On some generalizations of cyclic networks

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Zusammenfassung. Die ersten Abschnitte der Arbeit geben eine vollständige Beschreibung der endlichen zusammenhängenden gerichteten Graphen, die mindestens zwei Zyklen enthalten und in denen jeder Punkt und jede Kante entweder in einem Zyklus oder in (genau) zwei Zyklen liegt. Bezeichnen wir durch C_1 die Klasse dieser Graphen. Sei G ein Element von C_1 und k eine Zahl, die kleiner als die Längen der Zyklen von G ist; bezeichnen wir durch $\mathfrak{A}_k(G)$ den Graphen, dessen Punktmenge mit der Punktmenge von G übereinstimmt, so daß die Kante $\overrightarrow{AB}$ in $\mathfrak{A}_k(G)$ genau dann existiert, wenn $A \neq B$ und B aus A in G durch höchstens $k-1$ Kanten erreichbar ist. Sei C_2 die Klasse aller Graphen $\mathfrak{A}_k(G)$ wobei G die Elemente von C_1 durchläuft.

In den letzten Abschnitten wird es danach bestrebt, die in der früheren Arbeit [2] ausgearbeiteten Untersuchungen (über das Verhalten der Netzwerke mit einer speziellen graphentheoretischer Struktur) auf die in C_2 enthaltenen Graphen zu verallgemeinern. Es gelang nicht, alle erzielten Aussagen zu beweisen, folglich enthält die Arbeit auch unentschiedene Vermutungen (sowohl über die Struktur wie über das Verhalten).

§ 1. Introduction

In [2] certain cyclically symmetric networks were studied. These networks can be obtained in such a manner that we start with a single cycle and draw some additional edges in it.

Let us alter the mentioned procedure so that we start with a graph G satisfying the following four requirements (instead of being a cycle):

G is a finite connected directed graph,

to any edge e of G there exists at least one cycle containing e,

G contains at least two cycles,

whenever z_1, z_2, z_3 are three different cycles of G, then there exists no vertex lying in all of z_1, z_2, z_3.

The collection of these graphs G will be called the class C_1. We shall define a class of graphs (the class C_2) by adding edges to any graph in C_1 in an appriopriate manner. In §§ 3—4 we study the graph-theoretical structure of the members of the classes C_1, C_2; in §§ 6—8 the behaviour of the networks of type C_2 is analyzed. Since I did not succeed in solving all the arising problems, the paper also contains conjectures besides the propositions verified.

§ 2. Some graph-theoretical definitions

We shall always consider finite graphs having at least one edge. "Graph" will mean a *directed* one unless (rarely) we speak of a non-directed tree explicitly. Self-loops are (in general) permitted. Among the graphs containing parallel edges with the same orientation (especially, at least two self-loops on the same vertex), only the two graphs seen on Fig. 1 are allowed (cf. Remark 2 at the end of § 3).

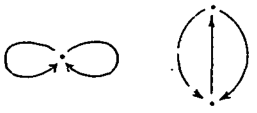

Fig. 1

Let G be a directed graph. A sequence

$$A_0, e_1, A_1, e_2, A_2, \ldots, e_n, A_n \tag{1}$$

consisting of the vertices $A_0, A_1, \ldots, A_n$ and the edges $e_1, e_2, \ldots, e_n$ of G (alternatively) is called a *directed edge sequence* (of length n) if each e_i $(1 \leqq i \leqq n)$ goes from A_{i-1} to A_i. If, in addition, $A_0, A_1, \ldots, A_n$ are different vertices, then (1) is a *path.* If $A_0, A_1, \ldots, A_{n-1}$ are different but $A_0 = A_n$, then (1) is a *cycle*. Let $Z(A)$ be the number of cycles of G which contain the vertex A; let $Z(e)$ be the number defined for the edge e analogously. We denote by M_G the minimal cycle length that occurs in the graph G.

In case of undirected graphs (or if the orientation of the edges is disregarded), the concepts analogous to path and cycle are called *chain* and *circuit*, respectively.

Let A be a vertex of the directed graph G, assume that A is incident to exactly k edges oriented *towards* A and to exactly l edges oriented *outwards from* A. Then we say that the *indegree* of A is k, the *outdegree* of A is l, and the *degree* of A the ordered pair (k, l). — If G is undirected, then the *degree* $d(P)$ of the vertex P is the number of edges incident to P.

Let H be a subgraph of G. If H contains all the vertices (but, possibly, not all the edges) of H, then we say that H is an *e-subgraph* of G. The subgraph H of G is called a *p-subgraph* of G if the following condition is satisfied: whenever A and B are contained in H and the edge e of G is incident to A and B, then e is contained in H too.[1] For each subgraph H of G, there exists exactly one graph $\mathfrak{S}_G(H)$ such that $\mathfrak{S}_G(H)$ is a p-subgraph of G and H is an e-subgraph of $\mathfrak{S}_G(H)$.

Let G be a directed graph fulfilling[2] $M_G \geqq 3$ and k be a number such that $2 \leqq k < M_G$. Let us form a graph H conforming to the following two rules:

the vertex set of H equals to the vertex set of G.

the (directed) edge $\overrightarrow{AB}$ $(A \neq B)$ exists in H if and only if in G there is a path of the length $<k$ from A to B.

The obtained graph H is denoted by $\mathfrak{A}_k(G)$. Obviously, G is an e-subgraph of $\mathfrak{A}_k(G)$ and $\mathfrak{A}_2(G) = G$ is always true.

Let C be a class consisting of directed graphs. Then we denote by $\mathfrak{A}(C)$ the class of all the graphs $\mathfrak{A}_k(G)$ where G runs through the members of C and, for any G, k runs through the numbers satisfying $2 \leqq k < M_G$.

[1] If e is a self-loop, then the same vertex is considered as A as well as B.

[2] The condition $M_G \geqq 3$ means that G contains neither self-loops nor (oppositely oriented) parallel edge pairs.

Now we introduce two classes of connected directed graphs. Let C_1 consist of all the graphs having at least two cycles and satisfying the inequalities

$$1 \leqq Z(A) \leqq 2 \quad \text{and} \quad 1 \leqq Z(e) \leqq 2$$

identically.[3,4] Let C_2 be[5] $\mathfrak{A}(C_1)$.

§ 3. The structure of the graphs in C_1

Construction I. The construction consists of four steps.

Step 1. Let T be a non-directed tree with at least one edge. For each vertex P of T, we denote by $e_1^{(P)}, e_2^{(P)}, \ldots, e_{d(P)}^{(P)}$ the edges incident to P (in an arbitrary manner). (Evidently, every edge gets *two* notations.)

Step 2. Let us form a directed graph G_1 by what follows: the vertices of G_1 correspond one-to-one with the edges of T; if the vertex A of G_1 corresponds to the edge $e_p^{(P)} = e_q^{(Q)}$ of T, then edges go from A to the vertices corresponding to $e_{p+1}^{(P)}$ and $e_{q+1}^{(Q)}$ and only to these vertices (in case $p = d(P)$, $e_1^{(P)}$ plays the role of $e_{p+1}^{(P)}$).

Step 3. Choose a subset V' of the set of vertices of G_1 arbitrarily. For any element A of V', perform the following procedure:

Replace A by two vertices A' and A'';
if an edge had gone to A, then let it go to A',
if an edge had gone from A, then let it go from A'';
finally, supplement the graph with a new edge leading from A' to A''.

Evidently, this process can be carried out for all the vertices in V' simultaneously. We denote the resulting graph by G_2. (See Fig. 2.)

Step 4. Instead of any edge of G_2, we draw a *path* of arbitrary length ($\geqq 1$). (Of course, the inner vertices of these paths have the degree $(1, 1)$.) We denote the resulting graph by G.

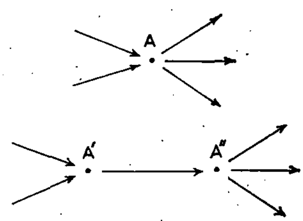

Fig. 2

Theorem 1. Any graph G arising by Construction I belongs to the class C_1.

Proof. First we show that G is connected. It is sufficient to verify that G_1 is connected because Steps 3, 4 cannot spoil the connectedness. Let A, B be two vertices of G_1. If the edges e_A, e_B of T, corresponding to A and B (resp.), are adjacent, then A and B can clearly be joined by a chain. — Let now A, B be arbitrary vertices

[3] The word "identically" means that the conditions are required for each vertex A and for each edge e, respectively.

[4] These four inequalities do not form an independent system: if $Z(A) \leqq 2$ and $Z(e) \geqq 1$ are true, then also $Z(A) \geqq 1$ and $Z(e) \leqq 2$ hold.

[5] Our present notation differs from that of [2]: the graph, denoted by $G(n; 1, 2, \ldots, k)$ in [2], is now denoted by $\mathfrak{A}_{k+1}(z_n)$, where z_n is the cycle of length n.

of G_1. There exists a chain in T with edges

$$e_A = e_1, e_2, \ldots, e_k = e_B.$$

Let the vertices of G_1 corresponding to these edges (respectively) be

$$A = A_1, A_2, \ldots, A_k = B.$$

We have shown that A_i, A_{i+1} can be joined by a chain (for each i, $1 \leqq i < k$); this implies that the same holds for A and B.

Let A be an arbitrary vertex of G_1 and e_A be the corresponding edge of T. There exist two vertices P, Q of T such that $e_A = e_p^{(P)} = e_q^{(Q)}$ (where p, q are suitable numbers). A is of degree (2, 2) by Step 2, and, moreover, A is a cut-vertex since any chain going from $e_{p+1}^{(P)}$ to $e_{q+1}^{(Q)}$ passes through A. These considerations imply that a sequence of vertices of G_1 determines a cycle if and only if it corresponds to the edge sequence

$$e_1^{(P)}, e_2^{(P)}, \ldots, e_{d(P)}^{(P)}$$

for some vertex P of T. Hence $Z(A)=2$, $Z(e)=1$ are identically satisfied in G_1.

Step 3 of the construction does not alter the number of cycles and the identical validity of $Z(A)=2$. For an edge e of G_2, either $Z(e)=2$ or $Z(e)=1$ holds according as e is a new edge (i.e. going from an A' to an A'') or not.

Step 4 does not modify the number of cycles, either. Denote by e' an edge of G_2, let A be an arbitrary inner vertex and e be an arbitrary edge of the path (in G) replacing e' by virtue of Step 4. We have obviously $Z(A)=Z(e)=Z(e')$. If A is a vertex of G_2, then $Z(A)=2$ holds in G as well as in G_2. Thus $1 \leqq Z(A) \leqq 2$ and $1 \leqq Z(e) \leqq 2$ are identically satisfied in G.

Lemma 1. Assume that the graph G satisfies $Z(A)=2$ and $Z(e)=1$ identically. Then any two cycles of G have at most one vertex in common.

Proof. Let z_1, z_2 be two cycles containing (at least) two common vertices. Let A be a common vertex such that the edges of z_1 and z_2, starting from A, are different. Let us pass from A on z_1 to the first other common vertex $B(\neq A)$, then let us pass from B to A on z_2. Thus we have got a third cycle containing A; this, however, contradicts $Z(A)=2$.

Theorem 2. Every graph G belonging to the class C_1 may be produced by Construction I.

Proof. Let G be contained in C_1. The condition $Z(A) \geqq 1$ implies that any vertex of G has a positive outdegree and a positive indegree. Neither the outdegree nor the indegree of a vertex A can exceed 2, because if e.g. the indegree were $k(>2)$, then each of the k edges starting from A could be extended to a cycle, hence $Z(A) \geqq \geqq k > 2$ would follow; this is a contradiction.

Thus the degree of any vertex of G is either (1, 1) or (2, 1) or (1, 2) or (2, 2). There is at least one vertex whose degree differs from (1, 1) (otherwise G would be a single cycle).

In what follows, we shall define a decomposition procedure for G that consists of four steps corresponding to Steps 4., 3., 2., 1. of Construction I, respectively.

Step 1. If A is of degree (1, 1), then we delete A and contract the two edges incident to A into one edge. This can be performed for all the vertices with degree

$(1, 1)$ simultaneously (without essential difficulties). Let us denote the resulting graph[6] by G_1'. It is clear that $G_1' \in C_1$, and, furthermore, that only the degrees $(2, 1)$, $(1, 2)$, $(2, 2)$ may occur in G_1'.

Now we establish three lemmas on G_1' (the proof of Theorem 2 will be continued later).

Lemma 2. Let e be an edge of G_1', going from A to B. Then $Z(e)=2$ if and only if $d(A)=(2, 1)$ and $d(B)=(1, 2)$.

Proof. The sufficiency is trivial. Conversely, suppose $Z(e)=2$; if the outdegree of A is 2, then $Z(A)\geqq 3$; if the indegree of B is 2, then $Z(B)\geqq 3$.

Lemma 3. Let e, A, B be as in Lemma 2. Then $Z(e)=1$ if and only if $d(A)$ is either $(1, 2)$ or $(2, 2)$ and $d(B)$ is either $(2, 1)$ or $(2, 2)$.

Proof. First we show that each of the following four statements leads to a contradiction:

(a) $d(A)=(2, 1)$ and $d(B)=(2, 1)$
(b) $d(A)=(1, 2)$ and $d(B)=(1, 2)$
(c) $d(A)=(2, 1)$ and $d(B)=(2, 2)$
(d) $d(A)=(2, 2)$ and $d(B)=(1, 2)$.

Indeed, (a) implies $Z(e')\geqq 3$ for the single edge e' going out from B, (c) implies $Z(B)\geqq 3$; (b) and (d) can be disproved analogously (by interchanging A and B).

Since the possibilities (a)—(d) and the ones of Lemma 2 are excluded, only those allowed in Lemma 3 remain.

Lemma 2 implies immediately.

Lemma 4. If $Z(e)=Z(e')=2$ for two different edges e, e' of G_1', then e and e′ are not adjacent.

Proof of Theorem 2 (continued).

Step 2. Consider the graph G_1' (resulting by Step 1), and choose an edge e of G_1' satisfying $Z(e)=2$. Contract the two vertices A, B incident to e into one vertex (i.e. delete e, A and B, and introduce a new vertex C so that any edge ($\neq e$) which has been incident to A or B will now be incident to C).[7] This process can be performed for all the edges fulfilling $Z(e)=2$ simultaneously (by Lemma 4). Let the resulting graph be denoted by G_2'. Obviously, $G_2' \in C_1$, and, moreover, $d(A)=(2, 2)$, $Z(A)=2$, and $Z(e)=1$ are identically valid in G_2'.

Step 3. Consider G_2', and define an undirected graph T' in the following manner: the vertices of T' correspond in a one-to-one way to the cycles of G_2'; two vertices P, Q of T' are joined by an edge if and only if the corresponding cycles of G_2' have a vertex in common.

Next we state two lemmas on T'. The first of them follows from $Z(A)=2$ (holding in G_2') and Lemma 1 at once:

[6] If parallel edges with the same orientation do not occur in G, then either the same holds for G_1' or G_1' is one of the graphs of Fig. 1.

[7] If there has been an edge e' going from B to A (of course, satisfying $Z(e')=1$), then e' will become a self-loop of the new vertex C.

Lemma 5. Let us assign to any edge e of T' the (unique) common vertex of the two cycles in G_2' corresponding to the vertices incident to e. This assignment is a one-to-one correspondence between all the edges of T' and all the vertices of G_2'.

Lemma 6. T' is a tree.

Proof. First we show that T' has no circuit. Assume that t is a circuit of minimal length in T', let t consist of

$$P_1, e_1, P_2, e_2, \ldots, P_k, e_k, P_1 \qquad (k \geqq 3)$$

(i.e. $P_1, P_2, \ldots, P_k$ are the vertices and $e_1, e_2, \ldots, e_k$ are the edges of t, passed through as they follow). Let

$$A_1, A_2, \ldots, A_k$$

be the vertices of G_2' corresponding to

$$e_1, e_2, \ldots, e_k$$

(resp.) and

$$z_1, z_2, \ldots, z_k$$

be the cycles of G_2' corresponding to

$$P_1, P_2, \ldots, P_k$$

(resp.). Let us form a directed edge sequence in G_2' so that we pass

on z_1 from A_k to A_1, afterwards

on z_2 from A_1 to A_2,

on z_3 from A_2 to A_3,

...

finally, on z_k from A_{k-1} to A_k.

This sequence z is a cycle (otherwise t cannot be minimal). Thus $Z(A_i) \geqq 3$ $(1 \leqq i \leqq k)$, which is a contradiction.

We are going to show that T' is connected. Suppose the contrary. The disconnectedness of T' implies (by Step 3) that G_2' is either disconnected or has an edge e fulfilling $Z(e) = 0$. Both alternatives are contradictory (the first one is because the connectedness of G_2' is equivalent to the connectedness of G, by Steps 1, 2).

Proof of Theorem 2 (final part). The proof is completed by noting that the decomposition procedure, described in this proof (together with Lemmas 2—6), is an exact counterpart of Construction I.

Remark 1. To a vertex A of the graph G_1 (produced by Step 2 of Construction I) a self-loop is incident exactly if the edge in T, corresponding to A, is a final edge in T (i.e. it is incident to a vertex of degree 1). A graph G produced by Construction I contains no self-loop (i.e. cycle of length 1) exactly if each self-loop of G_1 is eliminated either in Step 3 or in Step 4.

Remark 2. It is easy to see that if a connected directed graph G satisfying $Z(e) \geqq 1$ identically has two parallel edges with the same orientation, then either G is one of the graphs of Fig. 1 or G has a vertex A such that $Z(A) \geqq 3$. This fact justifies the agreement posed in the fourth sentence of § 2.

Remark 3. A graph G_1 (produced by Step 2 of Construction I) contains a pair of oppositely oriented parallel edges (i.e. a cycle of length 2) exactly if T has a vertex of degree 2. A graph G produced by Construction I does not contain a pair of oppositely oriented parallel edges exactly ife ach pair of this property of edges of G_1 is eliminated either in Step 3 or in Step 4 (of course, the possibility that G_1 contains no such pair is included).

Problem. How to describe all connected directed graphs fulfilling $1 \leqq Z(e) \leqq 2$ identically?[8]

§ 4. Some conjectures on the class C_2

By definition, each graph G contained in the class C_2 has at least one e-subgraph G' such that $G = \mathfrak{A}_k(G')$ where k is a suitable number fulfilling $k < M_{G'}$. It is an open problem whether or not the statement of unicity of this presentation holds. This problem would be solved in the affirmative sense if a method were given for constructing G' from G such that the resulting graph G' is the unique e-subgraph such that $G = \mathfrak{A}_k(G')$. In this § some conjectures related to this question will be exposed. The unicity statement is formulated in Conjecture 3.

In what follows, we shall make use of two further classes of connected directed graphs. Let C_3 contain a graph G if and only if G has an automorphism α such that α permutes the vertices of G cyclically and there exists an edge from A to $\alpha(A)$ for any vertex A.[9] Let G belong to the class C_4 exactly if the following assertion is fulfilled: whenever

$$G = \mathfrak{A}_k(G'), \quad k < M_{G'} \quad \text{and} \quad G' \in C_1$$

are satisfied for G', k, and z is a cycle of G', then[10, 11] $\mathfrak{S}_G(z) = \mathfrak{A}_k(z)$.

[8] This condition implies the identical fulfilment of $Z(A) \geqq 1$. Fig. 3 shows a graph in which $Z(e) = 1$ for each edge and $Z(A) = 3$ for some vertex.

[9] The relation $G \in C_3$ holds exactly if the vertices of G can be labelled by the numbers $1, 2, \ldots, n$ such that

$$G = G(n; 1, m_2, m_3, \ldots, m_k),$$

where n is the number of vertices of G, k is a suitable number and, on the right-hand side, the notation means that an edge $\overrightarrow{A_i A_j}$ exists exactly if $i-j$ is congruent to one of $1, m_2, \ldots, m_k$ modulo n.

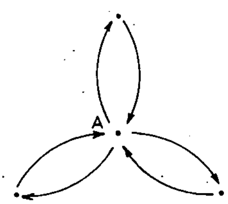

Fig. 3

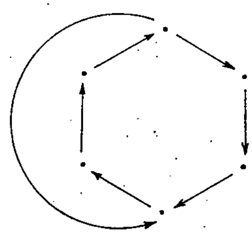

Fig. 4

[10] The inclusion $\mathfrak{S}_k(z) \supseteqq \mathfrak{A}_k(z)$ is trivially satisfied; we require now the converse inclusion. It is obvious that $\mathfrak{A}_k(z) \in C_3$.

[11] Let G' be the graph of Fig. 4 and z be the longer cycle of G'. Evidently $\mathfrak{A}_3(G') \notin C_4$. If all the cycles of G' are of the same length, then $\mathfrak{A}_k(G') \in C_4$.

Let G be a graph in C_2. If a p-subgraph G_1 of G belongs to the class C_3, then we say that G_1 is a C_3-subgraph of G. If G_1 is a C_3-subgraph of G and there exists no C_3-subgraph G_2 of G such that $G_1 \subset G_2 \subset G$, then we say that G_1 is a maximal C_3-subgraph of G.

Conjecture 1. Let G be a graph contained in C_2. Let G' be an e-subgraph of G and k be a natural number such that

$$G' \in C_1, \quad k < M_{G'} \quad \text{and} \quad G = \mathfrak{A}_k(G').$$

If G_1 is a C_3-subgraph of G, then there exists a cycle z of G' such that z contains all the vertices of G_1.

Conjecture 2. Let G, G' be two graphs as in Conjecture 1. Assume that G is contained in C_4. A p-subgraph G_1 of G is a maximal C_3-subgraph of G if and only if there exists a cycle z of G' such that $G_1 = \mathfrak{S}_G(z)$.

Proposition 1. If Conjecture 1 holds, then so does Conjecture 2 as well.

Proof. Let z be a cycle of G'. $\mathfrak{S}_G(z)$ is a C_3-subgraph of G in consequence of $G \in C_4$. Let G_2 be a proper C_3-subgraph of G such that $\mathfrak{S}_G(z) \subset G_2$. Conjecture 1 implies the existence of a cycle z' of G' containing all the vertices of G_2. The vertex set of z is a proper subset of the vertex set of z'; this contradiction shows that $\mathfrak{S}_G(z)$ is a maximal C_3-subgraph, thus the sufficiency statement of Conjecture 2 is proved.

Conversely, let G_1 be an arbitrary maximal C_3-subgraph of G. Consider $\mathfrak{S}_G(z)$, where z is the cycle whose existence is stated in Conjecture 1. $\mathfrak{S}_G(z)$ is a C_3-subgraph of G by $G \in C_4$. The maximality of G_1 implies $G_1 = \mathfrak{S}_G(z)$.

Conjecture 3. Suppose $G \in C_2$. Then there exists exactly one pair (G', k) (consisting of an e-subgraph G' of G and of a natural number k) such that $G = \mathfrak{A}_k(G')$.

Proposition 2. If Conjecture 2 holds and $G \in C_2 \cap C_4$, then the conclusion of Conjecture 3 is valid for G.

Proof. Let G', G'' be two e-subgraphs of G and k_1, k_2 be natural numbers such that

$$k_1 < M_{G'}, \quad k_2 < M_{G''}, \quad G' \in C_1, \quad G'' \in C_1, \quad G = \mathfrak{A}_{k_1}(G') = \mathfrak{A}_{k_2}(G'').$$

Conjecture 2 implies the equivalence of the following three assertions (i), (ii), (iii) for a p-subgraph G_1 of G:

(i) the vertices of G_1 coincide with the vertices of a cycle of G',

(ii) G_1 is a maximal C_3-subgraph of G,

(iii) the vertices of G_1 coincide with the vertices of a cycle of G''.

Hence the vertex sets of the cycles of G' coincide with the vertex sets of the cycles of G''. Let z' be a cycle of G' and z'' be a cycle of G'' such that z', z'' contain precisely the same vertices; let A be a vertex of z' (and of z''). We shall label the vertices (in question) as they follow A on z' or on z''. From A, edges (of G) go to the first, second, ..., k_1-th vertices (and only to these) of z'; analogously, from A edges go to the first, second, ..., k_2-th vertices (and only to these) of z''. This implies $z' = z''$ and $k_1 = k_2$, thus also $G' = G''$ (because $Z(e) \geqq 1$ is identically satisfied in G' and in G'').

§ 5. Some lemmas

Let A, B be two vertices of a graph G and a be a directed edge sequence from A to B. It is well-known that we can select a path a_1 from a such that a_1 leads from A to B, too; more precisely, a_1 may be constructed by iterating the method that we omit a cycle out of a directed edge sequence (unless it is a path). This fact will be used sometimes in this §.

Lemma 7. Let A, B be two vertices of a connected graph G. If $Z(e) \geqq 1$ is identically satisfied in G, then there exists a path a of G such that the beginning vertex of a is A and the end vertex of a is B.[12]

Proof. First we show that the conclusion is satisfied by some directed edge sequence. Since G is connected, there exists a chain whose vertices are

$$A = A_0, A_1, A_2, \ldots, A_{m-1}; A_m = B$$

where m is the length of the chain. For every subscript i $(0 \leqq i < m)$ there exists either the edge $\overrightarrow{A_i A_{i+1}}$ or the edge $\overrightarrow{A_{i+1} A_i}$.

Suppose that there exists a directed edge sequence b from A to A_i $(0 \leqq i < m)$, we shall prove the analogous statement for A, A_{i+1}. If $\overrightarrow{A_i A_{i+1}}$ does exist, then the existence of the required sequence is obvious. If $e = \overrightarrow{A_{i+1} A_i}$ exists, then let c be the path which originates from a cycle containing e by deleting e. b and c form together a directed edge sequence from A to A_{i+1}. We can select a path from the directed edge sequence constructed above between A and B. This completes the proof.

In the subsequent lemmas, we consider a graph G belonging to C_1 and we denote by d the greatest common divisor of the lengths of all cycles of G. For any pair A, B of vertices of G, the number of cycles containing both A and B is either 0 or 1 or 2.

Lemma 8. Let G be a graph belonging to C_1 and A, B be two vertices of G. Denote by $\pi(A, B)$ the number of paths going from A to B. The following three assertions are true:

(a) *If there is at most one cycle containing both A and B, then*

$$\pi(A, B) = 1.$$

(b) *If there exist two cycles containing both A and B, then*

$$\text{either} \quad \pi(A, B) = 2 \quad \text{and} \quad \pi(B, A) = 1,$$

$$\text{or} \quad \pi(A, B) = 1 \quad \text{and} \quad \pi(B, A) = 2.$$

(c) *Suppose that the first alternative of* (b) *holds. Let l_1, l_2 be the lengths of the paths leading from A to B and l_3 be the length of the path going from B to A. Then*

$$l_1 \equiv l_2 \equiv -l_3 \pmod{d}.$$

[12] If A and B coincide, then a path of length 0 fulfils the conclusion.

Remark. The assertions (a), (c) hold also in symmetrized form (by interchanging A and B).

Proof. We use induction with respect to the number of cycles of G. If G has two cycles, then the lemma is evidently valid.

Assume that the number of cycles is m and the lemma is true for the graphs having at most $m-1$ cycles. We shall rely upon Construction I without any explicit reference (this is justified by Theorem 2). Let z be a cycle of G such that z corresponds to a vertex of degree 1 of T. G has exactly one cycle z_1 such that z, z_1 have at least one vertex in common. The vertices of z can be denoted (uniquely) by

$$F_1, F_2, \dots, F_t, \quad D_1, D_2, \dots, D_w$$

such that z passes through the vertices in this ordering and exactly $F_1, F_2, \dots, F_t$ are the common vertices with z_1. Also z_1 passes over the F's according to increasing subscripts. (See Fig. 5.) $t=1$ is possible. $t=1$ implies that the degree of F_1 is (2, 2), $t>1$ implies that the degree of F_1 is (2, 1) and the degree of F_t is (1, 2); in both cases, all the remaining vertices of z are of degree (1, 1).

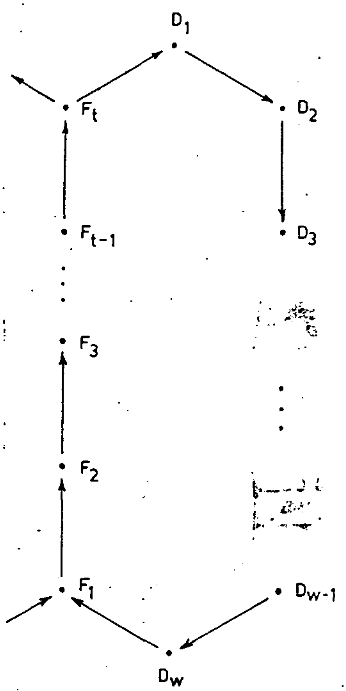

Fig. 5

Denote by G_1 the graph resulting if $D_1, D_2, \dots, D_w$ (and the edges incident to them) are deleted. Clearly $G_1 \in C_1$.

We distinguish six cases with respect to the situation of A and B. (The cases arising when A, B are interchanged are not treated separately.)

Case 1: neither A nor B occurs in z. Then the connectibility of A and B is the same in G as in G_1.

Case 2: $A=D_i$ and $B=D_j$ (where $1 \leqq i < j \leqq w$). Then (a) is trivially fulfilled.

Case 3: $A=F_i$ and $B=F_j$ ($1 \leqq j < i \leqq t$). We have

$$\pi_1(A, B) = \pi_1(B, A) = 1$$

for the function π_1 defined in G_1, hence

$$\pi(A, B) = 2 \quad \text{and} \quad \pi(B, A) = 1,$$

i.e. the first alternative of (b) holds. Let l_1, l_2 be the lengths of the paths from A to B along z_1, z, respectively; let l_1^*, l_2^* be the lengths of z_1, z (resp.); let l_3 be the length of the path from B to A. Then

$$l_1 + l_3 = l_1^*, \quad l_2 + l_3 = l_2^*,$$

hence, on the one hand, $d \mid l_1^* = l_1 + l_3$, thus

$$l_1 \equiv -l_3 \pmod d;$$

on the other hand, $l_1 - l_2 = l_1^* - l_2^*$. Since both of l_1^*, l_2^* are multiples of d, the same holds for their difference, thus

$$l_1 \equiv l_2 \pmod d.$$

Also (c) is verified.

Case 4: $A = D_i$ and $B = F_j$ $(1 \leqq i \leqq w,\ 1 \leqq j \leqq t)$. (a) is trivially fulfilled.

Case 5: A does not occur in z and $B = F_i$ $(1 \leqq i \leqq t)$. (a) follows from the induction hypothesis.

Case 6: A does not occur in z and $B = D_i$ $(1 \leqq i \leqq w)$. Because $\pi(A, F_t) = 1$ by Case 5, it is clear that $\pi(A, B) = 1$. — Analogously, $\pi(F_1, A) = 1$, hence $\pi(B, A) = 1$. The proof is completed.

Lemma 8 implies immediately

Lemma 9. Let G be a graph belonging to C_1 and A, B be two vertices of G. If a, b, c are three directed edge sequences such that both of a, b lead from A to B and c goes from B to A, then

$$l_1 \equiv l_2 \equiv -l_3 \pmod d,$$

where l_1, l_2, l_3 are the lengths of a, b, c respectively.

§ 6. Some notions concerning the behaviour of networks

We recall the continuous model of the behaviour of a network,[13] exposed in Section 3 of [1]. The subsequent treatment is — essentially — an extension of that of [2]. The mentioned behaviour may be shortly summarized as follows:

(1) To any vertex A_i a function $\alpha_i(t)$ is assigned. The domain of α_i is either the (real) interval $[0, \infty)$ or an interval $[0, T'_{\max})$ where $T'_{\max}$ is some positive number (common for the vertices). The range of α_i is the interval $[0, 1]$.

(2) For any number t lying in the domain of the functions α_i (where $1 \leqq i \leqq n$, n is the number of vertices), if the edge $\overrightarrow{A_j A_k}$ exists and $\alpha_j(t) = 1$, then $\alpha_k(t) = 0$.

(3) The initial values $\alpha_i(0)$ of the functions are assumed to fulfil the requirement posed in (2) (with 0 as t).

(4) If the value of the function α_i is less than 1, then it increases linearly unless it must be 0 in consequence of (2).

(5) If the value of the function α_i is 1, then it remains constantly 1 unless it must be 0 in consequence of (2).

(6) If $\overrightarrow{A_j A_k}$ exists and the function α_j, α_k reach the value 1 at some instant t_0 simultaneously, then (t_0 is denoted by $T'_{\max}$ and) the functions are not defined for numbers $t \geqq t_0$.

If the functioning of a network is defined at an instant t, then the vector

$$\mathfrak{B} = \langle \alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t) \rangle$$

is called the *state* of the network at t. Let us form the state $\mathfrak{D}$ of the network at

[13] By a network we mean a graph (without self-loops) together with numerical functions, depending on the time, assigned to the vertices in a one-to-one manner.

the instant $t+t'$ (where t' is a non-negative number and this new state $\mathfrak{D}$ is formed from $\mathfrak{B}$ in agreement with the above rules (1)—(6)); $\mathfrak{D}$ is denoted by $\mathfrak{B}[+t']$ too.

The state $\mathfrak{B}$ is called *cyclic* if there exists a positive t such that $\mathfrak{B}=\mathfrak{B}[+t]$; each suitable t is called a *period* of the network with the initial state $\mathfrak{B}$. $\mathfrak{B}$ is called *steady* if $\mathfrak{B}=\mathfrak{B}[+t]$ is true for every positive t. Any steady state is obviously cyclic. By a *proper cyclic state* a non-steady cyclic state is meant. If $\mathfrak{B}$ is a proper cyclic state, then clearly there exists a positive number t_0 such that $\mathfrak{B}=\mathfrak{B}[+t]$ holds exactly if $t=gt_0$ where g can be $0, 1, 2, 3, \dots$.

In the remaining part of this §, the concept of regular state will be introduced. Let us consider a graph G' belonging to the class C_1. Denote by d the greatest common divisor of the lengths of the cycles of G'. We define a partition Π of the vertex set of G' in the following manner: let $A\equiv B \pmod{\Pi}$ be true exactly if there exists a path a (of length $\geqq 0$) such that the beginning vertex of a is A, the end vertex of a is B and the length of a is a multiple of d. We have to show that Π is an equivalence.

Lemma 10. The relation Π is reflexive, symmetric and transitive.

Proof. The reflexivity is evident since paths of length 0 are allowed.

Next we prove the symmetry. Suppose $A\equiv B \pmod{\Pi}$. There exists a path a from A to B and a path b from B to A by Lemma 7. The length of a is a multiple of d by the supposition; either the fact that a, b form together a cycle or Lemma 8 (c) implies that also the length of b is a multiple of d; consequently, $B\equiv A \pmod{\Pi}$.

Finally, we show the transitivity. Assume $A\equiv B \pmod{\Pi}$ and $B\equiv C \pmod{\Pi}$. There exists a path a from A to B and a path b from B to C such that the lengths of a and b are multiples of d. Hence C can be reached from A on a directed edge sequence whose length is $\equiv 0$ modulo d. By Lemmas 7, 8, 9, the same holds for the path(s) leading from A to C (and there exists such a path).

Lemma 11. Let A, B, C, D be four vertices of G'. If $A\equiv B \pmod{\Pi}$, and there exists an edge from A to C, and there exists an edge from B to D, then $C\equiv D \pmod{\Pi}$.

Proof. Lemmas 7, 9 and the definition of Π imply the existence of two paths a, b such that a goes from C to A, the length of a is $\equiv -1 \pmod{d}$, b goes from A to B, the length of b is $\equiv 0 \pmod{d}$. Hence the directed edge sequences, going from C to D, are of length congruent with $-1+0+1=0 \pmod{d}$, thus $C\equiv D \pmod{\Pi}$. The proof is completed.

Since Lemma 11 is valid and $Z(A)\geqq 1$ is identically satisfied, it is easy to see that there are d equivalence classes modulo Π and we can label these classes by

$$E_1, E_2, \dots, E_d \tag{6.1}$$

so that if an edge comes from a vertex in E_i $(1\leqq i\leqq d)$, then it terminates at an element of E_{i-1} (where, of course, E_d plays the role of E_{1-1}). This enumeration of the classes is unique apart from cyclic translation.

Let us consider a network $G(\in C_2)$, having the vertices $A_1, A_2, \dots, A_n$, and an e-subgraph $G'(\in C_1)$ of G such that $G=\mathfrak{A}_k(G')$ and $2\leqq k<d$.[14] A state

$$\mathfrak{B}=\langle \alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)\rangle$$

[14] Throughout the following parts of the paper, this terminology will be used extensively.

of G will be called *regular*[15] (at the instant t) if it satisfies the following three conditions:

(a) If $A_i \equiv A_j \pmod{\Pi}$, then $\alpha_i(t) = \alpha_j(t)$.

(b) If $\alpha_i(t) = 1$ for the vertices A_i lying in a class E_j (occurring in (6. 1)), then $\alpha_{i'}(t) = 0$ for every

$$A_{i'} (\in E_{j-1} \cup E_{j-2} \cup \dots \cup E_{j-k+1})$$

where the expressions $j-1, j-2, \dots$ are meant modulo d.

(c) If $0 \leqq \alpha_i(t) < 1$, $A_i \in E_j$ and $\alpha_m(t) < 1$ for every

$$A_m (\in E_{j+1} \cup E_{j+2} \cup \dots \cup E_{j+k-1}),$$

then $\alpha_i(t) < \alpha_{i'}(t)$ for each $A_{i'} (\in E_{j-1})$, where the expressions $j+1, j+2, \dots$ and $j-1$ are again viewed modulo n.[16]

By comparing the notions of cyclic and regular states with how these concepts had been introduced in [2], one can ascertain that the cyclic states were defined in precisely the same manner and the regularity was introduced in an almost full analogy (the difference is motivated by the modification of the graph-theoretic structure).

§ 7. The cyclicity of regular states

Consider a network $G (\in C_2)$ (as in the definition of the regular state). Suppose that we start with a regular state of G at the instant 0. The behaviour of G may be studied in detail in analogy to the discussion in § 2 of [2]. In studying a function α_i assigned to a vertex A_i, the only modification here is that now the sequence of sets

$$H_1^{(i)}, H_2^{(i)}, H_3^{(i)}, \dots$$

must be considered, where $H_h^{(i)}$ consists of the vertices from which A_i is reachable by a path of length h (instead of the vertex sequence

$$P_{i+1}, P_{i+2}, P_{i+3}, \dots$$

in [2]); clearly, any set $H_h^{(i)}$ is a subset of E_{j+k} (where h can be 1, 2, 3, ..., and j is determined by $A_i \in E_j$), thus any two vertices lying in a common $H_h^{(i)}$ have the same initial value (by the requirement (a) in the definition of the regular state). The discussion and inferences, being in analogy with the respective parts of [2], lead to the following statements:

Proposition 3. If we start with a regular state at the instant 0 *and* $A_{i_1} \in E_j$, $A_{i_2} \in E_{j+k}$, *then*

$$\alpha_{i_1}(\tau) = \alpha_{i_2}(0)$$

(*the expression* $j+k$ *is meant* mod d).

Denote by g the least common multiple of d and k.

Proposition 4. Any regular state is cyclic, $g\tau/k$ *is a suitable period.*

[15] The regularity of a state depends on which e-subgraph of G is distinguished as G'. If Conjecture 3 is valid, then this dependence is apperent only.

[16] If $A_i \in E_j$, $\alpha_i(t) = 1$ and $A_{i'} \in E_{j-k}$, then both $\alpha_{i'}(t) = 0$ and $\alpha_{i'}(t) > 0$ are permitted.

Proposition 5. If $\mathfrak{B}$ is a regular state, then the state $\mathfrak{B}[+t]$ is regular for each non-negative t.

Proposition 6. Let $\mathfrak{B}$ be a regular state. $\mathfrak{B}$ is steady if and only if k is a divisor of d and there exists a number j such that $1 \leqq j \leqq d/k$ and the equivalence

$$\alpha_i(0)=1 \Leftrightarrow A_i \in E_j \cup E_{j+k} \cup E_{j+2k} \cup E_{j+3k} \cup \cdots \cup E_{j+d-k}$$

holds.

§ 8. On the regularity of cyclic states

Our Proposition 4 is an exact analogon of Proposition 2 of the article [2]. For the networks of the type investigated in [2], the converse statement is true as well: only the regular states are cyclic ([2], Proposition 8). Now we are going to make some considerations (without any claim for completeness) on the question whether or not a similar assertion concerning the networks lying in C_2 holds.

First we characterize the steady states (without presupposing the regularity). Let A, B be two vertices of a graph G; we say that A is k-reachable from B if there exists a path of length $\leqq k$ from B to A ($A=B$ is permitted).

Proposition 7. Let G, k, G' have the same meaning as in the definition of the regular state. Let $\mathfrak{B}$ be a state of the network G (at the instant 0). Denote by H the set of the vertices A_i satisfying $\alpha_i(0)=1$. The state $\mathfrak{B}$ is steady if and only if the following three conditions are fulfilled:

(i) *$A_i \notin H$ implies $\alpha_i(0)=0$ for all the vertices A_i of G.*

(ii) *If $A \in H$, $B \in H$ and A is $(k-1)$-reachable from B in G', then $A=B$.*

(iii) *To any vertex A of G there exists a vertex $B(\in H)$ such that A is $(k-1)$-reachable from B in G'.*[17]

Proof. If (i), (ii), (iii) are fulfilled, then at each vertex of G outside H at least one edge of G coming from an element of H terminates (by the operation $\mathfrak{A}_k$), hence all the initial values $\alpha_i(0)$ remain unchanged.

Assume that one of (i), (ii), (iii) is not satisfied. If (ii) were not true, then state in question would not be permitted. If either (i) or (iii) were not valid, then a vertex A_i would exist such that $\alpha_i(t)$ would increase in an interval $[0, t')$ with an appropriate positive t'. Thus the state could not be steady. The proof is complete.

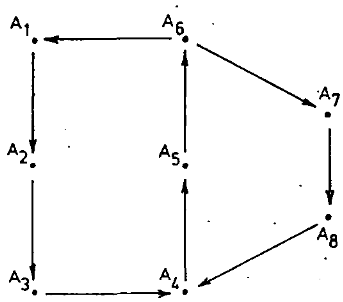

Fig. 6

[17] In case $A \in H$ the statement is satisfied with $B=A$ trivially.

Consider the graph $G=\mathfrak{A}_3(G')$, where G' is the graph on Fig. 6. If we put

$$\beta_1=\beta_2=\beta_4=\beta_5=\beta_7=\beta_8=0, \quad \beta_3=\beta_6=1 \quad (\text{where } \beta_i=\alpha_i(0))$$

then we get a steady state (since (i), (ii), (iii) are fulfilled) that is not regular. Thus the statement "any cyclic state is regular" does not hold. However, it can be expected that all non-regular cyclic states are steady, or (equivalently):

Conjecture 4. Any proper cyclic state of a network of type C_2 is regular.

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