Tree transformations and the semantics of loop-free programs

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In memory of László Kalmár

Alagić [1975] gave a category-theoretic treatment of natural state transformations which generalized the work of Thatcher [1970], and so, in particular, gave an elegantly general perspective on tree transformations. Arbib and Manes [1977] modified Alagić's approach to provide a somewhat more concrete category-theoretic approach to what they called *process transformations,* which they showed to embrace recursion theory, bottom-up tree transformations and linear systems. Section 1 of the present note specializes the theory of process transformations to show how pure bottom-up tree transformations may be expressed in category-theoretic form. Section 2 then shows how this formulation may provide insight into the semantics of loop-free programs. Later papers will consider the effect of loops. Necessary category-theoretic background may be found in Arbib and Manes [1975], especially Chapter 7 and Section 10.1.

1. Bottom-up tree transformations: A category-theoretic characterization

We first recall the 'machines in a category' approach to tree automata (i.e. Ω-algebras).

1. Definition. An *operator domain* Ω is a sequence $(\Omega_n | n \in \mathbf{N})$ of (possibly empty) disjoint sets. An *Ω-algebra* is a pair (Q, δ) where Ω is a set and $\delta = (\delta_n)$ is a sequence of maps $\delta_n \colon Q^n \times \Omega_n \to Q$. We write δ_ω for $\delta(-, \omega) \colon Q^n \to Q$ for $\omega \in \Omega_n$. Q is the *carrier* of the algebra.

Given Ω, we define a functor X_Ω: **Set** $\to$ **Set** by

$$QX_\Omega = \bigcup_{n \geq 0} Q^n \times \Omega_n \tag{2}$$

while, for $h \colon Q \to Q'$

$$hX_\Omega(q_1, \ldots, q_n, \omega) = (hq_1, \ldots, hq_n, \omega). \tag{3}$$

We now observe that an X_Ω-dynamics in the sense of Arbib and Manes [1974] — i.e. a map $QX_\Omega \to Q$ — is just an Ω-algebra, and that an X_Ω-dynamorphism is just an *Ω-homomorphism*, since the equation $\delta' \cdot hX_\Omega = h \cdot \delta$ which characterizes a map $h\colon Q \to Q'$ as a dynamorphism $h\colon (Q, \delta) \to (Q', \delta')$ unpacks to

$$h\delta_\omega(q_1, \dots, q_n) = \delta'_\omega(hq_1, \dots, hq_n) \quad \text{for} \quad \omega \in \Omega_n, \quad (q_1, \dots, q_n) \in Q^n.$$

Moreover, X_Ω is a recursion process (which is the same as an input process in the sense of Arbib—Manes), which means that there exists an Ω-algebra $(AX_\Omega^@, A\mu_0)$ equipped with an inclusion of generators $A\eta\colon A \to AX_\Omega^@$ such that for any Ω-algebra (Q, δ) we may extend each map $\tau\colon A \to Q$ uniquely to a homomorphism $r\colon (AX_\Omega^@, A\mu_0) \to (Q, \delta)$. $AX_\Omega^@$ is the carrier of the well-known free Ω-algebra generated by A, and may be defined by the usual inductive definition (Birkhoff [1935]):

$$A \subset AX_\Omega^@$$

$$\text{If} \quad \omega \in \Omega_n,\ t_1, \dots, t_n \in AX_\Omega^@, \quad \text{then} \quad \omega t_1 \dots t_n \in AX_\Omega^@. \tag{4}$$

Thus the elements of $AX_\Omega^@$ may be regarded as finite rooted trees, with nodes of outdegree n labelled by elements of Ω_n, save that some leaves (nodes of outdegree 0) may be labelled by elements of A. We abbreviate $X_\Omega^@$ to T_Ω. We may define

$$\begin{aligned} &A\eta\colon A \to AT_\Omega, \quad a \mapsto a \\ &A\mu_0\colon AT_\Omega X_\Omega \to AT_\Omega\colon (t_1, \dots, t_n, \omega) \mapsto \omega t_1 \dots t_n. \end{aligned} \tag{5}$$

If (Q, δ) is any Ω-algebra and $\tau\colon A \to Q$ is any map

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\tau"'] & AT_\Omega \arrow[d, "r"] & AT_\Omega X_\Omega \arrow[l, "A\mu_0"'] \arrow[d, "rX_\Omega"] \\ & Q & QX_\Omega \arrow[l, "\delta"] \end{tikzcd} \tag{6}$$

then the unique dynamorphic extension $r\colon AT_\Omega \to Q$ of τ is given by

$$\begin{aligned} r(a) &= \tau(a) \\ r(\omega t_1 \dots t_n) &= \delta_\omega(rt_1, \dots, rt_n). \end{aligned} \tag{7}$$

Note that this reduces to the dynamics $\delta\colon Q \times X_0 \to Q$ of a sequential machine if we take $\Omega_1 = X_0$ while $\Omega_n = \emptyset$ for $n \neq 1$.

Suppose that Ω and Σ are two operator domains. We consider 'bottom up' (i.e. working from the leaves to the root) transformations of trees in AT_Ω into trees in BT_Σ: (The following transformations are 'pure' in that no internal state is used in processing the trees. The more general definition is given in Arbib and Manes [1979].)

8. Definition. Given operator domains Ω and Σ, and sets A and B, a *bottom-up tree transformation* $(A, \Omega) \to (B, \Sigma)$ is given by a map $\alpha: A \to B$, together with a sequence $\beta = (\beta_n)$ of maps

$$\beta_n: \Omega_n \to \{1, \ldots, n\} T_\Sigma. \tag{9}$$

The *response* of (α, β) is $\gamma: A T_\Omega \to B T_\Sigma$ defined inductively by:

Basis step:

$$\gamma(a) = \alpha(a) \tag{10}$$

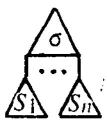
Induction step: To define

$$\gamma(\omega t_1 \ldots t_n), \quad \text{let} \quad \gamma(t_j) = s_j, \tag{11}$$

and let

$\beta(\omega) =$ 

Then

$\gamma(\omega t_1 \ldots t_n) =$ 

The following result in the style of the Yoneda Lemma (Mac Lane [1971]) allows us to view β as a natural transformation. (For an exposition of the concept of a natural transformation of functors, see Arbib and Manes [1975, Section 7.3].) This theorem is generalized in (Arbib and Manes [1977]).

12. Theorem. Let Ω be an operator domain, and let Y be any functor **Set**$\to$**Set**. Then there exists a canonical bijection

$$\frac{X_\Omega \xrightarrow{\beta} Y}{\Omega_n \xrightarrow{\beta_n} nY} \tag{13}$$

between natural transformations β and sequences (β_n) of functions. Mutually inverse passages are given by

$$\beta_n = \Omega_n \xrightarrow{k} n X_\Omega \xrightarrow{n\beta} nY \quad \text{where} \quad k(\omega) = (1, \ldots, n, \omega) \tag{14}$$

$$A\beta: A X_\Omega \to AY, \quad (a_1, \ldots, a_n, \omega) \mapsto (a_1, \ldots, a_n) Y \cdot \beta_n(\omega). \tag{15}$$

To explain the notation in (15), $(a_1, \ldots, a_n)$ is a function $g: n \to A$. Thus $(a_1, \ldots, a_n) Y$ is a function $gY: nY \to AY$.

Proof. To see that (15) describes a natural transformation, we must verify

$$\begin{tikzcd} A X_\Omega \arrow[r, "A\beta"] \arrow[d, "hX_\Omega"'] & AY \arrow[d, "hY"] \\ B X_\Omega \arrow[r, "B\beta"] & BY \end{tikzcd}$$

for arbitrary $h: A \to B$. But starting from $(g, \omega) \in A^n \times \Omega_n$, the upper path yields $hY \cdot gY(\beta_n(\omega))$ and the lower path yields $(hg) Y \cdot \beta_n(\omega)$ and these are equal since Y is a functor.

We now verify that (14) and (15) are inverse.

Now if $(\beta_n) \mapsto \beta \mapsto (\bar{\beta}_n)$, we have

$$\begin{aligned} \bar{\beta}_n(\omega) &= n\beta(1, \ldots, n, \omega) \\ &= n\beta(\mathrm{id}_n, \omega) \quad \text{for} \quad \mathrm{id}_n \in n^n \\ &= \mathrm{id}_n Y \cdot \beta_n(\omega) = \beta_n(\omega). \end{aligned}$$

Conversely, if $\beta \mapsto \beta_n \mapsto \bar{\beta}$, then for $g \in A^n$ we have the naturality square

$$\begin{tikzcd} nX_\Omega \arrow[r, "n\beta"] \arrow[d, "gX_\Omega"'] & nY \arrow[d, "gY"] \\ AX_\Omega \arrow[r, "A\beta"] & AY \end{tikzcd}$$

so that

$$\begin{aligned} (A\bar{\beta})(g, \omega) &= (gY)(\beta_n(\omega)) \\ &= (gY)(n\beta(\mathrm{id}_n, \omega)) \\ &= (A\beta)gX_\Omega(\mathrm{id}_n, \omega) \\ &= (A\beta)(g, \omega). \quad \square \end{aligned}$$

We thus conclude

16. Observation. A bottom-up tree transformation from Ω-trees to Σ-trees s equivalently given by a natural transformation

$$\beta: X_\Omega \to T_\Sigma$$

together with a map $\alpha: A \to B$. The *response* $\gamma: AT_\Omega \to BT_\Sigma$ is uniquely defined by the diagram

$$\begin{tikzcd} A \arrow[r, "A\eta^\Omega"] \arrow[d, "\alpha"'] & AT_\Omega \arrow[d, "\gamma"] & & AT_\Omega X_\Omega \arrow[ll, "A\mu_0^\Omega"'] \arrow[d, "\gamma X_\Omega"] \\ B \arrow[r, "B\eta^\Sigma"'] & BT_\Sigma & BT_\Sigma T_\Sigma \arrow[l, "B\mu^\Sigma"] & BT_\Sigma X_\Omega \arrow[l, "BT_\Sigma\beta"] \end{tikzcd} \tag{17}$$

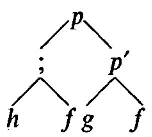
Proof. The left-hand square provides the basis step of the inductive definition of τ given in Definition (8), while the right-hand square expresses the way in which $\gamma(\omega t_1 \ldots t_n)$ depends on $\gamma(t_j)$ for $1 \leq j \leq n$. $\square$

2. Transforming loop-free flow diagrams

In this section, we capture the essential ideas of Reynolds' [1977] "Semantics of the domain of flow diagrams" by giving a succinct account of the relation between general flow diagrams and linear flow diagrams which provides the paradigm for the other relations discussed in that paper. We fix a set P of predicate symbols and a set F of function symbols. A general flow diagram may be represented by a Σ-tree where

$$\Sigma_0 = F, \quad \Sigma_1 = \emptyset, \quad \Sigma_2 = P \cup \{;\} \tag{18}$$

and we interpret the following element of $\emptyset T_\Sigma$



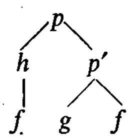
(19)

as "If the p-test yields true, execute h then f; whereas if the test yields false, carry out the p'-test, executing g if the outcome is true, f if the outcome is false."

A linear flow diagram is one in which we cannot compose arbitrary operations using ";", but instead apply one f at a time. They correspond to Ω-trees where

$$\Omega_0 = F\times\{0\}, \quad \Omega_1 = F\times\{1\}, \quad \Omega_2 = P \tag{20}$$

and (19) corresponds to the following element of $\emptyset T_\Omega$

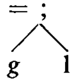


(21)

We now show that that transformation from linear flow diagrams (as represented by Ω-trees) to general flow diagrams (as represented by Σ-trees) is given by the tree transformation $\beta_n : \Omega_n \to \{1, \ldots, n\} T_\Sigma$ where

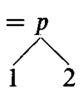
$$\beta_0(f, 0) = f$$

$\beta_1(g, 1) =$

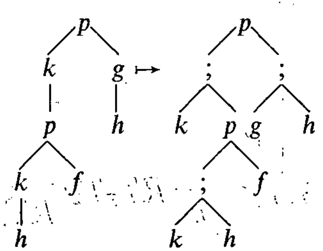


(22)

$\beta_2(p) =$



The response $\emptyset T_\Omega \to \emptyset T_\Sigma$ does indeed transform (21) into (19), and the reader may see that it also yields the following typical transformation:



(23)

Now Reynolds provides for each direct (resp., continuation) semantics for general flow diagrams a corresponding semantics for linear flow diagrams. But each semantics for a general (respectively linear) flow diagram is nothing more nor less than a Σ- (respectively Ω-) algebra. Any particular choice of a transformation of semantics which "preserves meaning" with respect to a particular transformation of flow diagrams is subsumed in the following result (which works just as well when T_Σ and T_Ω are replaced by arbitrary algebraic theories T_1 and T_2, see Manes [1976, Section 3.2]):

24. Proposition. Let Ω and Σ be operator domains, and let $\xi: RX_\Sigma \to R$ be a given Σ-algebra. Further, let the family of maps

$$\beta_n: \Omega_n \to \{1, \ldots, n\}T_\Sigma$$

define a tree transformation. Then there exists an Ω-algebra $\delta: RX_\Omega \to R$ such that the result of running δ on any Ω-tree equals the result of running ξ on the transformed Σ-tree.

Proof. By (13), β_n is equivalent to a natural transformation

$$\beta: X_\Omega \to T_\Sigma$$

yielding, in particular, the map

$$R\beta: RX_\Omega \to RT_\Sigma. \tag{25}$$

Now we define the run map $\xi^@: RT_\Sigma \to R$ of (R, ξ) by the diagram (compare (6))

$$\begin{tikzcd} R \arrow[r, "R\eta^\Sigma"] \arrow[dr, "\mathrm{id}_R"'] & RT_\Sigma \arrow[d, "\xi^@"] & RT_\Sigma X_\Sigma \arrow[l, "R\mu_0^\Sigma"'] \arrow[d, "\xi^@ X_\Sigma"] \\ & R & RX_\Sigma \arrow[l, "\xi"'] \end{tikzcd} \tag{26}$$

and we may then define an Ω-algebra (δ, R) by

$$\delta = RX_\Omega \xrightarrow{R\beta} RT_\Sigma \xrightarrow{\xi^@} R. \tag{27}$$

To show that δ has the claimed property, we must look at the response $\gamma: RT_\Omega \to RT_\Sigma$ of the tree transformation with $A = B = R$ and $\alpha = id_R$. Then (17) becomes:

$$\begin{tikzcd} R \arrow[r, "R\eta^\Omega"] \arrow[dr, "R\eta^\Sigma"'] & RT_\Omega \arrow[d, "\gamma"] & & RT_\Omega X_\Omega \arrow[ll, "R\mu_0^\Omega"'] \arrow[d, "\gamma X_\Omega"] \\ & RT_\Sigma & RT_\Sigma T_\Sigma \arrow[l, "R\mu^\Sigma"] & RT_\Sigma X_\Omega \arrow[l, "RT_\Sigma\beta"] \end{tikzcd} \tag{28}$$

We have to show that $\delta^{@}=RT_\Omega \xrightarrow{\gamma} RT_\Sigma \xrightarrow{\xi^{@}} R$ to complete the proof of the proposition. But this is immediate from the following diagram:

$$
\begin{tikzcd}
R \arrow[r, "R\eta^{\Omega}"] \arrow[dr, "R\eta^{\Sigma}"'] \arrow[ddr, bend right, "\mathrm{id}_R"'] & RT_\Omega \arrow[d, "\gamma"] \arrow[dl, phantom, "\mathrm{I}"] & & RT_\Omega X_\Omega \arrow[ll, "R\mu_0^{\Omega}"'] \arrow[d, "\gamma X_\Omega"] \arrow[dll, phantom, "\mathrm{II}"] \\
& RT_\Sigma \arrow[d, "\xi^{@}"] \arrow[dl, phantom, "\mathrm{III}"] & RT_\Sigma T_\Sigma \arrow[l, "R\mu^{\Sigma}"'] \arrow[d, "\xi^{@} T_\Sigma"] \arrow[dl, phantom, "\mathrm{IV}"] & RT_\Sigma X_\Omega \arrow[l, "RT_\Sigma\beta"'] \arrow[d, "\xi^{@} X_\Omega"] \arrow[dl, phantom, "\mathrm{V}"] \\
& R & RT_\Sigma \arrow[l, "\xi^{@}"'] \arrow[d, phantom, "\mathrm{VI}"] & RX_\Omega \arrow[l, "R\beta"'] \arrow[ll, bend left=60, "\delta"] \\
& & &
\end{tikzcd}
\tag{29}
$$

where I and II are just (28), III and IV extend (26), V is a naturality square for θ, and VI is the definition of δ. Thus $\xi^{@}\cdot\gamma$ satisfies the diagram which defines $\delta^{@}$ uniquely. □

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