)^*} \ \sum_{\substack{i_1,i_2,\ldots\in P_k \\ 1\le i_1,i_2,\ldots\le n_1+n_2}} \left(M_{p_2,\pi_1}\right)_{i_0,i_1} \left(M_{\pi_1,\pi_2}\right)_{i_1,i_2} \ldots, \quad i_0 \in Q_2,$$

$$\Big(\left(M^{\omega,k}\right)_{p_2}\Big)_{i_0} = 0, \quad i_0 \in Q_1.$$

As long as $\mathcal{P}$ remains in a state of Q_2, the contents of the pushdown tape is πp_1, $\pi \in \Gamma_2^*$. The transition from a state of Q_2 to a state of Q_1 is possible only in the following three situations:

(a) In the first step, the contents p_2 of the pushdown tape is replaced by p_1.

(b) The contents of the pushdown tape is pp_1, $p \in \Gamma_2$, and p is replaced by the empty word; so that after this replacement the contents is p_1.

(c) The contents of the pushdown tape is $p\pi p_1$, $p \in \Gamma_2$,$\pi \in \Gamma_2^+$, and p is replaced by the empty word. In this situation, no continuation of the computation of $\mathcal{P}$ is possible.

Since all the repeated states are states in Q_1, there must be a transition from a state of Q_2 to a state of Q_1.

As long as $\mathcal{P}$ remains in a state of Q_2 with πp_1, $\pi \in \Gamma_2^*$, on the pushdown tape, it simulates $\mathcal{P}_2$ up to situations (a) or (b). Then p_1 is the contents of the pushdown tape of $\mathcal{P}$, $\mathcal{P}$ is in a state of Q_1 and simulates $\mathcal{P}_1$, since there is no transition from a state of Q_1 to a state of Q_2.

Hence, we obtain, for $i_0 \in Q_2$,

$$
\begin{aligned}
&\Big(\big(M^{\omega,k}\big)_{p_2}\Big)_{i_0} \\
&= \sum_{\pi_1,\pi_2,\ldots\in\Gamma_1^+} \sum_{\substack{j_0,j_1,\ldots\in Q_1\\ (j_0,j_1,\ldots)\in P_k}} \left(M_{p_2,p_1}\right)_{i_0,j_0} \left(M_{p_1,\pi_1}\right)_{j_0,j_1} \left(M_{\pi_1,\pi_2}\right)_{j_1,j_2} \cdots + \\
&\sum_{t\geq 1} \sum_{\rho_1,\ldots,\rho_{t-1}\in\Gamma_2^+} \sum_{\rho_t\in\Gamma_2} \sum_{\pi_1,\pi_2,\ldots\in\Gamma_1^+} \sum_{i_1,\ldots,i_t\in Q_2} \sum_{\substack{j_0,j_1,\ldots\in Q_1\\ (j_0,j_1,\ldots)\in P_k}} \left(M_{p_2,\rho_1 p_1}\right)_{i_0,i_1} \\
&\qquad \left(M_{\rho_1 p_1,\rho_2 p_2}\right)_{i_1,i_2} \cdots \left(M_{\rho_t p_1,p_1}\right)_{i_t,j_0} \left(M_{p_1,\pi_1}\right)_{j_0,j_1} \left(M_{\pi_1,\pi_2}\right)_{j_1,j_2} \cdots \\
&= \sum_{j_0\in Q_1} \Big(\left(M_2\right)_{p_2,\varepsilon} P_2 I_1\Big)_{i_0,j_0} \bigg(\Big(M_1^{\omega,k}\Big)_{p_1}\bigg)_{j_0} + \sum_{t\geq 1} \sum_{\rho_1,\ldots,\rho_{t-1}\in\Gamma_2^+} \\
&\sum_{\rho_t\in\Gamma_2} \sum_{\pi_1,\pi_2,\ldots\in\Gamma_1^+} \sum_{i_1,\ldots,i_t\in Q_2} \sum_{\substack{j_0,j_1,\ldots\in Q_1\\ (j_0,j_1,\ldots)\in P_k}} \Big(\left(M_2\right)_{p_2,\rho_1}\Big)_{i_0,i_1} \Big(\left(M_2\right)_{\rho_1,\rho_2}\Big)_{i_1,i_2} \cdots \\
&\qquad \Big(\Big(\left(M_2\right)_{\rho_t,\varepsilon}\Big) P_2 I_1\Big)_{i_t,j_0} \Big(\left(M_1\right)_{p_1,\pi_1}\Big)_{j_0,j_1} \Big(\left(M_1\right)_{\pi_1,\pi_2}\Big)_{j_1,j_2} \cdots \\
&= \sum_{j_0\in Q_1} \sum_{t\geq 0} \Big(\left(M_2^{t+1}\right)_{p_2,\varepsilon} P_2 I_1\Big)_{i_0,j_0} \bigg(\Big(M_1^{\omega,k}\Big)_{p_1}\bigg)_{j_0} \\
&= \bigg(\left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \Big(M_1^{\omega,k}\Big)_{p_1}\bigg)_{i_0}.
\end{aligned}
$$

In the first equality, the first summand on the right side represents situation (a), while the second summand represents situation (b).

By definition,

$$\begin{aligned}\Big((M^*)_{p_2,\varepsilon}\Big)_{i_0,j} &= \sum_{t\geq 1}\ \sum_{\pi_1,\ldots,\pi_t\in(\Gamma_1\cup\Gamma_2)^*}\ \sum_{1\leq i_1,\ldots,i_t\leq n_1+n_2} (M_{p_2,\pi_1})_{i_0,i_1}\\ &\qquad (M_{\pi_1,\pi_2})_{i_1,i_2}\ldots(M_{\pi_t,\varepsilon})_{i_t,j}\,,\quad i_0\in Q_2, j\in Q_1\cup Q_2,\\ \Big((M^*)_{p_2,\varepsilon}\Big)_{i_0,j} &= 0,\quad i_0\in Q_1, j\in Q_1\cup Q_2.\end{aligned}$$

Observe that $\pi_1 = \pi p_1$, $\pi \in \Gamma_2^*$. To obtain the empty tape, $\mathcal{P}$ has to replace eventually p_1 by some $\pi' \in \Gamma_1^*$. But this is possible only in situations (a) or (b).

Hence, we obtain, for $i_0 \in Q_2, j \in Q_1$,

$$\begin{aligned}\Big((M^*)_{p_2,\varepsilon}\Big)_{i_0,j} &= \sum_{j_0\in Q_1}(M_{p_2,p_1})_{i_0,j_0}\Big((M^*)_{p_1,\varepsilon}\Big)_{j_0,j} +\\ &\quad \sum_{t\geq 1}\ \sum_{\rho_1,\ldots,\rho_{t-1}\in\Gamma_2^+}\ \sum_{\rho_t\in\Gamma_2}\ \sum_{i_1,\ldots,i_t\in Q_2}\ \sum_{j_0\in Q_1}(M_{p_2,\rho_1 p_1})_{i_0,i_1}\ldots\\ &\quad (M_{\rho_t p_1,p_1})_{i_t,j_0}\Big((M^*)_{p_1,\varepsilon}\Big)_{j_0,j}\\ &= \sum_{j_0\in Q_1}\Big((M_2)_{p_2,\varepsilon}P_2I_1\Big)_{i_0,j_0}\Big((M_1^*)_{p_1,\varepsilon}\Big)_{j_0,j} +\\ &\quad \sum_{t\geq 1}\ \sum_{\rho_1,\ldots,\rho_{t-1}\in\Gamma_2^+}\ \sum_{\rho_l\in\Sigma_2}\ \sum_{i_1,\ldots,i_t\in Q_2}\ \sum_{j_0\in Q_1}\Big((M_2)_{p_2,\rho_1}\Big)_{i_0,i_1}\ldots\\ &\quad \Big((M_2)_{\rho_{t-1},\rho_t}\Big)_{i_{t-1},i_t}\Big((M_2)_{\rho_t,\varepsilon}P_2I_1\Big)_{i_t,j_0}\Big((M_1^*)_{p_1,\varepsilon}\Big)_{j_0,j}\\ &= \Big((M_2)_{p_2,\varepsilon}P_2I_1(M_1^*)_{p_1,\varepsilon}\Big)_{j_0,j} +\\ &\quad \sum_{j_0\in Q_1}\sum_{t\geq 1}\Big((M_2^{t+1})_{p_2,\varepsilon}P_2I_1\Big)_{i_0,j_0}\Big((M_1^*)_{p_1,\varepsilon}\Big)_{j_0,j}\\ &= \sum_{t\geq 0}\Big((M_2^{t+1})_{p_2,\varepsilon}P_2I_1(M_1^*)_{p_1,\varepsilon}\Big)_{i_0,j}\\ &= \Big((M_2^*)_{p_2,\varepsilon}P_2I_1(M_1^*)_{p_1,\varepsilon}\Big)_{i_0,j}\,,\end{aligned}$$

and, for $i_0 \in Q_2, j \in Q_2$,

$$\Big((M^*)_{p_2,\varepsilon}\Big)_{i_0,j} = 0.$$

In the first equality, the first summand on the right side represents situation (a), while the second summand represents situation (b).

We obtain

$$\begin{aligned} I\left(M^*\right)_{p_2,\varepsilon} P &= \sum_{i\in Q_2}\sum_{j\in Q_1} (I_2)_i \left(\left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^*\right)_{p_1,\varepsilon}\right)_{i,j} (P_1)_j \\ &= I_2 \left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^*\right)_{p_1,\varepsilon} P_1 \end{aligned}$$

and

$$\begin{aligned} I\left(M^{\omega,k}\right)_{p_2} &= \sum_{i\in Q_2} (I_2)_i \left(\left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^{\omega,k}\right)_{p_1}\right)_i \\ &= I_2 \left(M_2^*\right)_{p_2,\varepsilon} P_2 I_1 \left(M_1^{\omega,k}\right)_{p_1}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathcal{P}\| &= I\left(M^*\right)_{p_2,\varepsilon} P + I\left(M^{\omega,k}\right)_{p_2} \\ &= I_2 \left(M_2^*\right)_{p_2,\varepsilon} P_2 \left(I_1 \left(M_1^*\right)_{p_1,\varepsilon} P_1 + I_1 \left(M_1^{\omega,k}\right)_{p_1}\right) \\ &= \|\mathcal{P}_2\| \|\mathcal{P}_1\|. \end{aligned}$$

□

Lemma 4.3. *Let S be a complete star-omega semiring and $\mathcal{P}_1$, $\mathcal{P}_2$ S'-ω-pushdown automata. Then there exists an S'-ω-pushdown automaton $\mathcal{P}$ such that $\|\mathcal{P}\| = \|\mathcal{P}_1\| + \|\mathcal{P}_2\|$.*

Proof. Let $\mathcal{P}_i = (n_i, \Gamma_i, I_i, M_i, P_i, p_i, k_i)$, $i = 1, 2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then we construct $\mathcal{P} = (n_1 + n_2, \Gamma, I, M, P, p_0, k_1 + k_2)$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{p_0\}$.

The matrix $M \in \left(S'^{(n_1+n_2)\times(n_1+n_2)}\right)^{\Gamma^*\times\Gamma^*}$ is defined as follows. Let, for $\pi_1, \pi_2 \in \Gamma_1^*$, $(\pi_1, \pi_2) \neq (\varepsilon, \varepsilon)$,

$$(M_1)_{\pi_1,\pi_2} = \begin{pmatrix} a_{\pi_1,\pi_2} & b_{\pi_1,\pi_2} \\ c_{\pi_1,\pi_2} & d_{\pi_1,\pi_2} \end{pmatrix},$$

where the blocks are indexed by $\{1, \dots, k_1\}, \{k_1 + 1, \dots, n_1\}$, and, for $\pi_1, \pi_2 \in \Gamma_2^*$, $(\pi_1, \pi_2) \neq (\varepsilon, \varepsilon)$,

$$(M_2)_{\pi_1,\pi_2} = \begin{pmatrix} a_{\pi_1,\pi_2} & b_{\pi_1,\pi_2} \\ c_{\pi_1,\pi_2} & d_{\pi_1,\pi_2} \end{pmatrix},$$

where the blocks are indexed by $\{1, \dots, k_2\}, \{k_2 + 1, \dots, n_2\}$.

Then, we define, for $\pi \in \Gamma_1^*$,

$$M_{p_0,\pi} = \begin{pmatrix} a_{p_1,\pi} & 0 & b_{p_1,\pi} & 0 \\ 0 & 0 & 0 & 0 \\ c_{p_1,\pi} & 0 & d_{p_1,\pi} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

for $\pi \in \Gamma_2^*$,

$$M_{p_0,\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{p_1,\pi} & 0 & b_{p_1,\pi} \\ 0 & 0 & 0 & 0 \\ 0 & c_{p_1,\pi} & 0 & d_{p_1,\pi} \end{pmatrix};$$

for $p \in \Gamma_1, \pi \in \Gamma_1^*$,

$$M_{p,\pi} = \begin{pmatrix} a_{p,\pi} & 0 & b_{p,\pi} & 0 \\ 0 & 0 & 0 & 0 \\ c_{p,\pi} & 0 & d_{p,\pi} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

and for $p \in \Gamma_2, \pi \in \Gamma_2^*$,

$$M_{p,\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{p,\pi} & 0 & b_{p,\pi} \\ 0 & 0 & 0 & 0 \\ 0 & c_{p,\pi} & 0 & d_{p,\pi} \end{pmatrix}.$$

Here the blocks are indexed by $\{1, \dots, k_1\}$, $\{k_1 + 1, \dots, k_1 + k_2\}$,$\{k_1 + k_2 + 1, \dots, k_2 + n_1\}$,$\{k_2 + n_1 + 1, \dots, n_1 + n_2\}$.

The initial state vector $I \in S'^{1 \times (n_1+n_2)}$ and the final state vector $P \in S'^{(n_1+n_2) \times 1}$ are defined by

$$I = \Big(((I_1)_i)_{1 \le i \le k_1}, ((I_2)_i)_{1 \le i \le k_2}, ((I_1)_i)_{k_1+1 \le i \le n_1}, ((I_2)_i)_{k_2+1 \le i \le n_2} \Big),$$

and

$$P = \Big(((P_1)_i)_{1 \le i \le k_1}, ((P_2)_i)_{1 \le i \le k_2}, ((P_1)_i)_{k_1+1 \le i \le n_1}, ((P_2)_i)_{k_2+1 \le i \le n_2} \Big)^\top,$$

with the same block indexing as before.

We have to prove that

$$\|\mathcal{P}\| = \|\mathcal{P}_1\| + \|\mathcal{P}_2\| = (I_1 M_1^* P_1 + I_2 M_2^* P_2) + (I_1 M_1^{\omega,k_1} + I_2 M_2^{\omega,k_2}).$$

The proof of this claim is as follows.

We obtain, for $1 \le i \le n_1 + n_2$,

$$\Big(\big(M^{\omega,k_1+k_2} \big)_{p_0} \Big)_i = \sum_{\pi_1,\pi_2,\ldots \in \Gamma^+} \sum_{\substack{(i_1,i_2,\ldots) \in P_{k_1+k_2} \\ 1 \le i_1,i_2,\ldots \le n_1+n_2}} (M_{p_0,\pi_1})_{i,i_1} (M_{\pi_1,\pi_2})_{i_1,i_2} \cdots$$

For $1 \le i \le k_1$ and $k_1 + k_2 + 1 \le i \le n_1 + k_2$, and by deleting the 0-block rows and the corresponding 0-block columns, we obtain

$$\begin{aligned}
&\Big(\big(M^{\omega,k_1+k_2}\big)_{p_0}\Big)_i \\
&= \sum_{\pi_1,\pi_2,\ldots\in\Gamma_1^+} \sum_{\substack{(i_1,i_2,\ldots)\in P_{k_1} \\ 1\le i_1,i_2,\ldots\le n_1}} \begin{pmatrix} a_{p_1,\pi_1} & b_{p_1,\pi_1} \\ c_{p_1,\pi_1} & d_{p_1,\pi_1} \end{pmatrix}_{i,i_1} \begin{pmatrix} a_{\pi_1,\pi_2} & b_{\pi_1,\pi_2} \\ c_{\pi_1,\pi_2} & d_{\pi_1,\pi_2} \end{pmatrix}_{i_1,i_2} \cdots \\
&= \sum_{\pi_1,\pi_2,\ldots\in\Gamma_1^+} \sum_{\substack{(i_1,i_2,\ldots)\in P_{k_1} \\ 1\le i_1,i_2,\ldots\le n_1}} \Big((M_1)_{p_1,\pi_1}\Big)_{i,i_1} \Big((M_1)_{\pi_1,\pi_2}\Big)_{i_1,i_2} \cdots \\
&= \Big(\big(M_1^{\omega,k_1}\big)_{p_1}\Big)_{i'},
\end{aligned}$$

where $i' = i$ if $1 \le i \le k_1$ and $i' = i - k_2$ if $k_1 + k_2 + 1 \le i \le n_1 + k_2$.

A similar proof yields, for $k_1 + 1 \le i \le k_1 + k_2$ and $n_1 + k_2 + 1 \le i \le n_1 + n_2$, and by deleting the 0-block rows and the corresponding 0-block columns,

$$\Big(\big(M^{\omega,k_1+k_2}\big)_{p_0}\Big)_i = \Big(\big(M_2^{\omega,k_2}\big)_{p_2}\Big)_{i'},$$

where $i' = i - k_1$ if $k_1 + 1 \le i \le k_1 + k_2$ and $i' = i - n_1$ if $n_1 + k_2 + 1 \le i \le n_1 + n_2$.

By similar arguments, we obtain, for $1 \le i, j \le k_1$ and $k_1 + k_2 + 1 \le i, j \le n_1 + k_2$,

$$\Big((M^*)_{p_0,\varepsilon}\Big)_{i,j} = \Big((M_1)^*_{p_1,\varepsilon}\Big)_{i',j'},$$

where $i' = i$ if $1 \le i \le k_1$, $i' = i - k_2$ if $k_1 + k_2 + 1 \le i \le n_1 + k_2$, $j' = j$ if $1 \le j \le k_1$, and $j' = j - k_2$ if $k_1 + k_2 + 1 \le j \le n_1 + k_2$,

and for $k_1 + 1 \le i, j \le k_1 + k_2$ and $n_1 + k_2 + 1 \le i, j \le n_1 + n_2$,

$$\Big((M^*)_{p_0,\varepsilon}\Big)_{i,j} = \Big((M_2)^*_{p_2,\varepsilon}\Big)_{i',j'},$$

where $i' = i - k_1$ if $k_1 + 1 \le i \le k_1 + k_2$, $i' = i - n_1$ if $n_1 + k_2 + 1 \le i \le n_1 + n_2$, $j' = j - k_1$ if $k_1 + 1 \le j \le k_1 + k_2$, and $j' = j - n_1$ if $n_1 + k_2 + 1 \le j \le n_1 + n_2$.

Hence, we obtain

$$\begin{aligned}
IM^*P = {} & \sum_{\substack{1\le i\le k_1 \\ k_1+k_2+1\le i\le k_2+n_2}} \sum_{\substack{1\le j\le k_1 \\ k_1+k_2+1\le j\le k_2+n_1}} I_i M^*_{i,j} P_j + \\
& \sum_{\substack{k_1+1\le i\le k_1+k_2 \\ k_2+n_1+1\le i\le n_1+n_2}} \sum_{\substack{k_1+1\le j\le k_1+k_2 \\ k_2+n_1+1\le j\le n_1+n_2}} I_i M^*_{i,j} P_j \\
= {} & \sum_{1\le i\le n_1} \sum_{1\le j\le n_1} (I_1)_i\,(M_1^*)_{i,j}\,(P_1)_j + \\
& \sum_{1\le i\le n_2} \sum_{1\le j\le n_2} (I_2)_i\,(M_2^*)_{i,j}\,(P_2)_j \\
= {} & I_1 M_1^* P_1 + I_2 M_2^* P_2,
\end{aligned}$$

$$
\begin{aligned}
&IM^{\omega,k_1+k_2}\\
&= \sum_{\substack{1\le i\le k_1\\ k_1+k_2+1\le i\le k_2+n_1}} I_1\left(\left(M^{\omega,k_1+k_2}\right)_{p_0}\right)_i + \sum_{\substack{k_1+1\le i\le k_1+k_2\\ k_2+n_1+1\le i\le n_1+n_2}} \left(I_1\left(M^{\omega,k_1+k_2}\right)_{p_0}\right)_i\\
&= \sum_{1\le i\le n_1} (I_1)_i\left(\left(M_1^{\omega,k_1}\right)_{p_1}\right)_i + \sum_{1\le i\le n_2} (I_2)_i\left(\left(M_2^{\omega,k_2}\right)_{p_2}\right)_i\\
&= I_1\left(M_1^{\omega,k_1}\right)_{p_1} + I_2\left(M_2^{\omega,k_2}\right)_{p_2},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{P}\| &= IM^*P + IM^{\omega,k_1+k_2}\\
&= \left(I_1M_1^*P_1 + I_1\left(M_1^{\omega,k_1}\right)_{p_1}\right) + \left(I_2M_2^*P_2 + I_2\left(M_2^{\omega,k_2}\right)_{p_2}\right)\\
&= \|\mathcal{P}_1\| + \|\mathcal{P}_2\|.
\end{aligned}
$$

□

Proof of Theorem 4.1. We have only to prove implication (*iii*) ⇒ (*iv*). Since $s, s_k, t_k, 1 \le k \le m$, are in $\mathfrak{Alg}\,(S')$, there exist, by Theorem 6.8 of Kuich [11], S'-pushdown automata $\mathcal{P}_s$, $\mathcal{P}_{s_k}$, $\mathcal{P}_{t_k}$ with behaviors $\|\mathcal{P}_s\| = s$, $\|\mathcal{P}_{s_k}\| = s_k$, $\|\mathcal{P}_{t_k}\| = t_k$.

By Lemma 4.1, we can construct S'-ω-pushdown automata $\mathcal{P}'_k$ with behaviors $\|\mathcal{P}'_k\| = t_k^\omega$, $1 \le k \le m$; by Lemma 4.2 S'-ω-pushdown automata $\mathcal{P}_k$ with behaviors $\|\mathcal{P}_k\| = s_k t_k^\omega$, and by Lemma 4.3 an S'-ω-pushdown automaton $\mathcal{P}'$ with behavior $\|\mathcal{P}'\| = \sum_{1\le k\le m} s_k t_k^\omega$. Again by Lemma 4.3, we can construct an S'-ω-pushdown automaton $\mathcal{P}$ with behavior $\|\mathcal{P}\| = \left(s, \sum_{1\le k\le m} s_k t_k^\omega\right)$. □

Algebraic expressions denoting formal power series in $S^{alg}\langle\langle\Sigma^*\rangle\rangle$, S a continuous commutative semiring and Σ an alphabet, are defined in Section 3.5 of Ésik, Kuich [10]. By help of Theorem 4.1 (iii) ω-algebraic expressions denoting pairs $(s, v) \in \omega$-$\mathfrak{Alg}\,(S')$, $S' = S\langle\Sigma \cup \{\varepsilon\}\rangle$, S a continuous star-omega semiring and Σ an alphabet, can easily be defined.

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