Use of Petri nets for performance evaluation

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Introduction

Petri nets [1], [2] have been found a simple and elegant formalism for the description of asynchronous systems with concurrent evolutions. According to the adopted interpretation, they can be used to model flow phenomena of information, of energy and of materials [3], [4] and [5]. However, this model is not complete enough for the study of system performances since no assumption is made about the firing of a transition as far as its duration and the moment at which it takes place after the transition has been enabled.

Timed Petri nets have been introduced by C. Ramchandani [6] by associating firing times to the transitions of Petri nets. He studied the steady state behavior and gave methods for calculating the throughput rate for certain classes of Petri nets. The results given in this paper are applicable to the class of pure [7] Petri nets and generalize, in some sense, those presented in [6]. The litterature on timed Petri nets is very poor: to the author's knowledge, the only works on this subjet are the Ramchandani's thesis and a paper by S. Ghosh [8] comparing the properties of boundedness and liveness for timed Petri nets and unrestricted Petri nets.

I. Definitions

Definition 1. A Petri net (PN) is a quadruple $\mathcal{N}=(P, T, \alpha, \beta)$ where

P is a set of *places*, $P\neq\emptyset$

T is a set of transitions, $T\neq\emptyset$, $P\cap T=\emptyset$

α: $P\times T\to\mathbf{N}$ forward incidence function

β: $P\times T\to\mathbf{N}$ backward incidence function

($\mathbf{N}$ represents the set of natural numbers: 0, 1, 2, 3, ...).

REPRESENTATION. To a PN one can associate a digraph the nodes of which are the places and the transitions, represented respectively by circles and dashes. There is a directed edge from the place p_s to the transition t_j iff $\alpha(p_s, t_j)=n_{s_j}\neq 0$. This edge is labeled by the value n_{sj}, called weight of the edge. There also is a directed edge from the transition t_r to the place p_w iff $\beta(p_w, t_r)=n_{w_r}\neq 0$. This edge is labeled by the weight n_{w_r}.

Definition 2. Let $\mathcal{N}=(P, T, \alpha, \beta)$ be a PN. We adopt the following notations: For $t \in T$, ${}^{\bullet}t=\{p \in P | \alpha(p, t) \neq 0\}$ and $t^{\bullet}=\{p \in P | \beta(p, t) \neq 0\}$.
For $p \in P$, ${}^{\bullet}p=\{t \in T | \beta(p, t) \neq 0\}$ and $p^{\bullet}=\{t \in T | \alpha(p, t) \neq 0\}$.
We call ${}^{\bullet}t$ $(t^{\bullet})$ *set of input (output) places* of t and by analogy, ${}^{\bullet}p$ $(p^{\bullet})$ *set of input (output) transitions* of p. These notations are extended to subsets of T and P; for example, if $P_1 \subset P$ then, ${}^{\bullet}P_1 = \bigcup_{p_k \in P_1} {}^{\bullet}p_k$.

Definition 3. A marking M of a PN $\mathcal{N}=(P, T, \alpha, \beta)$ is a mapping of P into $\mathbf{N}: P \xrightarrow{M} \mathbf{N}$. When $|P|=n$, one can represent a marking M by a vector $M \in \mathbf{N}^n$ such that its i-th entry $m_i=M(p_i)$.

Definition 4. A transition t of a PN is *enabled* for a marking M iff:

$$\forall p \in {}^{\bullet}t \Rightarrow \alpha(p, t) \leqq M(p).$$

Definition 5. Let $\mathcal{M}_t$ be the set of markings for which a transition t of a PN is enabled. The *firing* of the transition $t(F(t))$ is a mapping of $\mathcal{M}_t$ into the set of the markings $\mathcal{M}$ defined as follows: if $F(t)[M_i]=M_j$ then

$$M_j(p) = \begin{cases} M_i(p), \forall p \notin {}^{\bullet}t \cup t^{\bullet}, \\ M_i(p)-\alpha(p, t), \forall p \in {}^{\bullet}t-({}^{\bullet}t \cap t^{\bullet}), \\ M_i(p)+\beta(p, t), \forall p \in t^{\bullet}-({}^{\bullet}t \cap t^{\bullet}), \\ M_i(p)+\beta(p, t)-\alpha(p, t), \forall p \in {}^{\bullet}t \cap t^{\bullet}. \end{cases}$$

Definition 6. Let $\mathcal{N}=(P, T, \alpha, \beta)$ be a PN and M_0 one of its markings. Consider a sequence of transitions $\sigma=t_{j_1} t_{j_2} \dots t_{j_s}$. We say that σ is a *simulation sequence* or a *firing sequence* from M_0 iff there exists a sequence of markings $M_1, M_2, M_3, \dots, \dots, M_s$ such that $F(t_{j_i})[M_{i-1}]=M_i$ for $i=1, 2, 3, \dots, s$. We note that $M_0 \xrightarrow{\sigma} M_s$. M_s is the marking *attained* by applying σ from M_0. We denote by $\vec{M}_0$ the set of markings that can be attained from M_0. The *firing vector* of σ is a vector $R (R \in \mathbf{N}^m, m=|T|)$ such that its k-th entry is equal to the number of occurrences of the transition t_k in σ.

Definition 7. An *ordinary* PN is a PN $\mathcal{N}=(P, T, \alpha, \beta)$, such that $\alpha: P \times T \to \{0, 1\}$ and $\beta: P \times T \to \{0, 1\}$. A *marked graph* is an ordinary PN such that $\forall p \in P$, $|{}^{\bullet}p| \leqq 1$ and $|p^{\bullet}| \leqq 1$. A *state graph* is an ordinary PN such that $\forall t \in T$, $|t^{\bullet}| \leqq 1$ and $|{}^{\bullet}\tau| \leqq 1$.

Definition 8. Take a PN and one of its markings M_0.
We say that a *place p is bounded for* M_0 iff $\exists k \in \mathbf{N}$ such that $\forall M \in \vec{M}_0$, $M(p)<k$. A PN is *bounded for* M_0 iff all its places are bounded. We say that a *transition t is live* for M_0 iff for every marking M, $M \in \vec{M}_0$, there exists a sequence σ, $\sigma \in T^*$ such that σt is a firing sequence from M. A net having all its transitions live for a marking M_0 is called *live for* M_0.

Definition 9. A *pure* PN is a PN such that $\forall t \in T$, $\{{}^{\bullet}t\} \cap \{t^{\bullet}\}=\emptyset$.
For a pure PN $\mathcal{N}=(P, T, \alpha, \beta)$ $(|P|=n, |T|=m)$ one can define the matrices:

$$C = [c_{ij}]_{n \times m} \quad \text{with} \quad c_{ij} = \begin{cases} \beta(p_i, t_j) & \text{if} \quad \beta(p_i, t_j) \neq 0, \\ -\alpha(p_i, t_j) & \text{if} \quad \alpha(p_i, t_j) \neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

C is called *incidence matrix* of the net [7].

$$C^+ = [c_{ij}^+]_{n\times m} \quad \text{with} \quad c_{ij}^+ = \begin{cases} \beta(p_i, t_j) & \text{if} \quad \beta(p_i, t_j) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$C^- = [c_{ij}^-]_{n\times m} \quad \text{with} \quad c_{ij}^- = \begin{cases} \alpha(p_i, t_j) & \text{if} \quad \alpha(p_i, t_j) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK. $C = C^+ - C^-$.

II. Timed Petri nets

II. 1. Definitions

Definition 10. A *timed Petri net* (TPN) consists in giving:
a Petri net $\mathcal{N} = (P, T, \alpha, \beta)$,
$T = (\tau_1, \tau_2, \ldots, \tau_i, \ldots)$ an increasing sequence of real numbers called *time base*,
a mapping ν: $P \times T \to T$ such that $\forall (p, \tau_i) \in P \times T$: $\nu(p, \tau_i) = \tau_j \Rightarrow \tau_j \geqq \tau_i$.

SIMULATION RULES

a) A marker in a TPN may be in one of the two following states: *available* or *unavailable*. Initially each place p contains $M_0(p)$ available markers.

b) A transition t is enabled iff every place $p_s (p_s \in {}^{\bullet}t)$ contains $\alpha(p_s, t)$ available markers at least.

c) The firing of a transition t has to take place instantaneously as soon as t is enabled. It consits in removing $\alpha(p_s, t)$ available markers from each place p_s $(p_s \in {}^{\bullet}t)$ and in placing $\beta(p_w, t)$ markers in each place $p_w \in t^{\bullet}$.

d) A marker remains unavailable in a place p_s during the time interval between the instant of its arrival τ_i and the instant $\nu(p_s, \tau_i)$; then it becomes available.

REMARK. According to the above definition, firings in a TPN take place only at moments of T.

In what follows we study the behaviour of pure TPN's such that $\forall (p_i, \tau) \in \in P \times T$: $\forall (p_i, \tau) - \tau = z_i =$constant. That is, each marker is delayed by z_i in the place independently of the instant of its arrival.

Definition 11. Take a TPN and let $M_1 M_2 \ldots M_s$ be the markings attained successively from an initial marking M_0 by applying a firing sequence $\sigma = t_{i_0}, t_{i_1}, \ldots, \ldots, t_{i_{s-1}}$ and $\tau_{i_0}, \tau_{i_1}, \ldots, \tau_{i_{s-1}}$ the moments of firing of the transitions $t_{i_0}, t_{i_1}, \ldots, t_{i_{s-1}}$ respectively. The marking of the net at a moment τ_{i_k} will be by definition the marking of the net in the interval $\tau_{i_{k-1}} \leqq \tau' < \tau_{i_k}$. Generally, the marking of a TPN with $T = (\tau_0, \tau_1, \tau_2, \ldots, \tau_i, \ldots)$ at a moment $\tau_i \in T$ will be the marking of the net at the interval $\tau_{i-1} \leqq \tau < \tau_i$, $i = 1, 2, 3, \ldots$ The marking at τ_0 corresponds to the initial marking. For a TPN with n places we define a general temporal variable $Q^t(\tau) = [q_1(\tau), q_2(\tau), \ldots, q_n(\tau)]$ such that $\forall \tau_i \in T$, $Q(\tau_i) = M$ where M is the marking at the moment τ_i (Q^t denotes the transpose of a matrix Q). The variable $Q(\tau)$ will be called *charge variable*.

Let M_s be a marking attained from a marking M_0 by applying a sequence σ, $(M_0 \xrightarrow{\sigma} M_s)$ in a PN defined by its incidence matrix C. Then

$$M_s = M_0 + CR \tag{I}$$

where $R \in \mathbf{N}^m$, $m = |T|$, is the firing vector of σ. Equation (I) can be written for a TPN

$$Q(\tau) = Q(\tau_0) + CR(\tau). \tag{II}$$

Let us suppose now, that $\tau \neq \tau_0$ and put $\Delta\tau = \tau - \tau_0$. We have from (II)

$$\frac{\Delta Q(\tau)}{\Delta \tau} = \frac{Q(\tau) - Q(\tau_0)}{\Delta \tau} = C\frac{R(\tau)}{\Delta \tau} = CI(\tau) \Rightarrow \frac{\Delta Q(\tau)}{\Delta \tau} = CI(\tau) \tag{III}$$

where

$\frac{Q(\tau)}{\Delta\tau}$ is a vector representing the mean variation of the charge of the net in the interval $\Delta\tau$,

the k-th entry of the vector I, $i_k = \frac{r_k(\tau)}{\Delta\tau}$ represents the mean frequence of firing of the transition t_k during $\Delta\tau$.

The vector $I(\tau)$ will be called *current vector*, and evidently $\forall \tau_j \in T$, $I(\tau_j) > 0$.

II. 2. Description of the behavior for constant currents

II. 2.1 — **General case**. We are interested in the cases of functioning with constant currents for which the total charge of the net remains bounded. This amounts to searching for solutions of the equation

$$CI = 0 \quad (I > 0). \tag{IV}$$

Those solutions correspond to cyclic firing sequences in the net as it is shown in [6]. We give additional relations that the current vector I must satisfy in terms of the initial marking and of the delays associated to the places.

Definition 12. Let C be a matrix of order $n \times m$ on Q. We denote by $\mathscr{C}$ ($\mathscr{C}^t$) the set of non negative solutions of $CX = 0$ ($C^t X = 0$). A *generator* of $\mathscr{C}(\mathscr{C}^t)$ is a set of vectors $\{X_j\}_{j=1}^s$ with $X_j \in \mathbf{N}^m$ ($X_j \in \mathbf{N}^n$) such that any element X_0 of $\mathscr{C}(\mathscr{C}^t)$ could be expressed as the linear combination of elements of $\{X_j\}_{j=1}^s$ with non-negative rational coefficients, i.e., $X_0 = \sum_{j=1} \lambda_j X_j$, where λ_j are non-negative rational numbers.

If we assign constant currents to the transitions of a bounded TPN, we have a periodic functioning, and let $Q(\tau_{k_0})$, $Q(\tau_{k_1})$, ..., $Q(\tau_{k_s})$ be the successive markings of the net during a period. Then, the mean value $\bar{Q}$ of the charge variable $Q(\tau)$ is given by

$$\bar{Q} = \frac{Q(\tau_{k_0}) + Q(\tau_{k_1}) + Q(\tau_{k_2}) + \ldots + Q(\tau_{k_s})}{s+1}.$$

If we multiply this last equation by $J_0^t (J_0^t \in \mathscr{C}^t)$, we obtain

$$J_0^t \bar{Q} = J_0^t Q(\tau_0). \tag{Va}$$

But the mean value $\bar{q}_w$ of the charge of a place p_w satisfies the inequality

$$\bar{q}_w \geqq z_w C_w^+ I$$

where C_w^+ is the w-th line of the matrix C^+, and the product $C_w^+ I$ represents the mean frequence of the arrivals of markers at the place p_w.

Let Z be the following square matrix of order n:

$$Z = \begin{bmatrix} z_1 & 0 & 0 & 0 & 0 \dots 0 \\ 0 & z_2 & 0 & 0 & 0 \dots 0 \\ 0 & 0 & z_3 & 0 & 0 \dots 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \dots z_n \end{bmatrix}$$

Then, the set of the inequalities $\{\bar{q}_w \geqq z_w C_w^+ I\}_{w=1}^n$ can be written in the form

$$\bar{Q} \geqq ZC^+I = ZC^-I. \tag{Vb}$$

Let J_0^t be a positive solution of $J^t C = 0$. From (Va) and (Vb) one can obtain

$$J_0^t Q(\tau_0) \geqq J_0^t ZC^+I = J_0^t ZC^-I. \tag{Vc}$$

This last inequality establishies a relation between the initial marking, the current vector and the delays associated to the places of a TPN.

Let $\mathscr{J} = \{J_1^t, J_2^t, \dots, J_k^t\}$ be a generator of $\mathscr{C}^t$. If $J_0^t \in \mathscr{C}^t$ then any inequality $J_0^t Q(\tau_0) \geqq J_0^t ZC^+ I$ can be expressed as a linear combination of the set of inequalities $\{J_s^t Q(\tau_0) \geqq J_s^t ZC^+ I\}_{s=1}^k$.

The relations

$$CI = 0 \quad (I > 0) \tag{IV}$$

$$\{J_s^t Q(\tau_0) \geqq J_s^t ZC^+ I\}_{s=1}^k \tag{V}$$

describe the functioning of a timed Petri net for constant currents.

II. 2.2 — **Functioning of TPN at its natural rate.**

Definition 13. Given a TPN by its incideme matrix C and its delay matrix Z. We say that it functions at its *natural rate* for a given current vector I_0 iff I_0 satisfies the equations $CI = 0$ (IV) and $\{J_s^t Q(\tau_0) = J_s^t ZC^+ I\}_{s=1}^k$ (VI), where $\{J_s^t\}_{s=1}^k$ is a generator of $\mathscr{C}^t$.

Proposition 1. There exist at most n linearly independent equations describing the functioning at natural rate of a TPN with n places.

Proof. Suppose that the rank of C is equal to ϱ. Then (IV) contains at most ϱ linearly independent equations. Also, the dimension of the space of the solutions of $J^t C = 0$ is $n - \varrho$. Thus (VI) has at most $n - \varrho$ linearly independent equations, and consequently there exist at most n linearly independent equations in the system (IV) and (VI).

Example 1. Consider the TPN of Figure 1. We want to calculate the current vectors (if there exist any) corresponding to functionings at natural rate. $Q(\tau_0)$ and Z are supposed given.

$$C=\begin{bmatrix}-1 & -1 & 1 & 1\\ 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0\\ -3 & 1 & 0 & 0\\ 3 & -1 & 0 & 0\end{bmatrix}$$

Solution of $CI=0$: we find $i_2=i_4=3i_1$, $i_1=i_3$

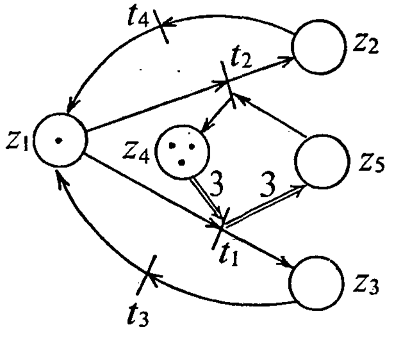

Figure 1

A generator of $\mathscr{C}^t$ is $\{J_1^t=[1\ 1\ 1\ 0\ 0],\ J_2^t=[0\ 0\ 0\ 1\ 1]\}$. Thus

$$J_1^tZC^+I=J_1^tQ_0 \Rightarrow q_{0_1}+q_{0_2}+q_{0_3}=z_1i_3+z_1i_4+z_2i_2+z_3i_1$$

and

$$J_2^tZC^+I=J_2^tQ_0 \Rightarrow q_{0_4}+q_{0_5}=i_2z_4+3i_1z_5$$

The condition for the existence of a solution is

$$\frac{q_{0_1}+q_{0_2}+q_{0_3}}{4z_1+3z_2+z_3}=\frac{q_{0_4}+q_{0_5}}{3(z_4+z_5)}\ (\alpha).$$

In this case

$$i_1=\frac{q_{0_4}+q_{0_5}}{3(z_4+z_5)},\quad i_2=\frac{q_{0_4}+q_{0_5}}{z_4+z_5}.$$

Suppose now that we have $z_1=z_2=z_3=z_4=z_5=1$ and $Q_0^t=[1\ 0\ 0\ 3\ 0]$. Then the equation (α) is not verified and there is no functioning at natural rate. The inequalities (V) give

$$q_{0_1}+q_{0_2}+q_{0_3}\geqq z_1(i_3+i_4)+z_2i_2+z_3i_1 \Rightarrow 1\geqq 8i_1,$$

$$q_{0_4}+q_{0_5}\geqq i_2z_4+3i_1z_5 \Rightarrow 3\geqq 6i_1$$

yielding

$$i_{1\max}=\min\left\{\frac{1}{8},\ \frac{1}{2}\right\}=\frac{1}{8}\ \text{ and }\ i_{2\max}=\frac{3}{8}.$$

III. Solution of $CI=0$ and $J^tC=0$ decomposition

In this section we present some results relative to the properties of non-negative solutions of $CI=0$ and $J^tC=0$, where C is the incidence matrix of a PN. Many authors have used linear equations for the study of the properties of PN's ([6], [7], [9], [10], [11], [12] and [14]). In particular, a part of the results on the decomposition of PN's exposed in this section have been developped independently by Memmi [10], Crespi—Reghizzi and Mandrioli [9] and the author [14]. Also, similar results, in a less restrained context, are well known since several years (see, for example [13]). Our contribution consists in making evident the ralations between the structure of the net (decomposability into consistent and invariant components) and the solutions of $CI=0$ and $J^tC=0$. We borrowed the terms "consistent" and "invariant" from [6] and [7], respectively, and the term "support" from FulKerson [13], although it is used in a slightly different sense. This study is limited to pure and strongly connected PN's. Pureness is imposed by the fact that we use the incidence matrix for representing PN's and strong connexity by the fact that it is a necessary condition for a net to be bounded [6]. In what follows, the term "PN" denotes a strongly connected and pure PN.

Definition 14. For a PN $\mathcal{N}=(P, T, \alpha, \beta)$ a *subnet* of $\mathcal{N}$ is a PN $\mathcal{N}_1=(P_1, T_1, \alpha_1, \beta_1)$ such that $P_1 \subset P$, $T_1 \subset T$, moreover $\alpha_1\colon P_1 \times T_1 \to \mathrm{N}$ and $\beta_1\colon P_1 \times T_1 \to \mathrm{N}$ are restrictions of α and β, respectively.

Definition 15. The *union* of two subnets $\mathcal{N}_1=(P_1, T_1, \alpha_1, \beta_1)$ and $\mathcal{N}_2=(P_2, T_2, \alpha_2, \beta_2)$ of a PN $\mathcal{N}=(P, T, \alpha, \beta)$ is a subnet $\mathcal{N}_3=(P_3, T_3, \alpha_3, \beta_3)$ of $\mathcal{N}$ with $P_3=P_1 \cup P_2$ and $T_3=T_1 \cup T_2$.

Definition 16. Let $\mathcal{N}=(P, T, \alpha, \beta)$ be a PN and $\mathcal{S}=\{\mathcal{N}_i=(P_i, T_i, \alpha_i, \beta_i)\}_{i=1}^k$ a set of subnets of $\mathcal{N}$. $\mathcal{N}$ is *covered* by S or S is a *decomposition* of $\mathcal{N}$ if

$$P = \bigcup_{i=1}^{k} P_i \quad \text{and} \quad T = \bigcup_{i=1}^{k} T_i.$$

III. 1. Non-negative solutions of $CI=0$. Decomposition into consistent components

Definition 17. Let $\mathcal{N}=(P, T, \alpha, \beta)$ be a PN. Then a set $T_1 \subset T$ defines a *t-complete subnet* $\mathcal{N}_1=(P_1, T_1, \alpha_1, \beta_1)$ of $\mathcal{N}$ if $P_1={}^{\bullet}T_1=T_1^{\bullet}$.

Proposition 2. Let C be the incidence matrix of a PN and $I_0 \in \mathcal{C}$. Then the set $F_1=\{t_j | i_{0_j} \neq 0\}$ defines a t-complete subnet of the net having C as incidence matrix.

Proof. Consider the subnet with $T_1=\{t_j | i_{0_j} \neq 0\}$ and $P_1={}^{\bullet}T_1 \cup T_1^{\bullet}$. Then each place p of P_1 has at least one input transition or one output transition (by construction of the set P_1). Suppose that a place p_w ($p_w \in P_1$) has the input transitions $t_{i_1}, t_{i_2}, \ldots, t_{i_r}$ but no output transition in the subnet defined by P_1 and T_1. Then we have $\sum_j i_{0_{i_j}} \beta(p_w, t_{i_j})=0$ where $i_{0_{i_j}}$ and $\beta(p_w, t_{i_j})$ are positive rational numbers which is absurde. Thus p_w must have an output transition belonging to T_1. In the same manner one can prove that if a place p_w has an output transition belonging to T_1 then it has an input transition belonging to T_1.

Definition 18. Let C be the incidence matrix of a PN $\mathcal{N}$. A *consistent component* of $\mathcal{N}$ is any t-complete subnet $\mathcal{N}_1$ defined by the set of transitions corresponding to the positive entries of a vector I_1 ($I_1 \in \mathscr{C}$). $\mathcal{N}_1$ is the *support* of I_1, (we note $\mathcal{N}_1 = S(I_1)$). If there exists I_0 such that $S(I_0) = \mathcal{N}$ then we say that $\mathcal{N}$ is *consistent*.

Definition 19. Let a PN be given with an initial marking M. A firing sequence σ is a *cyclic* firing sequence from M iff $M \xrightarrow{\sigma} M$.

Proposition 3 [6]. A PN having a live and bounded marking is concistent.

Proposition 4. [6]. Let $\mathcal{N}_1 = (P_1, T_1, \alpha_1, \beta_1)$ be a consistent component of a net $\mathcal{N}$. Then $\mathcal{N}_1$ has a marking M from which there exists a cyclic firing sequence $\sigma = t_{k_1}\, t_{k_2} \ldots t_{k_s}$ such that $\bigcup_{j=1}^{s} t_{k_j} = T_1$. Inversely, each cyclic firing sequence $\sigma = t_{k_1}\, t_{k_2} \ldots t_{k_s}$ in $\mathcal{N}$, defines a consistent component of $\mathcal{N}$ having $T_1 = \bigcup_{j=1}^{s} t_{k_j}$ as set of transitions.

Proposition 5. The union of two consistent components of a net is a consistent component.

Proof. Let I_1 and I_2 be two elements of $\mathscr{C}$ defining two consistent components $S(I_1)$ and $S(I_2)$. Then $I_1 + I_2 \in \mathscr{C}$ defines the consistent component $S(I_1) \cap S(I_2)$.

Definition 20. Let $I_1 \in \mathscr{C}$ where C is the incidence matrix of a PN $\mathcal{N}$. Then $S(I_1)$ is an *elementary* consistent component of $\mathcal{N}$ iff there exists no I_2 ($I_2 \neq 0$, $I_2 \in \mathscr{C}$) such that $S(I_2) \not\subset S(I_1)$. A vector I_1 defining an elementary consistent component $S(I_1)$ is called *elementary vector* of $\mathscr{C}$.

Proposition 6. If C is the incidence matrix of a PN and I_1 and I_2 are two elementary vectors of $\mathscr{C}$ such that $S(I_1) = S(I_2)$ then I_1 and I_2 are linearly dependent.

Proof. Suppose that T_1 and I_2 are linearly independent and $S(I_1) = S(I_2)$. Let $\lambda = \min_{i_{2_j} \neq 0} \left\{ \frac{i_{1_j}}{i_{2_j}} \right\}$ and $I_3' = I_1 - \lambda I_2$. Then $I_3' \neq 0$ and there exists a scalar μ such that $I_3 = \mu I_3' \in \mathscr{C}$. We have $S(I_3) \not\subset S(I_1)$. Thus $S(I_1)$ is not elementary.

Proposition 7. Every consistent PN $\mathcal{N}$ can be decomposed into a set of elementary consistent components.

Proof. Let $I_0 \in \mathscr{C}$, $S(I_0) = \mathcal{N}$ and suppose that $\mathcal{N}$ is not elementary. Then, there exists a consistent component $\mathcal{N}_1 (\mathcal{N}_1 \not\subset \mathcal{N})$ and I_1 such that $S(I_1) = \mathcal{N}_1$. Let $\lambda = \min_{i_{1_j} \neq 0} \left\{ \frac{i_{0_j}}{i_{1_j}} \right\}$ and $I_2' = I_0 - \lambda I_1$. Then it is easy to verify that there exists a scalar μ such that $I_2 = \mu I_2' \in \mathscr{C}$ and if $\mathcal{N}_2 = S(I_2)$ then $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$.

Corollary 1. The set of elementary vectors of $\mathscr{C}$ is a generator.

Definition 21. Let a PN $\mathcal{N}$ with incidence matrix C be given and S a set of elementary vectors of $\mathscr{C}$. Then S is a *t-base* of $\mathcal{N}$ iff S is a generator (of $\mathscr{C}$) of minimal cardinality.

Proposition 8. Let $B=[I_1, I_2, \ldots, I_s]$ be a matrix of order $m\times s$ such that $\{I_j\}_{j=1}^s$ is a t-base of a PN. Then the rank of B is less than or equal to $m-\varrho$ where ϱ is the rank of the incidence matrix of the net. Furthermore, if the net is consistent then the rank of B is equal to $m-\varrho$.

Proof. The fact that rank $[B]\leqq m-\varrho$ is obvious because the space of the solutions of $CI=0$ is of dimension $m-\varrho$. In order to prove that rank $[B]=m-\varrho$, in the case of a consistent net, it is sufficient to prove that any solution I_0 of $CI=0$ can be expressed as the linear combination of $I_1, I_2, \ldots, I_s$ (columns of B). If $I_0>0$ then this is always possible according to corollary 1. If not, one can obtain from I_0, a vector $\tilde{I}$ ($\tilde{I}>0$) such that $\tilde{I}=\sum_{j=1}^{s} \beta_j I_j + I_0$ where the β_j's are non-negative rational numbers. But $C\tilde{I}=0$ and $\tilde{I}$ defines a consistent component. Thus, according to the Corollary 1, we can write $\tilde{I}=\sum_{j=1}^{s} \gamma_j I_j$. This gives $I_0=\sum_{j=1}^{s} (\gamma_j-\beta_j) I_j$.

REMARK. $CB=0$. For any current vector $I\in\mathscr{C}$, $I=BI_b$, the k-th entry of I_b being the "loop current" associated to the elementary component corresponding to the k-th column of B.

III. 2. Non-negative solutions of $J^tC=0$. Decomposition into invariant components

The following definitions and propositions are dual of those in III. 1.

Definition 22. Let $\mathscr{N}=(P, T, \alpha, \beta)$ be a PN. Then a set P_1 ($P_1\subset P$) defines a *p-complete* subnet $\mathscr{N}_1=(P_1, T_1, \alpha_1, \beta_1)$ of $\mathscr{N}$ if $T_1={}^{\bullet}P_1=P_1^{\bullet}$.

Proposition 9. Let C be the incidence matrix of a PN and $J_0^t\in\mathscr{C}^t$. Then the set $P_1=\{p_i | j_{0_i}\neq 0\}$ defines a p-complete subnet of the net having C as incidence matrix.

Definition 23. Let C be the incidence matrix of a PN $\mathscr{N}$. An *invariant component* pf $\mathscr{N}$ is any p-complete subnet $\mathscr{N}_1$ defined by the set of places corresponding to the oositive entries of a vector J_1^t ($J_1^t\in\mathscr{C}^t$). $\mathscr{N}_1$ is the *support* of J_1^t, (we note $\mathscr{N}_1=S(J_1^t)$). If there exists J_0^t such that $S(J_0^t)=\mathscr{N}$ then we say that $\mathscr{N}$ is *invariant*.

Proposition 10. The union of two invariant components is an invariant component.

Definition 24. Let $J_1^t\in\mathscr{C}^t$ where C is the incidence matrix of a PN $\mathscr{N}$. Then $S(J_1^t)$ is an *elementary* invariant component of $\mathscr{N}$ iff there exists no J_2^t ($J_2^t\neq 0$, $J_2^t\in\mathscr{C}^t$) such that $S(J_2^t)\not\subset S(J_1^t)$. A vector J_1^t defining an elementary invariant component $S(J_1^t)$ is called *elementary vector* of $\mathscr{C}^t$.

Proposition 11. If C is the incidence matrix of a PN and J_1^t and J_2^t are two elementary vectors of $\mathscr{C}^t$ with $S(J_1^t)=S(J_2^t)$ then J_1^t and J_2^t are linearly dependent.

Proposition 12. Every invariant PN $\mathscr{N}$ can be decomposed into a set of elementary invariant components.

Corollary 2. The set of elementary vectors of $\mathscr{C}^t$ is a generator (of $\mathscr{C}^t$).

Definition 25. Let $\mathcal{N}$ be a PN with incidence matrix C and S a set of elementary vectors of $\mathscr{C}^t$. Then S is a *p-base* of $\mathcal{N}$ iff S is a generator (of $\mathscr{C}^t$) of minimal cardinality.

Proposition 13. Let $D^t=[J_1, J_2, \ldots, J_s]$ be a matrix of order $n\times s$ such that $\{J_i^t\}_{i=1}^s$ is a p-base of a PN. Then the rank of D is less than or equal to $n-\varrho$ where ϱ is the rank of the incidence matrix of the net. Furthermore, if the net is invariant then rank $[D]=n-\varrho$.

III. 3. Particular cases: state graphs and marked graphs

Proposition 14. Let C be the incidence matrix of a pure and strongly connected state graph with n places and m transitions. Then the following statements are well known:

a) rank $[C]=n-1$,

b) the space of solutions of $CI=0$ is of dimension $m-n+1$,

c) the space of solutions of $J^tC=0$ is of dimension 1 and the vector $J_0^t=[1\ 1\ 1\ldots 1]$ is a base of this space.

REMARKS. A t-base for a state graph is a circuit base,
C in Proposition 14 expresses the fact that any strongly connected state graph is an elementary invariant component.

Proposition 15. Let C be the incidence matrix of a pure strongly connected marked graph with n-places and m transitions. Then we have the dual of Proposition 14:

a) rank $[C]=m-1$,

b) the space of the solutions of $CI=0$ is of dimension 1 and the vector $I_0^t= =[1\ 1\ 1\ \ldots\ 1]$ is a base of this space,

c) the space of solutions of $J^tC=0$ is of dimension $n-m+1$.

REMARKS. A t-base for a marked graph is a circuit base,
b) in Proposition 15 expresses the fact that any strongly connected marked graph is an elementary consistent component.

IV. Resolution of the equations (IV) and (VI) with given $Q(\tau_0)$ and Z

In this section we show that the problem of determining the currents of a TPN for functioning at natural rate, when we know $Q(\tau_0)$ and Z may have either several solutions or no solution at all. The extreme cases correspond to state graphs and marked graphs.

Example 2. For the TPN of Figure 2 the system of the equations (IV) and (VI) generally has a unique solution for I. We have:

$$B^t=\begin{bmatrix}2 & 1 & 1 & 0 & 1 & 1\\ 2 & 1 & 1 & 1 & 2 & 0\end{bmatrix} \text{ and } D=\begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}.$$

If i_x and i_y are the currents associated respectively to the two elementary consistent components of the net, we have, $I=B\begin{vmatrix} i_x \\ i_y \end{vmatrix}$.

On the other hand, we have two equations expressing the charge conservation in the state graphs defined by the lines of D:

$$(i_x+i_y)2z_0=1 \quad \text{(we put } z_0=z_5=z_6\text{)}$$

and

$$2(i_x+i_y)z_1+2(i_x+i_y)z_2+(i_x+i_y)z_3+(2i_y+i_x)z_4=1.$$

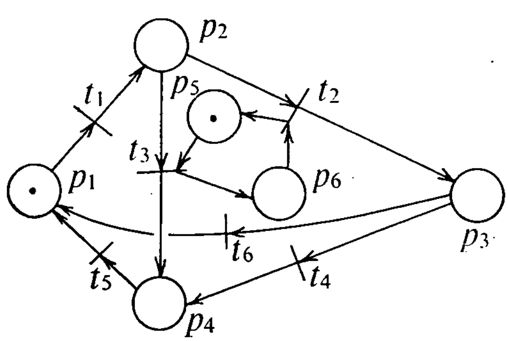

Figure 2

By resolving this system, we obtain

$$i_x=\frac{2(z_1+z_2+z_4)+z_3-2z_0}{2z_0z_4}, \quad i_y=\frac{2z_0-2(z_1+z_2)-z_3-z_4}{2z_0z_4}$$

where i_x and i_y must satisfy the inequalities $i_y>0$ and $i_x+i_y>0$. The second inequality is always verified, and the first gives the condition

$$z_0>\frac{2(z_1+z_2)+z_3+z_4}{2}.$$

TIMED MARKED GRAPHS. In this case we have $n\geqq m(n=|P|,\ m=|T|$, the equality is verified only if the marked graph is a circuit). Thus, the currents determined by solving m equations among the n equations (IV) and (VI) must satisfy the remaining $n-m$ equations in order to have a functioning at natural rate. If not, it is sufficient to search for the solutions of

$$\{J_r^t Q_0 \geqq J_r^t Z C^+ I\}_{r=1}^{n-m+1}$$

where $\{J_r^t\}_{r=1}^{n-m+1}$ is a p-base (base of circuits in this case) and $I^t=[ii\ldots i]$ a solution of $CI=0$.

The r-th inequality can be written in the form $\sum_{k_r} q_{0_i} \geqq (\sum_{k_r} z_i) i$, where $\sum_{k_r} q_{0_i}$ is the sum of the markers in the circuit K_r, $K_r = S(J_r')$ and $\sum_{k_r} z_i$ is the sum of the delays associated to the places of this circuit. Therefore,

$$i_{\max} = \min_{r=1}^{n-m+1} \{(\sum_{k_r} q_{0_i})/(\sum_{k_r} z_i)\}.$$

This result is given in [6].

TIMED STATE GRAPHS. In this case $m \geqq n$, and for I it is always possible to solve the system (IV) and (VI). One can construct a system having a unique solution for I by giving additional equations imposing a constant ratio between the currents of the transitions having the same input place. There exist exactly $m-n$ linearly independent equations of this kind for any state graph.

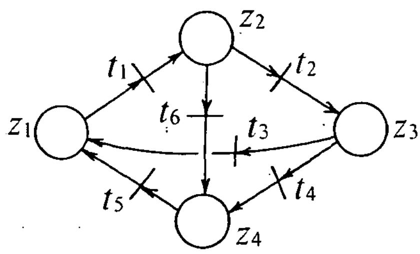

Figure 3

Example 3. Consider the timed state graph of Figure 3. The solution of $CI=0$ gives

$$i_1 = i_3 + i_4 + i_6,$$
$$i_2 = i_3 + i_4,$$
$$i_5 = i_6 + i_4.$$

The equation of conservation of the charge in the graph is

$$(i_6 + i_3 + i_4)(z_1 + z_2) + (i_3 + i_4) z_3 + (i_4 + i_6) z_4 = \sum_{j=1}^{4} q_{0_j}.$$

If we impose $\frac{i_2}{i_6} = \lambda_1$ and $\frac{i_3}{i_4} = \lambda_2$ we have

$$i_2 = \frac{\lambda_1 i_1}{\lambda_1 + 1}, \quad i_3 = \frac{\lambda_1 \lambda_2 i_1}{(1+\lambda_1)(1+\lambda_2)}, \quad i_4 = \frac{\lambda_1 i_1}{(1+\lambda_1)(1+\lambda_2)},$$

$$i_5 = \frac{(1+\lambda_1+\lambda_2) i_1}{(1+\lambda_1)(1+\lambda_2)}, \quad i_6 = \frac{i_1}{1+\lambda_1}.$$

We can now uniquely determine the currents in terms of Q_0, Z and parameters λ_1 and λ_2. For example, for i_1 we obtain

$$i_1 \left[(z_1 + z_2) + \frac{\lambda_1}{1+\lambda_1} z_3 + \frac{1+\lambda_1+\lambda_2}{(1+\lambda_1)(1+\lambda_2)} z_4 \right] = \sum_{j=1}^{4} q_{0_i}.$$

REMARK. One can construct a system having a unique solution for I, from the system of the equations (IV) and (VI) by imposing the additional constraint that the sum of the charge of each circuit of a base of circuits of the state graph is constant. In this case we have $(n-1)$ linearly independent equations from the system $CI=0$ and $m-n+1$ linearly independent equations by application of this constraint. Thus we have m equations describing the behaviour of the net. (The equation $[1\ 1\ 1\ \dots\ 1]\, Q_0=[1\ 1\ 1\ \dots\ 1]\ ZC^+I$ can be obtained as the linear combination of $n-m+1$ equations). The analogy with the electrical circuits is obvious. The $m-n+1$ equations express the application of the Kirchhoff's voltage law: the sum of $i_j \cdot z_j$ (voltage drops) for a circuit is equal to its total charge (electromotive force).

V. Applications

Example 4. *Producer-consumer system.* Consider the producer-consumer problem with a buffer of bounded capacity N_0. We suppose that the producer and the consumer do not try to access the buffer at the same time. The producer deposits items in the buffer as long as it is not full and the consumer does not try to take an item from the buffer when it is empty. Items are produced, deposited, taken and consumed one by one.

The TPN of Figure 4 describes the system producer-consumer with a possible initial marking. Interpretation of the delays associated to the places:

z_p means time of producing an item,
z_d means time of depositing an item,
z_t means time of taking an item,
z_c means time of consuming an item.

We suppose that the z_i' s associated to the other places are equal to zero. That is, the producer and the consumer are functioning at maximum speed: the producer is allowed to deposit an item right after having produced one and he always finds the access to the buffer free. Also, the consumer is allowed to take an item right after having consumed one and he always finds the access to the buffer free.

By solving the equation $CI=0$ we find that the same current i must be associated to all the transitions. Also, a cover by elementary invariant components (state graphs in this case) is given in Figure 5.

PROBLEM. Considering as initial marking the marking given in Figure 4, find conditions for functioning at natural rate.

The inequality (V) applied for SG1, SG2, SG3, SG4 gives, respectively,

$$i \leqq \frac{1}{z_p+z_d}, \quad i \leqq \frac{1}{z_d+z_t+z_s}, \quad i \leqq \frac{1}{z_c+z_t}, \quad i \leqq \frac{N_0}{z_d+z_t+z_a}$$

which yield: $i_{\max}=\min\left\{\frac{1}{z_p+z_d}, \frac{1}{z_d+z_t+z_s}, \frac{1}{z_c+z_t}, \frac{N_0}{z_d+z_t+z_a}\right\}.$

Conditions for functioning at natural rate are

$$z_s = z_p-z_t=z_c-z_d > 0 \quad \text{and} \quad N_0-1 = \frac{z_a-z_s}{z_p+z_d} = \frac{z_a-z_s}{z_c+z_t}$$

CONCLUSION. The producer's and consumer's periods must be equal: $z=z_p+z_d==z_c+z_t$. Also, z_s, the mean time between two successive accesses, is given by $z_s==z_p-z_t=z_c-z_d>0$. From $N_0-1=\frac{z_a-z_s}{z}$ we deduce that:

a) for $z_a < z_s$, a fLnctioning at natural rate is impossible,
b) if $z_a = z_s$, a minimum capacity $N_0 = 1$ is necessary,
c) if $z_a > z_s$, a minimum capacity of $N_0 = 1 + \frac{z_a - z_s}{z}$ is necessary.

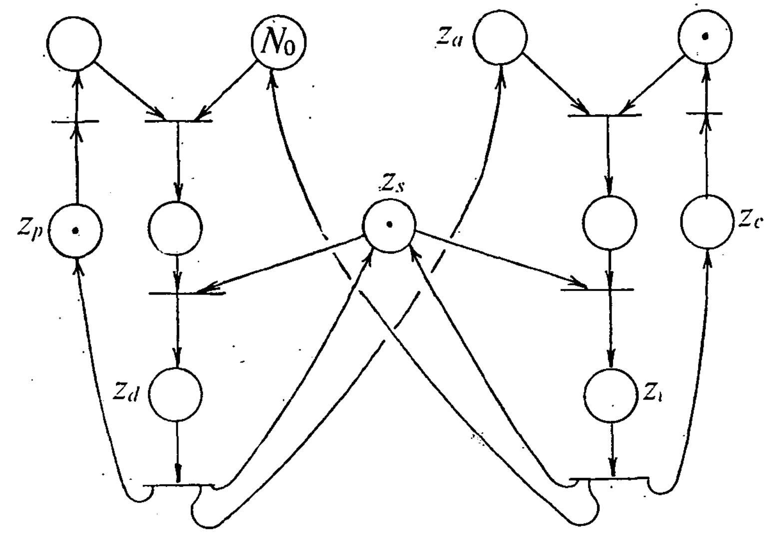

Figure 4

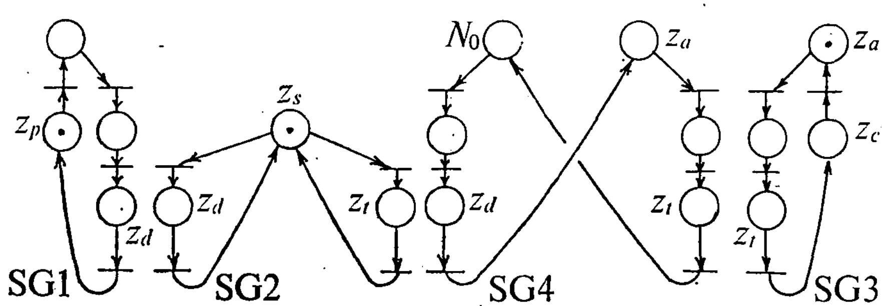

Figure 5

Example 5. *System of r producers and w consumers.* Let a system of r producers and w consumers be connected with a buffer of capacity N_0. The simultaneous access to the buffer is not allowed. We consider for the delays associated to the places the same notations as in the preceding example by adding an index in order to distinquish the producers and the consumers among them. Thus, z_{d_i} is the time for the deposit of an item by the i-th producer and z_{c_j} is the time of consuming an item by the j-th consumer. We consider the case in which procuders and consumers are functioning at maximum speed, which implies zero waiting times before the deposit or before taking an item (Figure 6).

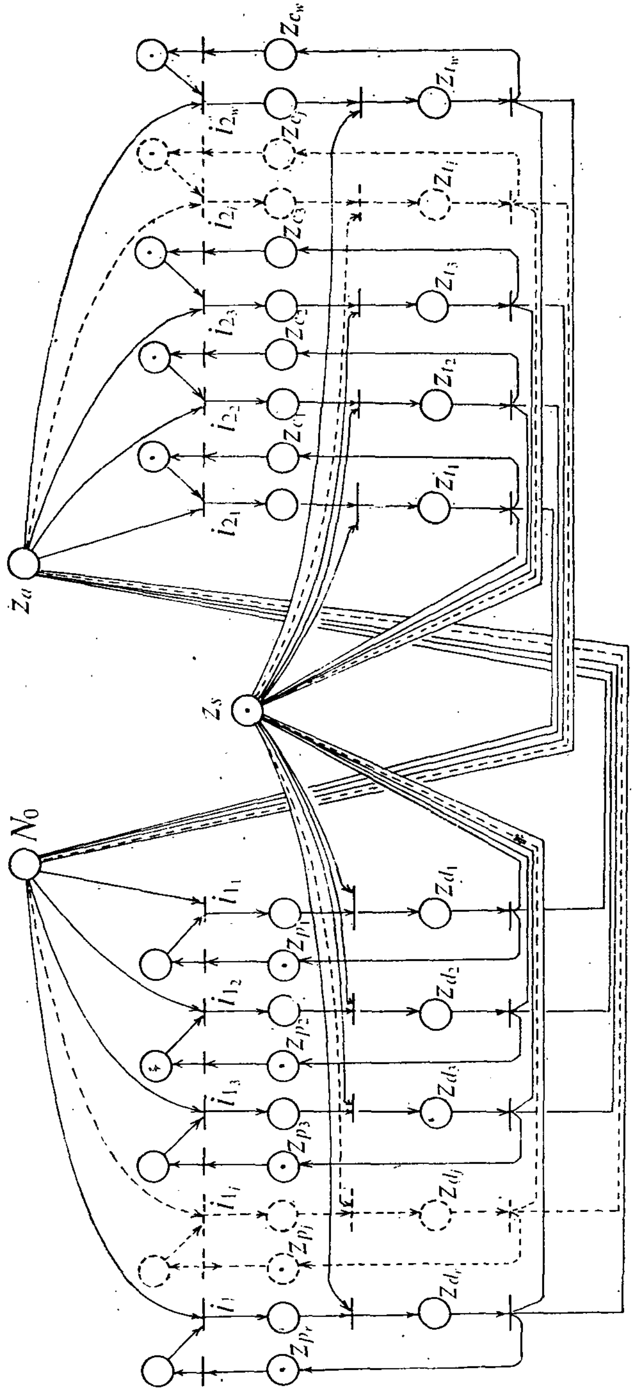

Figure 6

In Figure 7, we give a decomposition of the PN representing the system into elementary invariant components.

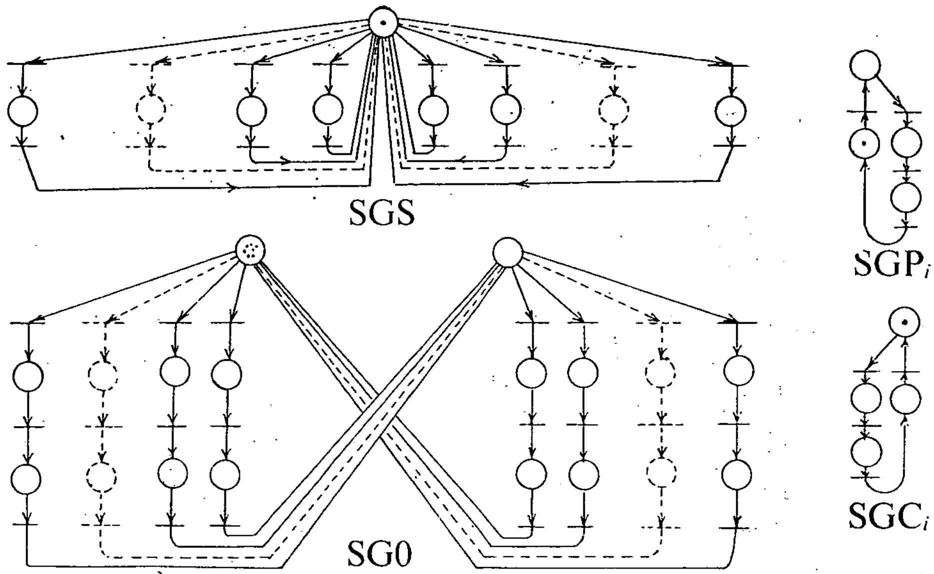

Figure 7

If i_{0_j} and i_{0_j} are the currents associated to the cycles of the j-th producer and j-th consumer, respectively, we have:

$$\left\{i_{1_j} \leqq \frac{1}{z_{d_j}+z_{p_j}}\right\}_{j=1}^{r} \quad \text{and} \quad \left\{i_{2_j} \leqq \frac{1}{z_{c_j}+z_{t_j}}\right\}_{j=1}^{w}.$$

Furthermore, $\sum_{j=1}^{r} i_{1_j} = \sum_{j=1}^{w} i_{2_j} = i_0$ where i_0 is the current throughout the buffer. Thus, $i_0 = \min\left\{\sum_{j=1}^{r} \frac{1}{z_{d_j}+z_{p_j}}, \sum_{j=1}^{w} \frac{1}{z_{c_j}+z_{t_j}}\right\}$.

The equation of conservation of the charge for SGS is:

$$\sum_{j=1}^{r} i_{1_j} z_{d_j} + \sum_{j=1}^{w} i_{2_j} z_{t_j} + i_0 z_s = 1 \Rightarrow z_s = \frac{1-\sum_{j=1}^{r} i_{1_j} z_{d_j} - \sum_{j=1}^{w} i_{2_j} z_{t_j}}{i_0}. \tag{a}$$

But,

$$\sum_{j=1}^{r} i_{1_j} z_{d_j} \leqq \sum_{j=1}^{r} \frac{z_{d_j}}{z_{d_j}+z_{p_j}} \quad \text{and} \quad \sum_{j=1}^{w} i_{2_j} z_{t_j} \leqq \sum_{j=1}^{w} \frac{z_{t_j}}{z_{c_j}+z_{t_j}}.$$

From the two preceding inequalities and (a) we get

$$z_s \geqq \frac{1}{i_0}\left(1 - \Sigma \frac{z_{d_j}}{z_{d_j}+z_{p_j}} - \Sigma \frac{z_{t_j}}{z_{t_j}+z_{c_j}}\right). \tag{b}$$

Finally, for SGO (Figure 7) we have

$$\sum_{j=1}^{r} i_{1_j} z_{d_j} + \sum_{j=1}^{w} i_{2_j} z_{t_j} + z_a i_0 = N_0 \Rightarrow 1 - i_0 z_s + i_0 z_a = N_0 \Rightarrow N_0 - 1 = (z_a - z_s) i_0.$$

From this last equation and the inequality (b) we obtain

$$z_a \geqq \frac{1}{i_0}\left(N_0 - \sum_{j=1}^{r} \frac{z_{d_j}}{z_{d_j} + z_{p_j}} - \sum_{j=1}^{w} \frac{z_{t_j}}{z_{t_j} + z_{c_j}}\right). \tag{c}$$

The inequalities (b) and (c) give least bounds for the mean time between two successive accesses to the buffer (z_s) and for the mean waiting time (z_a) of an item in the buffer.

Example 6. Consider the TPN of Figure 8. One could imagine that it represents the functioning of an enterprise of car location having customers of two types. Customers of type 1, whose number is N_1, have a mean location time z_1 and a mean time between two succesive demands for location z_{a_1}. Also, customers of type 2, whose number is N_2, have a mean location time z_2 and a mean time between two successive demands for location z_{a_2}. We suppose that the total number of cars of the enterprise is N_0 and that after location, a service of mean duration z_s is done to each car. We finally admit that a car ready for location waits during z_0 before a customer demands it.

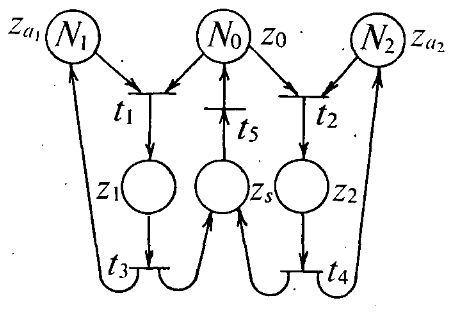

Figure 8

By solving $CI=0$, we get $i_1=i_3$, $i_2=i_4$, $i_5=i_1+i_2$. Furthermore, the resolution of $J^tC=0$ gives a decomposition into state graphs (Figure 9).

PROBLEM. Knowing N_1 and N_2 as well as the delays associated to the places, determine N_0 such that a functioning at natural rate will be possible.

The equations of charge conservation for SG1 and SG2 are, respectively,

$$i_1 = \frac{N_1}{z_1 + z_{a_1}} \quad \text{and} \quad i_2 = \frac{N_2}{z_2 + z_{a_2}}.$$

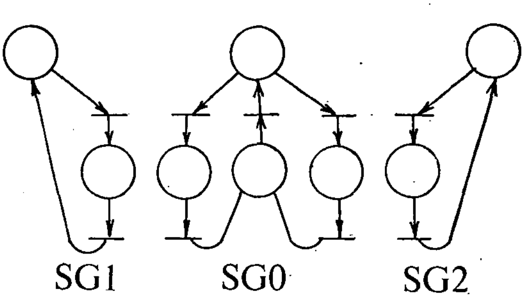

Figure 9

For SG0, we have

$$N_0 = (i_1+i_2)(z_0+z_s)+i_1z_1+i_2z_2 \Rightarrow N_0 = \frac{N_1(z_0+z_1+z_s)}{z_{a_1}+z_1}+N_2\frac{(z_0+z_s+z_2)}{z_{a_2}+z_2}.$$

N_0 is the minimum number of cars to satisfy the demands of the (N_1+N_2) customers.

Abstract

We study the behavior of pure timed Petri nets for constant current assignments. It is given a set of relations describing the behavior of a timed Petri net and it is shown that its maximum computation rate can be calculated by solving a set of n linear equations where n is the number of its places. These relations are established between the currents, the initial marking and the delays of the network. Also, in order to better understand and use these relations, we give some results on the decompositions of a Petri net, obtained by studying the types of solutions of the equations $CI=0$ and $J^tC=0$ where C is the incidence matrix of the net. It is shown, that every consistent (resp. invariant) Petri net can be decomposed into a set of consistent (invariant) "elementary" subnets. We finally give some examples in order to illustrate the use of timed Petri nets in the study of the dynamic behavior of the systems.

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References

[1] Petri, C. A., Communication with automata, Techn. Rep. n° RADC—TR—65—377 v. 1, Rome Air Develop. Center, N. Y., Jan. 1966.
[2] Holt, A. W., Information systems theory project, Applied Data Research Incorporated Princeton, N. J., AD. 676.972, Sept. 1968.
[3] Hack, M., Analysis of production schemata by Petri nets, MIT—Project MAC, TR—94, 1972.
[4] Petri, C. A., Concepts of net theory, Proc. of Symp. on Math. Foundations of Comp. Science, High Tatras, Sept. 1973, pp. 137—146.
[5] Petri, C. A., Interpretations of net theory, Interner Bericht 75—07, G. M. D. Bonn, July 1975.
[6] Ramchandani, C., Analysis of asynchronous concurrent systems by timed Petri nets, PhD. Thesis MIT, Sept. 1973.
[7] Lautenbach, K. and H. A. Schmid, Use of Petri nets for proving correctness of concurrent process systems, IFIP 1974, North Holland Publ. Co., 1974, pp. 187—191.
[8] Ghosh, S., Some comments on timed Petri nets, Journées sur les Réseaux de Petri, March 1977, Paris, pp. 151—163.
[9] Crespi-Reghizzi, S. and D. Mandrioli, Some algebraic properties of Petri nets, Alta Frequenza, v. 45, N° 2, pp. 130—137.
[10] Memmi, G., Semiflows and invariants. Applications in Petri nets theory, Journées sur les Réseaux de Petri, Paris, March 1977, pp. 145—150.
[11] Lien, Y. E., Termination properties of generalized Petri nets, S.I.A.M. J. Comput. v. 5, N° 2, June 1976, pp. 251—265.
[12] Murata, T. and R. W. Church, Analysis of marked graphs and Petri nets by matrix equations, R. R. N° M.D.C. 1.1.8 Univ. of Illinois, Nov. 1975.
[13] Fulkerson, D. R., Networks, frames, blocking systems, in Mathematics for decision sciences, v. 11 of Lectures in Applied Mathematics, 1968, the American Mathematical Society, Providence, Rhode Island.
[14] Sifakis, J., Etude du comportement permanent des réseaux de Petri temporisés, Journées sur les Réseaux de Petri, Paris, March 1977, pp. 165—185.

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