Cutting plane methods for solving nonconvex programming problems

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1. Introduction

It is well known that the solution of the programming problem

$$f(\mathbf{x}) \to \max \tag{1.1}$$

subject to

$$\mathbf{x} \in L,$$

— where L is a subset of the Euclidean n-space E^n and f is a scalar-valued function — can be very difficult unless L is convex and $f(\mathbf{x})$ is quasiconcave (see: [1], [2]). For special cases of (1. 1) efficient methods have been developed among which the so called "cutting plane" methods are of considerable importance (see: [3], [4], [5], [6]).

In this paper we want to apply the cutting plane idea — developed originally in [6] for quadratic objective function, in [5] and later but independently in [7] for convex objective function — to more general programming problems including such as

- maximizing a quasiconvex function over a convex polyhedron
- maximizing a quasiconvex function over the lattice points of a convex polyhedron
- mixed zero-one integer programming with convex objective function to be maximized
- fixed charge problems with convex objective function
- separable nonlinear programming with linear constraints
- general continuous nonlinear programming
- general pure integer programming.

2. A method for accelerating the full description method

Let the problem be the following[1]:

$$f(\mathbf{x}) \to \max \tag{2.1}$$

subject to

$$\mathbf{Ax} \leqq \mathbf{b},$$

[1] Throughout the paper $\mathbf{A}, \mathbf{B} \ldots$ denote matrices, $\mathbf{a}, \mathbf{b} \ldots$ denote vectors, $*$ stands for transposition and $\mathbf{e}_j$ is the j^{th} identity vector.

where $\mathbf{x} \in E^n$, $\mathbf{b} \in E^m$, A is an m by n matrix, $L = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}$ is nonempty and bounded, $f(\mathbf{x})$ is continuous and quasiconvex over the whole E^n. This latter means that for all $\mathbf{x}_1, \mathbf{x}_2$ and λ $(0 \leqq \lambda \leqq 1)$

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leqq \max \{f(\mathbf{x}_1), f(\mathbf{x}_2)\}. \tag{2. 2}$$

It is known ([2]) that among the global maximumpoints of (2. 1) there is at least one extreme point of L. This gives a basis to a method of solving (2. 1) called "full description method" ([8], [9]) which generates in some way all extreme points of L and then we can choose that (those) extreme point(s) which give(s) the maximal objective function's value. Unfortunately in cases of practical problems this method fails because of the large number of extreme points.

In this section we give a method which is based on an arbitrary variant of the full description method but it does not require usually the determination of all extreme points.

We shall call the method capable of leading us through all extreme points of L the "wandering method". (A realization of a "wandering method" is e.g. [8] and [9]). We call a point $\hat{\mathbf{x}}$ of the convex polyhedron L a nondegenerate basic solution if $\mathbf{A}$ and $\mathbf{b}$ can be partitioned in the following manner:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \quad \begin{array}{l} \mathbf{A}_1 \hat{\mathbf{x}} = \mathbf{b}_1, \\ \mathbf{A}_2 \hat{\mathbf{x}} < \mathbf{b}_2, \end{array} \tag{2. 3}$$

where $\mathbf{A}_1$ is nonsingular. All other basic solutions are called degenerate.

To begin with let us determine an extreme point of L, say $\mathbf{x}_0$. If $\mathbf{x}_0$ is degenerate, then applying the "wandering method" find a nondegenerate basic solution $\overline{\mathbf{x}}_0$. If $\mathbf{x}_0$ is nondegenerate, then $\overline{\mathbf{x}}_0 = \mathbf{x}_0$. Let the maximal objective function's value through the path leading from $\mathbf{x}_0$ to $\overline{\mathbf{x}}_0$ be $C_0' = C_0$. If all basic solutions of L are degenerate, then we have to determine all extreme points. In this case our method reduces to the full description method and $C_0 = \max\limits_{\mathbf{x} \in L} f(\mathbf{x})$.

Since $\overline{\mathbf{x}}_0$ is nondegenerate we can transform (2. 2) into the equivalent problem:

$$f(\overline{\mathbf{x}}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{O}, \tag{2. 4}$$

$$\mathbf{A}_3 \mathbf{y} \leqq \mathbf{b}_3,$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \quad \begin{array}{l} \mathbf{A}_1 \overline{\mathbf{x}}_0 = \mathbf{b}_1, \\ \mathbf{A}_2 \overline{\mathbf{x}}_0 < \mathbf{b}_2, \end{array} \quad \mathbf{y} = \mathbf{b}_1 - \mathbf{A}_1 \mathbf{x}, \tag{2. 5}$$

$$\mathbf{A}_3 = -\mathbf{A}_2 \mathbf{A}_1^{-1}, \quad \mathbf{b}_3 = \mathbf{b}_2 - \mathbf{A}_2 \overline{\mathbf{x}}_0.$$

The objective function of (2. 4) is also quasiconvex since $f(\mathbf{x})$ is assumed to be quasiconvex over the entire E^n. Since $\overline{\mathbf{x}}_0$ is nondegenerate $\mathbf{y} = \mathbf{O}$ in problem (2. 4) has exactly n adjacent extreme points:

$$\alpha_1 \mathbf{e}_1, \alpha_2 \mathbf{e}_2, \ldots, \alpha_n \mathbf{e}_n,$$

where $\alpha_j > 0$ $(j = 1, \ldots, n)$.

Let

$$\bar{C}_0 = \max\{C_0', \max_{1 \leqq j \leqq n} f(\bar{\mathbf{x}}_0 - \alpha_j \mathbf{A}_1^{-1}\mathbf{e}_j)\} \tag{2. 6}$$

and t_j the maximal number ($t_j = \infty$ is admitted) for which the inequality

$$f(\bar{\mathbf{x}}_0 - t\mathbf{A}_1^{-1}\mathbf{e}_j) \leqq \bar{C}_0 \qquad (j = 1, \dots n) \tag{2. 7}$$

holds $\left(t_j > 0 \text{ since } f(\bar{\mathbf{x}}_0 - \alpha_j \mathbf{A}_1^{-1}\mathbf{e}_j) \leqq \bar{C}_0\right)$.

Denote

$$\mathbf{t}^* = (1/t_1, \dots, 1/t_n).$$

(If $t_j = \infty$, then $1/t_j = 0$ by definition.)

We shall distinguish two cases:

(i) $|\mathbf{t}^* \mathbf{A}_1| \leqq T$,
(ii) $|\mathbf{t}^* \mathbf{A}_1| > T$,

where T is a fixed positive number.

In case (i) we consider the problem:

$$f(\bar{\mathbf{x}}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{O}, \tag{2. 8}$$

$$\mathbf{t}^* \mathbf{y} \leqq 1.$$

By (2. 7) it is clear that the global maximum of (2. 8) does not exceed $\bar{C}_0$. Therefore the cutting inequality

$$\mathbf{t}^* \mathbf{y} \geqq 1 \tag{2. 9}$$

and its transformation by (2. 5)

$$\mathbf{h}^* \mathbf{x} \leqq h_0, \tag{2. 10}$$

where $\mathbf{h}^* = \mathbf{t}^* \mathbf{A}_1$ and $h_0 = \mathbf{t}^* \mathbf{b}_1 - 1$ excludes a region of L, where $f(\mathbf{x}) \leqq \bar{C}_0$.

In case (ii) let

$$\mathbf{d}^* = (1/\alpha_1, \dots, 1/\alpha_n)$$

and consider the inequality

$$\mathbf{d}^* \mathbf{A}_1 \mathbf{x} \leqq \mathbf{d}^* \mathbf{b}_1 - 1. \tag{2. 11}$$

It can easily be seen that (2. 11) cuts off the simplex with vertices

$$\bar{\mathbf{x}}_0, \bar{\mathbf{x}}_0 - \alpha_1 \mathbf{A}_1^{-1}\mathbf{e}_1, \dots, \bar{\mathbf{x}}_0 - \alpha_n \mathbf{A}_1^{-1}\mathbf{e}_n.$$

Adjoining inequality (2. 10) or (2. 11) to the original constraints of (2. 1) we reduce the feasible set L. Let the new feasible set be $L_1 (L = L_0)$. Then the whole procedure can be repeated with the obvious modification that in Step $k+1$

$$C_k = \max\{\bar{C}_{k-1}, C_k'\}.$$

When computing $\bar{C}_k$ by (2. 6) we replace the index 0 by k and C_k' denotes the maximal objective function's value along the path leading to a nondegenerate basic solution in Step $k+1$.

It is clear that

$$L_0 \supset L_1 \supset \dots \supset L_k \supset \dots,$$

$$C_0 \leqq C_1 \leqq \dots \leqq C_k \leqq C_{k+1} \dots .$$

The procedure terminates if for some index $p \geqq 1$, $L_p = \emptyset$. Then $\mathbf{x}'$ is a solution of (2. 1) if $f(\mathbf{x}') = \bar{C}_{p-1}$.

We shall prove that the procedure terminates in finite number of steps. For the proof we need a simple lemma.

Lemma 1. If $\bar{\mathbf{x}}_k$ is the nondegenerate basic solution obtained in Step $k+1$ and cut (2. 10) is applied, then

$$|\bar{\mathbf{x}}_k - \mathbf{u}_k| \geqq T^{-1},$$

where $\mathbf{u}_k$ is the orthogonal projection of $\bar{\mathbf{x}}_k$ onto the hyperplane $\mathbf{h}^*\mathbf{x} = h_0$ $(\mathbf{h} \neq \mathbf{O})$.

Proof. By the definition of $\mathbf{h}$ and h_0 it follows that $\bar{\mathbf{x}}_k$ is on the hyperplane $\mathbf{h}^*\mathbf{x} = h_0 + 1$. Write Schwarz's inequality for $\mathbf{h}$ and $\bar{\mathbf{x}}_k - \mathbf{u}_k$

$$|\mathbf{h}^*(\bar{\mathbf{x}}_k - \mathbf{u}_k)| \leqq |\mathbf{h}|\,|\bar{\mathbf{x}}_k - \mathbf{u}_k|.$$

Since the left hand side equals 1 we get the desired inequality

$$|\bar{\mathbf{x}}_k - \mathbf{u}_k| \geqq |\mathbf{h}|^{-1} = |\mathbf{t}^*\mathbf{A}_1|^{-1} \geqq T^{-1}.$$

Theorem 1. There is an index $p \geqq 1$ for which $L_p = \emptyset$.

Proof. It is sufficient to prove that cut (2. 10) cannot be applied infinite times since (2. 11) cuts off a simplex and every polyhedron consists of finitely many simpleces.

Suppose on the contrary that (2. 10) occurs infinite times. Then the sequence of nondegenerate basic solutions determined in the steps when (2. 10) is used has at least one cluster point $\hat{\mathbf{x}}$ because L is bounded. Thus there is a neighbourhood $K(\hat{\mathbf{x}}, \varepsilon)$ of $\hat{\mathbf{x}}$ and an index r such that for a $k \geqq r$, $\bar{\mathbf{x}}_k \in K(\hat{\mathbf{x}}, \varepsilon)$. In the step when $\bar{\mathbf{x}}_k$ is cut off by inequality (2. 10) Lemma 1. assures that the distance of $\bar{\mathbf{x}}_k$ from the cutting plane is at least T^{-1}. Thus ε can be chosen so small that the entire $K(\hat{\mathbf{x}}, \varepsilon)$ lies on the infeasible side of the cutting plane. But this is a contradiction.

Remarks

1. If $f(\mathbf{x})$ is strictly convex that is for any $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < \lambda < 1$ the inequality $f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) < \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$ holds, then the method described above gives all global maximumpoints. We have never cut such points where the objective function's value equals the maximum obtained so far and since $f(\mathbf{x})$ is strictly convex every global (and local) maximumpoint is an extreme point of L.

2. It is clear that the procedure works well with an arbitrary extreme point as a starting solution in each step but it seems us more advantageous to start with a local vertex maximumpoint. (This is a point that has no adjacent extreme point of higher objective function's value.)

3. It is obvious that the efficiency of the method is greatly reduced if degeneration occurs frequently. Therefore it is of disadvantage if cut (2. 11) has to be applied many times since this cut increases the number of degenerate basic solutions. In the next section we give a variant of this method which is insensitive to degeneration.

4. By the construction of the method the number of constraints increases. But simultaniously some of the old constraints may become redundant. (We call a constraint redundant if there is no feasible point satisfying it as an equality.) For elimination of the redundant constraints the method proposed in [6] can be used.

5. The procedure can be simplified to a great extent if we are content with an "ε optimal" solution of problem (2. 1). We call $\mathbf{z}\in L$ an "ε optimal" solution if

$$\max_{\mathbf{x}\in L} f(\mathbf{x}) \leqq f(\mathbf{z})+\varepsilon, \quad \varepsilon>0.$$

In this case it is sufficient to find a local vertex maximumpoint in every step (not necessarily nondegenerate) and tranformation (2. 5) can be carried out with any basis associated with the extreme point in question. Inequality (2. 7) is changed by replacing the right hand side to $\bar{C}_0+\varepsilon$. Since $f(\mathbf{x})$ is continuous every t_j will be positive and cut (2.10) excludes a proper subset of the feasible region. Thus cut (2. 11) is not necessary and we do not need the "wandering method" too.

It is an open question whether this modified procedure terminates in finite number of steps.

3. Maximizing a quasiconvex function over the lattice points of a convex polyhedron

Let the problem be

$$f(\mathbf{x}) \to \max$$

subject to

$$\mathbf{Ax}\leqq\mathbf{b}, \tag{3. 1}$$

$$\mathbf{x}=\text{integer},$$

where

(i) $L=\{\mathbf{x}|\mathbf{Ax}\leqq\mathbf{b}\}$ is nonempty and bounded,
(ii) The entries of $\mathbf{A}$ and $\mathbf{b}$ are integers,
(iii) $f(\mathbf{x})$ is continuous and quasiconvex on E^n.

The method proposed for solving (3. 1) consists of iterational steps. In each step we reduce the feasible region. Denote the feasible set in Step k by L_k.

Step 0. Find a feasible point to (3. 1) with any method of integer programming. If there is no such point, then (3. 1) has no solution. Otherwise go to Step 1.

Step k.

a) Find a local (vertex) maximumpoint $\mathbf{x}_k$ of L_k $(L_1=L)$.

b) Do transformation (2. 5) and determine t_j as the maximal positive number satisfying the inequality

$$f(\mathbf{x}_k-t\mathbf{A}_{11}^{-1}\mathbf{e}_j) \leqq C_k+\varepsilon, \quad \varepsilon>0 \qquad (j=1, \ldots, n), \tag{3. 2}$$

where $\mathbf{A}_{11}$ is a nonsingular submatrix of $\mathbf{A}_1$ (we have not assumed nondegeneracy!) and C_k is the maximal objective function's value obtained so far on lattice points of L. Then we construct the vector $\mathbf{t}_k$ as in Section 2. and test the inequality $|\mathbf{t}_k^*\mathbf{A}_{11}|\leqq T$. If it is satisfied by $\mathbf{t}_k$ or $\mathbf{x}_k=$integer, then we reduce L_k by cut (2. 10). If $\mathbf{x}_k$ has at least one noninteger component and $|\mathbf{t}_k^*\mathbf{A}_{11}|>T$, or there is no positive t satisfying (3. 2) then reduce L_k by a Gomory cut (see [1] p. 272). Let L_{k+1} be the new feasible set and go to Step $k+1$.

The procedure terminates if for some $p\geqq 1$, $L_p=\emptyset$.

Theorem 2. After finite number of steps $L_p=\emptyset$ for some $p\geqq 1$.

Proof. Cut (2. 10) cannot be applied infinite many times by the reasoning in the proof of Theorem 1. and because the number of lattice points of L is finite. Furthermore the application of the Gomory cuts provides an integer point after finite number of steps ([1] p. 276).

Remarks

1. The procedure described above gives "only" an "ε optimal" solution which is always satisfactory in practical situations. But if $f(\mathbf{x})$ takes on integral values for any integer $\mathbf{x}$ (e.g. $f(\mathbf{x})$ is a polynomial with integer coefficients), then we can replace ε by 1 and determine at least one "true" optimal solution of (3. 1).

2. It is clear that this procedure can be used instead of the method proposed in Section 2. in almost all practical cases since the integrity stipulation is very week if we choose proper scale. In addition if in (2. 1) every extreme point of L is integer (e.g. (2. 1) is a transportation problem with integer parameters [11]), then the procedure of this section can be applied without changing the scale.

4. Mixed zero-one integer programming with convex objective function to be maximized

Let us consider the problem

$$F(\mathbf{x}) \to \max$$

subject to

$$\begin{gathered} 0 \leqq x_j \leqq 1, \quad x_j = \text{integer} \qquad (j=1, \dots, p), \quad p \geqq 1 \\ 0 \leqq x_j \leqq k_j \qquad (j = p+1, \dots, n), \\ \sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad (i=1, \dots, m), \end{gathered} \tag{4. 1}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $F(\mathbf{x})$ is convex on E^n.

For the solution of (4. 1) we can apply the full description method. The following theorem gives the basis for doing so.

Theorem 3. Among the optimal points of (4. 1) there is at least one extreme, point of L. (L denotes the set of points satisfying the conditions of (4. 1) ignoring the integrity stipulations.)

Proof. Let $\mathbf{z}$ be an optimal solution of (4. 1). Fix the first p components of $\mathbf{z}$ and consider the problem:

$$F(z_1, \dots, z_p, x_{p+1}, \dots, x_n) \to \max$$

subject to

$$\begin{gathered} 0 \leqq x_j \leqq k_j \qquad (j = p+1, \dots, n), \\ \sum_{j=p+1}^{n} a_{ij} x_j \leqq b_i - \sum_{j=1}^{p} a_{ij} z_j \qquad (i=1, \dots, m). \end{gathered} \tag{4. 2}$$

Let $\mathbf{y}$ be an optimal extreme point of (4. 2). (There is at least one such point since $(z_{p+1}, \dots, z_n)$ is a feasible point and $F(\mathbf{x})$ is convex.) Let

$$\mathbf{x}_0^* = (z_1, \dots, z_p, \mathbf{y}^*)$$

$\mathbf{x}_0$ is an optimal solution of (4. 1) since $F(\mathbf{x}_0) \geqq F(\mathbf{z})$. Suppose that $\mathbf{x}_0$ is not an extreme point. Then there are points $\mathbf{x}_1 \in L$, $\mathbf{x}_2 \in L$, $\mathbf{x}_1 \neq \mathbf{x}_2$ such that $\mathbf{x}_0 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$. Since the first p entries of $\mathbf{x}_0$ are 0 or 1 the first p components of $\mathbf{x}_1$ and $\mathbf{x}_2$ are equal. But the last $n-p$ components of $\mathbf{x}_1$ and $\mathbf{x}_2$ must coincide because $\mathbf{y}$ is an extreme point of (4. 2). This contradicts the assumption $\mathbf{x}_1 \neq \mathbf{x}_2$. Thus $\mathbf{x}_0$ is an optimal extreme point.

Our purpose is to apply the methods of Section 2. for (4. 1) to accelerate the full description method. The following theorem provides a continuous equivalent to problem (4. 1).

Theorem 4. Consider the programming problem:

$$F(\mathbf{x}) - \lambda \sum_{j=1}^{p} x_j(1-x_j) \to \max \qquad (\lambda > 0) \tag{4. 3}$$

subject to

$$\mathbf{x} \in L.$$

There exists a real number $\lambda_0 > 0$ so that for all $\lambda \geqq \lambda_0$ the set of optimal extreme points of (4. 1) and (4. 3) coincide.

Proof. Let

$$\delta_0 = \min_{x \in L'} \sum_{j=1}^{p} x_j(1-x_j),$$

where L' denotes the set of those extreme points of L which have at least one non-integer component among their first p components.

Let $\mathbf{x}^0(\lambda) = (x_1^0(\lambda), \dots, x_n^0(\lambda))^*$ be an arbitrary optimal extreme point of (4. 3). Then

$$F(\mathbf{z}) \leqq F(\mathbf{x}^0(\lambda)) - \lambda \sum_{j=1}^{p} x_j^0(\lambda)[1 - x_j^0(\lambda)], \tag{4. 4}$$

where $\mathbf{z}$ is an optimal extreme point of (4. 1). If one of the first p components of $\mathbf{x}^0(\lambda)$ is not integer, then from (4. 4) it follows

$$\frac{1}{\lambda}[F(x^0(\lambda)) - F(z)] \geqq \sum_{j=1}^{p} x_j^0(\lambda)[1 - x_j^0(\lambda)] \geqq \delta_0 > 0. \tag{4. 5}$$

Thus we see that if λ is sufficiently large, then $\mathbf{x}^0(\lambda)$ cannot have noninteger components among its first p entries. Consequently there is a λ_0 such that for $\lambda \geqq \lambda_0$ every optimal extreme point of (4. 3) is an optimal solution to (4. 1).

Conversely if $\mathbf{z}$ is an optimal extreme point of (4. 1), then $\mathbf{z}$ has to be optimal for (4. 3) because of (4. 5).

For practical computation we need an estimation for λ_0. Suppose that we are content with an "almost feasible" "ε optimal" solution of (4. 1). We call $\mathbf{y}$ "δ feasible" "ε optimal" ($\delta > 0$, $\varepsilon > 0$) solution of (4. 1) if $\mathbf{y}$ can violate the conditions

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad (i = 1, \dots m),$$

by no more than δ and $F(\mathbf{y}) \geqq F(\mathbf{z})-\varepsilon$ where $\mathbf{z}$ is an optimal solution of (4. 1). The following theorem provides an estimation for λ_0.

Theorem 5. Assume that

(i) for every $\mathbf{x}_1 \in T$, $\mathbf{x}_2 \in T$ where

$$T=\{\mathbf{x}|0\leqq x_j\leqq 1 \quad (j=1, \dots, p), \quad 0\leqq x_j\leqq k_j \quad (j = p+1, \dots, n)\}$$

the inequality

$$|F(\mathbf{x}_1)-F(\mathbf{x}_2)| \leqq M\,|\mathbf{x}_1-\mathbf{x}_2|^a$$

holds where M and a are positive constants,

(ii) $K_a \leqq F(\mathbf{z}) \leqq K_f$ where $\mathbf{z}$ is any feasible solution of (4. 1),

(iii) $\mathbf{A}=[a_{ij}]$ has no zero rows and columns.

If λ satisfies the inequality

$$\lambda \geqq \frac{K_f-K_a}{\alpha_0(1-\alpha_0)} \tag{4. 6}$$

then every vector obtained from an optimal extreme point of (4. 3) by rounding the first p components to the nearest integer is a "δ feasible" "ε optimal" solution of (4. 1) where

$$\alpha_0=\min(\hat{\alpha}, \bar{\alpha})$$

$$\hat{\alpha} \leqq \min\left\{\frac{1}{2}, \frac{1}{\sqrt{p}}\sqrt[a]{\frac{\varepsilon}{M}}\right\}; \quad \hat{\alpha}>0 \tag{4. 7}$$

$$\bar{\alpha}<\min\left\{\min_{1\leqq i\leqq m}\frac{\delta}{\sum_{j=1}^{p}|a_{ij}|}, \frac{1}{2}\right\}; \quad \bar{\alpha}>0. \tag{4. 8}$$

Proof. Let $\mathbf{x}^0(\lambda)$ be an optimal extreme point of (4. 3) with λ satisfying (4. 6) and $\mathbf{z}$ an optimal extreme point of (4. 1). Then by (4. 5), (4. 6) and assumption (ii) we obtain

$$\begin{aligned} x_j^0(\lambda)[1-x_j^0(\lambda)] &\leqq \sum_{j=1}^{p} x_j^0(\lambda)[1-x_j^0(\lambda)] \leqq \frac{1}{\lambda}\left[F(\mathbf{x}^0(\lambda))-F(\mathbf{z})\right] \leqq \\ &\leqq \frac{1}{\lambda}(K_f-K_a) \leqq \alpha_0(1-\alpha_0) \leqq \bar{\alpha}(1-\bar{\alpha}) \qquad (j=1, \dots, p). \end{aligned} \tag{4. 9}$$

This implies that one of the following inequalities holds

$$\begin{gathered} 0\leqq x_j^0(\lambda)\leqq\bar{\alpha} \\ 1-\bar{\alpha} \leqq x_j^0(\lambda) \leqq 1 \qquad (j=1, \dots, p). \end{gathered} \tag{4. 10}$$

Denote by $\bar{\mathbf{x}}(\lambda)$ the vector obtained from $\mathbf{x}^0(\lambda)$ by rounding the first p components to the nearest integer. By (4. 8) we get

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j(\lambda) \leqq b_i+\bar{\alpha}\sum_{j=1}^{p}|a_{ij}| \leqq b_i+\delta.$$

This means that $\bar{\mathbf{x}}(\lambda)=(\bar{x}_1(\lambda), \dots, \bar{x}_n(\lambda))$ is "δ feasible".

To prove the "ε optimality" we obtain by assumption (i), (4. 7) and (4. 10) the inequalities:

$$|F(\mathbf{z})-F(\overline{\mathbf{x}}(\lambda))| \leqq M|\mathbf{x}^0(\lambda)-\overline{\mathbf{x}}(\lambda)|^a \leqq M(\sqrt{p}\hat{\alpha})^a \leqq M\left(\sqrt{p}\,\frac{1}{\sqrt{p}}\sqrt[a]{\frac{\varepsilon}{M}}\right)^a = \varepsilon$$

which means that $\overline{\mathbf{x}}(\lambda)$ is "ε optimal".

Corollaries

1. If every a_{ij} and b_i is integer and $p=n$, then by choosing $\delta<1$, $\overline{\mathbf{x}}(\lambda)$ is a feasible solution of (4. 1). Furthermore if $F(\mathbf{x})$ takes on integral values for every integer $\mathbf{x}$, then by choosing $\varepsilon<1$ we obtain an optimal solution of (4. 1).
2. If $\mathbf{x}^0(\lambda)$ is a "Δ optimal" solution of (4. 3), then by changing (4. 6) to

$$\lambda \geqq \frac{K_f-K_a+\Delta}{\alpha_0(1-\alpha_0)}$$

we get a "δ feasible", "Δ+ε optimal" solution $\overline{\mathbf{x}}(\lambda)$ by rounding $\mathbf{x}^0(\lambda)$.

In the pure integer case δ and Δ+ε have to be chosen smaller then 1 in order to get an optimal solution of (4. 1).

For the solution of (4. 3) we can apply the methods proposed in Section 2. and 3. If $F(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j$, then (4. 1) is the mixed zero-one integer linear programming problem. In this case we can increase the efficiency of our cutting plane method by adjoining to the constraint set the inequality

$$\sum_{j=1}^{n} c_j x_j \geqq F_k+\Delta,$$

where F_k is the largest objective function's value obtained up to Step k. In the pure case Δ can be chosen 1 provided all the c_j-s are integer.

5. Fixed charge problems with convex objective function

The following problem occurs very frequently in economic applications:

A production vector has to be found which satisfies a number of linear constraints and minimizes a cost function composed of individual cost functions having a fixed cost at $x_j=0$. For $x_j>0$ the cost function is concave.

In mathematical terms the problem to be solved is the following:

$$f(\mathbf{x}) = -\sum_{j=1}^{n} f_j(x_j) \to \max$$

subject to

$$0\leqq x_j \leqq k_j \qquad (j=1,\ldots,n), \tag{5. 1}$$

$$\mathbf{x}\in L,$$

where L is a convex polyhedron and

$$f_j(x_j) = \begin{cases} 0 & \text{if } x_j = 0, \\ g_j(x_j) & \text{if } x_j > 0, \end{cases} \quad \lim_{x_j \to +0} g_j(x_j) = A_j \geqq 0 \qquad (j = 1, \dots, n),$$

$g_j(x_j)$ is a concave monotone increasing function.

We can formulate (5. 1) as a mixed zero-one integer programming problem in the following manner: (For convenience we suppose that $A_j > 0$ $(j = 1, \dots, p)$ and $A_j = 0$ $(j = p+1, \dots, n)$.)

$$F(\mathbf{x}, \xi) = -\sum_{j=1}^{p} [A_j \xi_j + g_j(x_j)] + \sum_{j=p+1}^{n} g_j(x_j) \to \max$$

subject to

$$\begin{gathered} 0 \leqq x_j \leqq k_j \qquad (j = 1, \dots, n), \\ \mathbf{x} \in L \\ x_j - k_j \xi_j \leqq 0 \\ 0 \leqq \xi_j \leqq 1, \quad \xi_j = \text{integer} \qquad (j = 1, \dots, p). \end{gathered} \tag{5. 2}$$

Since (5. 2) is of type (4. 1) the method proposed in Section 4. can be used for solving (5. 2). From computational point of view it is not indifferent that (5. 2) has p new variables. In this section we give a method for solving (5. 1) without increasing the number of variables.

Without any loss of generality we may assume that $p = n$. The following theorem asserts the existence of a continuous equivalent to (5. 1).

Theorem 6. Let us consider the problem:

$$\bar{f}(\mathbf{x}, \mathbf{r}) \to \max$$

subject to

$$\begin{gathered} 0 \leqq x_j \leqq k_j \qquad (j = 1, \dots, n), \\ \mathbf{x} \in L, \end{gathered} \tag{5. 3}$$

where $\mathbf{r}^* = (r_1, \dots, r_n)$ $(\mathbf{r} \geqq \mathbf{O})$ is a parameter vector and

$$\bar{f}(\mathbf{x}, \mathbf{r}) = -\sum_{j=1}^{n} \bar{f}_j(x_j, r_j),$$

$$\bar{f}_j(x_j, r_j) = \begin{cases} m_j x_j & \text{if } x_j \leqq r_j, \\ g_j(x_j) & \text{if } x_j > r_j, \end{cases} \qquad (j = 1, \dots, n),$$

$$m_j = \frac{g_j(r_j)}{r_j}.$$

There exists a positive vector $\mathbf{r}_0$ such that for all $\mathbf{r}$ $(0 < \mathbf{r} \leqq \mathbf{r}_0)$ the sets of optimal extreme points of (5. 1) and (5. 3) coincide.

Proof. Let $\mathbf{x}_0$ and $\mathbf{x}_1(\mathbf{r})$ be optimal solutions to (5. 1) and (5. 3) resp. Since both $f(\mathbf{x})$ and $\bar{f}(\mathbf{x}, \mathbf{r})$ are concave functions we may assume that $\mathbf{x}_0$ and $\mathbf{x}_1(\mathbf{r})$ are extreme points. Let $\mathbf{x}^* = (x_1, \ldots, x_n)$ and

$$s = \min_{\mathbf{x} \in L'} \left(\min_{1 \leqq j \leqq n} x_j \right) \quad (x_j > 0),$$

where L' denotes the extreme points of the common feasible region of (5. 1) and (5. 3). (We can disregard of the trivial case if $\mathbf{O} \in L'$ since in this case $\mathbf{x}_1(\mathbf{r}) = \mathbf{x}_0 = \mathbf{O}$ for any positive $\mathbf{r}$ by the monotonicity of the functions $g_j(x_j)$.) Let $\mathbf{r}_0$ be a positive vector satisfying $\mathbf{r}_0^* \mathbf{e}_j \leqq s$ $(j=1, \ldots, n)$. Then

$$\bar{f}(\mathbf{x}_1(\mathbf{r}_0), \mathbf{r}_0) = f(\mathbf{x}_1(\mathbf{r}_0)).$$

Since $\bar{f}_j(\mathbf{x}_1^*(\mathbf{r}_0)\mathbf{e}_j, \mathbf{r}_0^* \mathbf{e}_j) = 0$ if $\mathbf{x}_1^*(\mathbf{r}_0)\mathbf{e}_j = 0$ and $\bar{f}_j(\mathbf{x}_1^*(\mathbf{r}_0)\mathbf{e}_j, \mathbf{r}_0^* \mathbf{e}_j) = g_j(\mathbf{x}_1^*(\mathbf{r}_0)\mathbf{e}_j)$ if $\mathbf{x}_1^*(\mathbf{r}_0)\mathbf{e}_j \geqq s \geqq \mathbf{r}_0^* \mathbf{e}_j$ $(j=1, \ldots, n)$. But by the optimality of $\mathbf{x}_1(\mathbf{r}_0)$ it follows

$$f(\mathbf{x}_1(\mathbf{r}_0)) = \bar{f}(\mathbf{x}_1(\mathbf{r}_0), \mathbf{r}_0) \geqq \bar{f}(\mathbf{x}_0, \mathbf{r}_0) = f(\mathbf{x}_0)$$

which means that $\mathbf{x}_1(\mathbf{r}_0)$ is optimal to (5. 1). Conversely by the optimality of $\mathbf{x}_0$

$$\bar{f}(\mathbf{x}_0, \mathbf{r}_0) = f(\mathbf{x}_0) \geqq f(\mathbf{x}_1(\mathbf{r}_0)) = \bar{f}(\mathbf{x}_1(\mathbf{r}_0), \mathbf{r}_0)$$

which means that $\mathbf{x}_0$ is optimal to (5. 3) if $\mathbf{r} \leqq \mathbf{r}_0$.

Corollary. The objective function of (5. 3) is convex on E^n and therefore the method of Section 2. can be used to solve it.

The only difficulty is that we cannot give an a priori estimation on $\mathbf{r}_0$. Fortunately by a slight modification of the algorithm described in Section 2. we do not need the exact value of $\mathbf{r}_0$. There are only two places where changes have to be done:

1. In (2. 4) $f(\bar{\mathbf{x}}_0 - \mathbf{A}_\mathrm{I}^{-1}\mathbf{y})$ is defined only for those values of $\mathbf{y}$ where

$$\bar{\mathbf{x}}_0 - \mathbf{A}_\mathrm{I}^{-1}\mathbf{y} \geqq \mathbf{0}. \tag{5.4}$$

2. In the definition of t_j ((2. 7)) (5. 4) has also to be taken into consideration. Thus t_j is the maximal number for which the inequalities

$$\begin{aligned} f(\bar{\mathbf{x}}_0 - t\mathbf{A}_\mathrm{I}^{-1}\mathbf{e}_j) &\leqq \bar{C}_0 \\ \bar{\mathbf{x}}_0 - t\mathbf{A}_\mathrm{I}^{-1}\mathbf{e}_j &\geqq \mathbf{0} \end{aligned} \qquad (j=1, \ldots, n) \tag{5.5}$$

hold.

All other statements of Section 2. including Theorem 1. remain valid. Naturally our method can be combined with other methods e.g. approximative methods like [14] since any good approximative solution can serve as a starting point for the cutting plane method. Of course the difficulties caused by degeneration can be overcome by searching for an "ε optimal" solution.

6. Separable nonlinear programming with linear constraints

Nonlinear programming with general objective function is a rather undiscovered field of mathematical programming. There are methods based on the idea of approximation with polygons, [1], [15], algorithms applying "branch and bound" [16], [17] and full description methods [18]. We shall apply the cutting plane method of Section 2. for accelerating the full description method. We begin with the simple case of separable objective function and thereafter we discuss the general programming problem.

$$f(\mathbf{x}) = \sum_{j=1}^{n} f_j(x_j) \to \max$$

subject to

$$\mathbf{O} \leqq \mathbf{x} \leqq \mathbf{k} \qquad (\mathbf{k} \geqq \mathbf{O}) \tag{6. 1}$$

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b}.$$

Suppose that

(i) $L = \{\mathbf{x} | \mathbf{O} \leqq \mathbf{x} \leqq \mathbf{k},\ \mathbf{A}\mathbf{x} \leqq \mathbf{b}\} \neq \emptyset$,

(ii) for every x_j^1, x_j^2 satisfying $0 \leqq x_j^r \leqq \mathbf{k}^* \mathbf{e}_j$ $(r=1, 2)$ holds the inequality:

$$|f_j(x_j^1) - f_j(x_j^2)| \leqq M_j |x_j^1 - x_j^2| \qquad (j=1, \dots, n), \tag{6. 2}$$

where M_j is constant,

(iii) $f_j(x_j) \equiv -\infty$ for $x_j < 0$ and $x_j > \mathbf{k}^* \mathbf{e}_j$ $(j=1, \dots, n)$. Our purpose is to determine an "ε optimal" feasible solution $\bar{\mathbf{x}} \in L$.

The method for solving (6. 1) consists of iterational steps. To start with let us determine an extreme point of $L = L_0$, say $\mathbf{x}_0 = (x_1^0, \dots, x_n^0)^*$. Let us assume that we have a "good" approximative solution $\mathbf{y}_0 \in L$. ($\mathbf{y}_0$ can be e.g. a local maximum-point of (6. 1) which can be obtained by several local methods such as gradient methods, linear approximation e.t.c.)

Put $K_0 = f(\mathbf{y}_0)$ and define $\bar{f}(\mathbf{x})$ in the following manner:

$$\bar{f}(\mathbf{x}) = \sum_{j=1}^{n} \bar{f}_j(x_j),$$

where $\bar{f}_j(x_j)$ is convex, $\bar{f}_j(x_j) \geqq f_j(x_j)$ for all x_j, $\bar{f}_j(x_j^0) = f_j(x_j^0)$ $(j=1, \dots, n)$.

Because of Property (ii) $\bar{f}(\mathbf{x})$ is defined over the entire E^n. Since $\mathbf{x}_0$ is a vertex of L transformation (2. 5) can be carried out. Now consider the problem (see (2. 4))

$$\bar{f}(\mathbf{x}_0 - A_I^{-1}\mathbf{y}) \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{O} \tag{6. 3}$$

$$\mathbf{A}_3 \mathbf{y} \leqq \mathbf{b}_3.$$

By the definition of $\bar{f}(\mathbf{x})$

$$\bar{f}(\mathbf{x}_0) = f(\mathbf{x}_0).$$

Let t_j be the maximal number (but at most M, a large fixed positive number) satisfying

$$\bar{f}(\mathbf{x}_0 - t\mathbf{A}_I^{-1}\mathbf{e}_j) \leqq K_0 + \varepsilon \qquad (j=1, \dots, n). \tag{6. 4}$$

Each t_j is positive since $\bar{f}(\mathbf{x}_0) \leqq K_0$ and $\bar{f}(\mathbf{x})$ is continuous. (Since it is convex over E^n.) Let

$$\mathbf{t}^* = (1/t_1, \dots, 1/t_n).$$

Take a fixed positive number T and apply the cut

$$\mathbf{t}^* \mathbf{y} \geqq 1 \tag{6.5}$$

if $|\mathbf{t}^* A_1| \leqq T$. With cut (6. 5) we have excluded a region where

$$\bar{f}(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \leqq K_0 + \varepsilon$$

and since $\bar{f}(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \geqq f(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y})$ the relation

$$f(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \leqq K_0 + \varepsilon$$

holds for every $\mathbf{y}$ in the excluded region.

If $|\mathbf{t}^* \mathbf{A}_1| > T$, then we apply the cut

$$\bar{\mathbf{t}}^* \mathbf{y} \geqq 1, \tag{6.6}$$

where

$$\bar{\mathbf{t}}^* = (1/\alpha t_1, \ldots, 1/\alpha t_n)$$

and α is chosen so that $|\bar{\mathbf{t}}^* \mathbf{A}_1| = T$ is satisfied.

Of course in this case we can only guarantee that for all $\mathbf{y}$ in the excluded region

$$f(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \leqq \bar{f}(\mathbf{x}_0 - \mathbf{A}_1^{-1}\mathbf{y}) \leqq \max_{1 \leqq j \leqq n} \bar{f}(\mathbf{x}_0 - \alpha t_j \mathbf{A}_1^{-1} \mathbf{e}_j) = P_0.$$

The whole procedure is repeated for the reduced polyhedron L_1. We have seen in Section 2. (Theorem 1.) that after finite number of steps $L_p = \emptyset$ for some $p \geqq 1$.

Naturally if in the course of computations we arrive at a vector which gives higher objective function's value than K_0, then starting from this point we can find a better local maximumpoint with objective function's value $K_1 > K_0$ and replace K_0 by K_1.

After having arrived at the situation where $L_p = \emptyset$ the best solution $\mathbf{y}_r$ obtained so far satisfies the inequality

$$f(\mathbf{y}_r) \leqq \max_{0 \leqq k \leqq p-1} R_k,$$

where $R_k = K_k + \varepsilon$ if in Step k cut (6. 5) was used and $R_k = P_k$ if cut (6. 6) was applied. Thus if

$$\max_{0 \leqq k \leqq p-1} R_k = K_{k'} + \varepsilon \quad \text{for some} \quad 0 \leqq k' \leqq p-1 \tag{6.7}$$

then $\mathbf{y}_r$ is an "ε optimal" solution of (6. 1) if

$$f(\mathbf{y}_r) = K_k.$$

Let $Q = \{q_1, \ldots, q_r\}$ be the set of indices for which

$$P_{q_s} > \max_{0 \leqq k \leqq p-1} K_k + \varepsilon \qquad (s = 1, \ldots, r).$$

For each q_s there can be associated a problem:

$$f(\mathbf{x}_{q_s} - B_{q_s}^{-1}\mathbf{y}) \rightarrow \max$$

subject to

$$\mathbf{y} \geqq \mathbf{O}$$

$$\mathbf{A}_{q_s}\mathbf{y} \leqq \mathbf{b}_{q_s} \qquad (s=1, \dots, r), \tag{6.8}$$

$$\bar{\mathbf{t}}^*_{q_s}\mathbf{y} \leqq 1,$$

where $\mathbf{A}_{q_s}\mathbf{y} \leqq \mathbf{b}_{q_s}$, $\mathbf{y} \geqq \mathbf{O}$ defines L_{q_s-1} after having done transformation (2. 5), $\mathbf{B}_{q_s}$ is the matrix of transformation, $\bar{\mathbf{t}}_{q_s}$ is defined in (6. 6), $\mathbf{x}_{q_s}$ is the actual extreme point of L_{q_s-1}.

We shall decompose (6. 8) into d subproblems having the form:

$$f(\mathbf{x}_{q_s} - \mathbf{B}_{q_s}^{-1}\mathbf{y}) \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{O}$$

$$\mathbf{A}_{q_s}\mathbf{y} \leqq \mathbf{b}_{q_s} \qquad (l=1, \dots, d). \tag{6.9}$$

$$\bar{\mathbf{t}}^*_{q_s}\mathbf{y} = \frac{l}{d}$$

Let $\mathbf{s} \neq \mathbf{O}$ be an arbitrary feasible point of (6. 8) and $\mathbf{v}$ the intersection of the ray determined by $\mathbf{O}$ and $\mathbf{s}$ with the hyperplane $\bar{\mathbf{t}}^*_{q_s}\mathbf{y}=1$. Let further l be the index for which the inequality

$$\frac{l}{d} \leqq \bar{\mathbf{t}}^*_{q_s}\mathbf{s} \leqq \frac{l+1}{d}$$

holds. Denote by $\mathbf{r}$ the intersection of the ray $(\mathbf{O}, \mathbf{s})$ with hyperplane $\bar{\mathbf{t}}_{q_s}\mathbf{y} = \frac{l}{d}$. Since $\mathbf{r}$ and $\mathbf{s}$ are on the ray $(\mathbf{O}, \mathbf{v})$ they can be written in the following way:

$$\mathbf{r} = \lambda_r \mathbf{v},$$

$$\mathbf{s} = \lambda_s \mathbf{v},$$

where $\lambda_r = \frac{l}{d}$. Then the following relations hold:

$$|\mathbf{r}-\mathbf{s}| = |\lambda_r - \lambda_s|\,|\mathbf{v}| = \left|\frac{l}{d} - \lambda_s\right| |\mathbf{v}| = \left|\frac{l}{d} - \bar{\mathbf{t}}^*_{q_s}\mathbf{s}\right| |\mathbf{v}| \leqq \left|\frac{l}{d} - \frac{l+1}{d}\right| |\mathbf{v}| = \frac{1}{d}|\mathbf{v}|.$$

Since

$$|\mathbf{v}| \leqq \max_{1 \leqq j \leqq n} \bar{\mathbf{t}}^*_{q_s}\mathbf{e}_j \leqq M,$$

d can be chosen so large that $|\mathbf{r}-\mathbf{s}| \leqq \delta$ for given $\delta > 0$. But because of property (ii) if δ is small enough, then

$$|f(\mathbf{r}) - f(\mathbf{s})| \leqq \frac{\varepsilon}{2}.$$

This means that if we can solve problem (6. 9) for each l, then the objective function's value of an "$\varepsilon/2$ optimal" solution of problem (6. 9) cannot differ from the optimum of (6. 8) by more than ε. But the feasible set of (6. 9) is of lower dimension than that of (6. 8).

For solving subproblems (6. 9) we can apply the same procedure as for (6. 1). It is clear that after finite number of steps either situation (6. 7) occurs or the dimension of the subproblems reduces to zero. In both cases we obtain at least one "ε optimal" solution of (6. 1).

Of course the right hand side of inequality (6. 4) may increase by discovering new better solutions and those subproblems of type (6. 8) where P_{q_s} does not exceed the best objective function's value obtained so far can be dropped.

To illustrate the method let us take a numerical example:

$$f(x_1, x_2) = -(x_1-1)^3+x_2-1 \rightarrow \max.$$

Subject to

$$\begin{aligned} 0 \leqq x_1 &\leqq 2 \\ 0 \leqq x_2 &\leqq 2 \\ -15x_1+10x_2 &\leqq 2 \\ -3x_1+4x_2 &\leqq 2. \end{aligned} \tag{6. 10}$$

First of all determine a local maximumpoint. For this purpose we can use e.g. the method of Zangwill [19]. If we start from $x_1=2$, $x_2=2$, then this method leads us to the local maximumpoint $x_1 = \frac{3}{2}$, $x_2 = \frac{13}{8}$ where the objective function's value $f\left(\frac{3}{2}, \frac{13}{8}\right) = K_1 = \frac{1}{2}$.

According to the method proposed in this section we have to start with an arbitrary extreme point. Let this be $x_1=2$, $x_2=2$. The construction of the functions $\bar{f}_1^{(1)}(x_1)$ and $\bar{f}_2^{(1)}(x_2)$ is an elementary task. (The upper index denotes the number of iterations.)

$$\bar{f}_1^{(1)}(x_1) = -3x_1+5,$$

$$\bar{f}_2^{(1)}(x_2) = x_2-1.$$

The matrix of the transformation and its inverse is the following:

$$\mathbf{A}_1 = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1/4 & 3/4 \end{bmatrix}.$$

We have to find the maximal positive solutions to the inequalities: (The admissable error $\varepsilon=0,1$)

$$\bar{f}^{(1)}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} - t\begin{bmatrix} 0 \\ 1/4 \end{bmatrix}\right) = \bar{f}_1^{(1)}(2)+\bar{f}_2^{(1)}\left(2-\frac{1}{4}t\right) \leqq \frac{1}{2}+\frac{1}{10} = \frac{3}{5},$$

$$\bar{f}^{(1)}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} - t\begin{bmatrix} 1 \\ 3/4 \end{bmatrix}\right) = \bar{f}_1^{(1)}(2-t)+\bar{f}_2^{(1)}\left(2-\frac{3}{4}t\right) \leqq \frac{3}{5}.$$

The solutions are $t_1^{(1)}=\infty$, $t_2^{(1)}=4/15$. Thus the cutting inequality obtained in the first step is

$$[0,\ 15/4]\begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqq [0,\ 15/4]\begin{bmatrix} 2 \\ 2 \end{bmatrix} - 1$$

or briefly

$$x_1 \leqq \frac{26}{15}.$$

In the second iterational step let our starting extreme point be $x_1 = \frac{26}{15}$, $x_2 = \frac{9}{5}$. Since $f\left(\frac{26}{15}, \frac{9}{5}\right) < \frac{1}{2}$, $K_2 = K_1 = \frac{1}{2}$. The matrix of transformation remains unchanged but $\bar{f}^{(2)}(\mathrm{x})$ will be different from $\bar{f}^{(1)}(\mathrm{x})$. By simple computation we get

$$\bar{f}_1^{(2)}(x_1) = -\frac{121}{75}x_1 + \frac{8107}{3375},$$

$$\bar{f}_2^{(2)}(x_2) = x_2 - 1.$$

Consider the inequalities

$$\bar{f}^{(2)}\left(\begin{bmatrix} \frac{26}{15} \\ \frac{9}{5} \end{bmatrix} - t\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}\right) = \bar{f}_1^{(2)}\left(\frac{26}{15}\right) + \bar{f}_2^{(2)}\left(\frac{9}{5} - \frac{1}{4}t\right) \leqq \frac{3}{5},$$

$$\bar{f}^{(2)}\left(\begin{bmatrix} \frac{26}{15} \\ \frac{9}{5} \end{bmatrix} - t\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}\right) = \bar{f}_1^{(2)}\left(\frac{26}{15} - t\right) + \bar{f}_2^{(2)}\left(\frac{9}{5} - \frac{3}{4}t\right) \leqq \frac{3}{5}.$$

The maximal positive solutions $t_1^{(2)} = \infty$, $t_2^{(2)} = \frac{196\,800}{344\,729}$. The cutting inequality

$$x_1 \leqq \frac{482\,658}{344\,729} < \frac{3}{2}.$$

To make the calculations simple we take the less sharp cut

$$x_1 \leqq \frac{3}{2}.$$

Our starting solution in the third iterational step is: $x_1 = \frac{3}{2}$, $x_2 = \frac{13}{8}$. The matrix of transformation is also unchanged and $K_3 = K_2 = K_1$.

$$\bar{f}_1^{(3)}(x_1) = -\frac{3}{3}x_1 + 1,$$

$$\bar{f}_2^{(3)}(x_2) = x_2 - 1.$$

Consider the inequalities:

$$\bar{f}^{(3)}\left(\begin{bmatrix}\frac{3}{2}\\ \frac{13}{8}\end{bmatrix} - t\begin{bmatrix}0\\ \frac{1}{4}\end{bmatrix}\right) = \bar{f}_1^{(3)}\left(\frac{3}{2}\right) + \bar{f}_2^{(3)}\left(\frac{13}{8} - \frac{1}{4}t\right) \leqq \frac{3}{5},$$

$$\bar{f}^{(3)}\left(\begin{bmatrix}\frac{3}{2}\\ \frac{13}{8}\end{bmatrix} - t\begin{bmatrix}1\\ \frac{3}{4}\end{bmatrix}\right) = \bar{f}_1^{(3)}\left(\frac{3}{2} - t\right) + \bar{f}_2^{(3)}\left(\frac{13}{8} - \frac{3}{4}t\right) \leqq \frac{3}{5}.$$

The maximal positive solutions are: $t_1^{(3)} = \infty$, $t_2^{(3)} = \infty$. Thus $L_3 = \emptyset$ which means that $x_1 = \frac{3}{2}$, $x_2 = \frac{13}{8}$ is an "0,1 optimal" solution of (6. 10). (Throughout the calculations we have assumed M and T very large.)

7. The solution of general continuous nonlinear programming problems

As a first step of generalization let us drop the separability stipulation for $f(\mathbf{x})$. That is we consider the problem

$$f(\mathbf{x}) \to \max$$

subject to

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b}, \tag{7. 1}$$

where

(i) $L = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}$ is nonvoid and bounded,

(ii) for every closed, bounded, convex set $C \subset E^n$ there exists a constant M_c such that for all $\mathbf{x}_1, \mathbf{x}_2 \in C$

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leqq M_c |\mathbf{x}_1 - \mathbf{x}_2|. \tag{7. 2}$$

The method proposed to solve (7. 1) is very similar to the method of Section 6. Since we have used the separability of the objective function in the construction of $\bar{f}(\mathbf{x})$ we define $\bar{f}(\mathbf{x})$ for (7. 1) in an other way. Let

$$\bar{f}_c(\mathbf{x}) = M_c |\mathbf{x} - \mathbf{x}_0| + f(\mathbf{x}_0), \tag{7. 3}$$

where $\mathbf{x}_0$ is the starting extreme point, M_c is the constant belonging to a closed, bounded, convex set C (see (7. 2)). It is easy to prove that $\bar{f}_c(\mathbf{x})$ is convex and if $f(\mathbf{x})$ is continuously differentiable on E^n, then

$$M_c \geqq \max_{\mathbf{x} \in C} |f'(\mathbf{x})|, \tag{7. 4}$$

where $f'(\mathbf{x})$ is the gradient vector of $f(\mathbf{x})$.

Since in the definition of $\bar{f}_c(\mathbf{x})$ the set C is involved we have to modify the procedure of determining t_j. In this case t_j is the maximal number for which the relations

$$\bar{f}_c(\mathbf{x}_0 - t\mathbf{A}_1^{-1}\mathbf{e}_j) \leqq K_0 + \varepsilon$$
$$\mathbf{x}_0 - t\mathbf{A}_1^{-1}\mathbf{e}_j \in C \qquad (j=1, \dots, n) \tag{7.5}$$

hold.

All other steps of the method of Section 6. remain unchanged.

It is clear that the efficiency of a particular cut depends greatly on the choice of C. Theoretically we ought to choose C to minimize $|\mathbf{t}^*\mathbf{A}_1|$. But this is a very difficult problem solving. Instead of solving this problem we propose choosing a region depending on one parameter (e.g. a ball with radius λ, a cube with edge-length λ etc.) and to solve the one variable minimization problem.

Now we are able to treat the general continuous nonlinear programming problem:

$$f(\mathbf{x}) \to \max$$

subject to

$$\mathbf{x} \in L,$$
$$g_k(\mathbf{x}) \leqq 0 \qquad (k=1, \dots, p), \tag{7.6}$$

where L is a bounded convex polyhedral set and the functions $f(\mathbf{x}), g_1(\mathbf{x}), \dots, g_p(\mathbf{x})$ are continuously differentiable over E^n.

By using the idea of Fiacco and McCormick [20] we reduce (7. 6) to (7. 1) and then we apply the method of cutting planes.

Problem (7. 6) can always be transformed into the following problem:

$$-\exp z \to \max$$

subject to

$$\mathbf{y} \in S,$$
$$h_k(\mathbf{y}) = 0 \qquad (k=1, \dots, p), \tag{7.7}$$
$$\varphi(\mathbf{y}) - z = 0,$$

where S is a convex polyhedron.

We shall search for a "(δ, ϱ) solution" $(\delta > 0, \varrho > 0)$ of (7. 7). A point $(\mathbf{y}_0, z_0)$ is called "(δ, ϱ) solution" of (7. 7) if

$$\mathbf{y}_0 \in S,$$
$$|h_k(\mathbf{y}_0)| \leqq \delta \qquad (k=1, \dots, p), \tag{7.8}$$
$$z_0 \geqq \bar{z} - \varrho,$$

where $\bar{z}$ is optimal to (7. 7).

Consider the following problem:

$$F(\mathbf{y}, z, a_t) = -\exp z - a_t \left[\sum_{k=1}^{p} h_k^2(\mathbf{y}) + (\varphi(\mathbf{y}) - z)^2 \right] \to \max \tag{7.9}$$

subject to

$$\mathbf{y} \in S,$$

where a_t is a positive parameter.

(7. 9) can always be solved since S is bounded. Let $(\mathbf{y}_t, z_t)$ be an "ε_t optimal" solution of (7. 9) $(\varepsilon_t > 0)$.

Theorem 7. If $\lim_{t\to\infty} a_t = \infty$ and $\lim_{t\to\infty} \varepsilon_t = 0$, then every cluster point of the sequence $\{(\mathbf{y}_t, z_t)\}$ is an optimal solution of (7. 7).

Proof. Let $(\bar{\mathbf{y}}, \bar{z})$ be an optimal solution of (7. 7). By the definition of $(\mathbf{y}_t, z_t)$ the following inequalities hold

$$-\exp z_t - a_t\left[\sum_{k=1}^{p} h_k^2(\mathbf{y}_t) + (\varphi(\mathbf{y}_t) - z_t)^2\right] \geqq$$

$$\geqq -\exp \bar{z} - a_t\left[\sum_{k=1}^{p} h_k^2(\bar{\mathbf{y}}) + (\varphi(\bar{\mathbf{y}}) - \bar{z})^2\right] - \varepsilon_t = -\exp \bar{z} - \varepsilon_t.$$

Hence

$$0 \leqq h_k^2(y_t) \leqq \frac{1}{a_t}[-\exp z_t + \exp \bar{z} + \varepsilon_t] \leqq \frac{1}{a_t}[\exp \bar{z} + \varepsilon_t] \qquad (k=1, \ldots, p). \quad (7.10)$$

From (7. 10) we get for any cluster point $(\hat{\mathbf{y}}, \hat{z})$

$$h_k(\hat{\mathbf{y}}) = 0 \qquad (k=1, \ldots, p),$$

which means that $\hat{\mathbf{y}}$ is feasible. By the same reasoning we obtain

$$h_k(\hat{\mathbf{y}}) - \hat{z} = 0.$$

Also from (7. 10)

$$\exp z_t \leqq \exp \bar{z} + \varepsilon_t$$

which means that if $\varepsilon_t \to 0$, then $z_t \to \bar{z}$.

Corollary. Let us assume that we know lower and upper bounds for $\bar{z}$.

$$N \leqq \bar{z} \leqq M. \quad (7.11)$$

Then from (7. 10)

$$h_k^2(\mathbf{y}_t) \leqq \frac{1}{a_t}(\exp \bar{z} + \varepsilon_t) \leqq \frac{1}{a_t}(\exp M + \varepsilon_0) \quad (7.12)$$

$$(\varepsilon_0 \geqq \varepsilon_t;\, t=1, 2, \ldots) \quad (k=1, \ldots, p),$$

$$|h_k(\mathbf{y}_t)| \leqq \sqrt{\frac{\exp M + \varepsilon_0}{a_t}}.$$

If we want $|h_k(\mathbf{y}_t)| \leqq \delta$ to hold, then a_t has to be chosen to satisfy

$$a_t \geqq \frac{\exp M + \varepsilon_0}{\delta^2}. \quad (7.13)$$

Furthermore from (7. 10)

$$\exp z_t - \exp \bar{z} \leqq \varepsilon_0$$

$$\exp \{(z_t - \bar{z}) + \bar{z}\} - \exp \bar{z} \leqq \varepsilon_0$$

$$\exp \bar{z}[\exp (z_t - \bar{z}) - 1] \leqq \varepsilon_0$$

and using inequality (7. 11) we obtain the estimation

$$z_t-\bar{z} \leqq \log\left(\frac{\varepsilon_0}{\exp \bar{z}}+1\right) \leqq \log\left(\frac{\varepsilon_0}{\exp N}+1\right).$$

If we want $z_t-\bar{z} \leqq \varrho$ to hold, then we have to choose ε_0 to satisfy

$$\varepsilon_0 \leqq (\exp \varrho-1)\exp N. \tag{7. 14}$$

Summing up. If we find an "ε_0 optimal" solution to (7. 9) where ε_0 satisfies (7. 14) and a_t satisfies (7. 13), then this solution is a "(δ, ϱ) solution" of (7. 7).

To solve (7. 7) we can apply the cutting plane method described in this section.

8. General pure integer programming

Let us consider the problem

$$f(\mathbf{x}) \to \max$$

subject to

$$\mathbf{x} \in L, \tag{8. 1}$$

$$\mathbf{x} = \text{integer},$$

where L is a polyhedron and $f(\mathbf{x})$ satisfies Property (ii) in Section 7.

The method of Section 6. and 7. can be modified to be able to solve (8. 1) too. The main steps of the procedure are as follows:

Step 0. Find a feasible point (if there is any) of (8. 1) with an integer programming algorithm.

Step k. Take an extreme point $\mathbf{x}_k$ of L_k $(L=L_1)$. Denote by K_k the maximal objective function's value obtained so far on integer points of L. Let $\mathbf{A}_1$ be the matrix of transformation and T a fixed positive number.

Case 1. $\mathbf{x}_k$ is noninteger, $f(\mathbf{x}_k) \leqq K_k$.

If $|\mathbf{t}^*\mathbf{A}_1| \leqq T$, then apply cut (6. 5).

If $|\mathbf{t}^*\mathbf{A}_1| > T$, then make a Gomory cut or construct subproblems (6. 9).

Case 2. $\mathbf{x}_k$ is noninteger, $f(\mathbf{x}_k) > K_k$.

Make a Gomory cut.

Case 3. $\mathbf{x}_k$ is integer.

Apply cut (6. 5).

It can easily be proved along the lines of the proof of Theorem 1, Theorem 2 and Section 6. that these procedures converge in finite number of steps to an "ε optimal" solution of (8. 1).

9. Computational considerations

For the various algorithms contained in the previous sections concrete computational experiences are available only for application of the cutting plane method to the pure zero-one integer linear programming. Detailed description of test problems and results will be reported elsewhere. However we can mention in advance that finding the optimal solution needs much less computational effort than verifying the optimality. We think that an optimal solution of zero-one integer linear programming problems up to 120 variables can be obtained by the cutting plane method within acceptable time interval with the best computers available in Hungary. It may happen however that we cannot make sure that this is the optimal solution.

There are special problems where existence theorems assure that there is at least one integer feasible solution and every feasible point is a solution of the problem (e.g. finding an equilibrium point of a bimatrix game [21]). In these cases the cutting plane method seems to be able to solve the problem completely.

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