# Systems of linear equations over a bounded chain

K. PEEVA

## § 1. Introduction

The equivalence, reduction and minimization are classical problems for the theory of abstract automata. They are completely studied for deterministic, non-deterministic and stochastic automata (8). These problems are still open for fuzzy automata because there does not exist a polynomial time algorithm for solving systems of linear equations over a bounded chain.

Let $\mathbf{L}=(L, \vee, \wedge, 0, 1)$ be a bounded chain (5) with underlying linearly ordered set $L$, the greatest element 1 and the smallest element 0. We shall write $\mathbf{L}$ instead of $\mathbf{L}=(L, \vee, \wedge, 0, 1)$.

For a given set $D$ we denote by $|D|$ its cardinality.

Let $\mathbf{L}$ be given. Let $I\neq\emptyset$, $J\neq\emptyset$ be sets of indices. We write $B\in L^{I\times J}$ for the matrix $B=(b_{ij})$ where $b_{ij}=b(i,j)$ is the $(i,j)$-th entry of a map $b$: $I\times J\to L$.

Let $J$ be finite set and $A=(a_{ij})\in L^{I\times J}$, $B=(b_{jk})\in L^{J\times K}$ be given. The matrix $C=A\cdot B=(c_{ik})\in L^{I\times K}$ is called a *product* of $A$ and $B$ if

$$c_{ik} = \bigvee_{p=1}^{|J|} (a_{ip}\wedge b_{pk}) \text{ for each } i\in I,\ k\in K.$$

Obviously $c_{ik}\in\{a_{ip}\colon p\in J\}\cup\{b_{pk}\colon p\in J\}$.

Let $A=(X, Q, Y, M)$ be a fuzzy automaton (7, 9) with input alphabet $X$, state set $Q$, output alphabet $Y$ and set of the step-behaviour matrices

$$M = \{M(x/y)\colon x\in X, y\in Y\}.$$

Each $M(x/y)=(m(x/y)_{qq'})\in L^{|Q|\times|Q|}$ and $m(x/y)_{qq'}$ is the grade of membership of a transition to state $q'$ under input $x$ assuming the output is $y$ and the start state is $q$. If $X$, $Q$, $Y$ are finite sets then $A$ is finite fuzzy automaton.

For any set $D$ we write $D^*$ for the free monoid on $D$ with the empty word $e\in D^*$ as the identity element. For $(u, v)\in X^*\times Y^*$ we write $(u/v)$ if the number of the letters in $u$ is equal to that of the letters in $v$.

Let $A=(X, Q, Y, M)$ be a finite automaton. The expression

$$M(u/v) = M(x_1/y_1)\ldots M(x_k/y_k),\quad u = x_1\ldots x_k\in X^*,\quad v = y_1\ldots y_k\in Y^*$$

defines the operation of $A$ for the pair of words $(u/v)$. Let

$$M(u/v) = \big(m(u/v)_{qq'}\big).$$

For the given automaton $A$ let us consider its behaviour matrix

$$B^* = \big(b(u/v)_q\big), \quad q \in Q, \quad (u/v) \in X^* \times Y^*,$$

where $b(u/v)_q = \bigvee_{q' \in Q} m(u/v)_{qq'}$ is the grade of membership of the output $v$ upon the input $u$ when the start state is $q$. The matrix $B^*$ is semi-infinite with $|Q|$ rows.

It is well-known (7, 9) that there exists a finite submatrix $B$ of $B^*$ with linearly independent columns. For the problems of equivalence, reduction and minimization of fuzzy automata the main question is how to compute $B$ from $B^*$. That means for any column in $B^*$ we have to answer whether it is a $\vee$—$\wedge$-linear combination of the previous columns in $B^*$. If we can solve systems of linear equations over a bounded chain then we can compute $B$ from $B^*$.

In this paper the attention is concentrated on computing a solution of the system of linear equations over **L**. The main result is (see Algorithm 3 and the Theorem corresponding to it) that there exists a polynomial time algorithm for solving a system of linear equations over a bounded chain. An extension of this result for some semirings is given in (6).

We would like to remark that the classical methods (5) for solving systems of linear equations over a field are not useful here because **L** is not a field. Since the conjugate matrix for a given matrix in **L** does not exist in general, the ideas of (1) can not be applied. As our problem is essentially different from the extremal linear programing (10) these results can not be implemented.

Further we shall use without explicit explanation the concept of computational complexity as described in (3) and the properties of chains according to (5).

## § 2. Linear equations over L

In order to determine the general solution of the system we consider first a linear equation in **L**.

By $A \cdot X = b$ we denote the following linear equation

$$(a_1 \wedge x_1) \vee \ldots \vee (a_n \wedge x_n) = b \tag{1}$$

with coefficients $A = (a_j) \in L^{\{1\} \times J}$, unknowns $X = (x_j) \in L^{J \times \{1\}}$ and a constant $b \in L$. Here $\{1\}$ stands for the singleton set and we assume $|J| = n \in \mathbf{N}$.

The matrix $X^0 = (x_j^0) \in L^{J \times \{1\}}$ is a *point solution* of (1) if and only if $A \cdot X^0 = b$ holds. If there exists $X^0$ with $A \cdot X^0 = b$ then the equation (1) is called *solvable*, otherwise it is *unsolvable*. An $n$-tuple $(X_1, \ldots, X_n)$ of intervals $X_i \subseteq L$ is called an *interval solution* of (1) if every $n$-tuple $(x_1, \ldots, x_n)$ with $x_i \in X_i$ is a point solution of (1) and $(X_1, \ldots, X_n)$ is maximal with respect to this property.

Let the equation (1) be given and

$$S = \{j \in J \colon a_j < b\}, \quad E = \{j \in J \colon a_j = b\}, \quad G = \{j \in J \colon a_j > b\}$$

**Proposition 1.** The equation (1) is solvable if and only if $E \cup G \neq \emptyset$ and the interval solutions are the $n$-tuples $(X_1, \ldots, X_n)$ where for each $j \in J$ either

(i) there exists a $k \in G$ such that

$$X_j = \begin{cases} \{b\} & \text{if } j = k, \\ [0, b] & \text{if } j \in G \setminus \{k\}. \\ L & \text{if } j \in S \cup E, \end{cases}$$

or (ii) there exists a $k \in E$ such that

$$X_j = \begin{cases} [b, 1] & \text{if } j = k, \\ L & \text{if } j \in S \cup E \setminus \{k\}, \\ [0, b] & \text{if } j \in G. \end{cases}$$

The number of the interval solutions for (1) is equal to the cardinality of the set $E \cup G$ (which does not exceed $n$). We can compute the interval solutions of (1) in a polynomial time.

## § 3. Systems of linear equations

In this section systems of linear equations are solved by taking appropriate intersections of interval solutions of the single equations.

By $A \cdot X = B$ we denote the system of linear equations of the form

$$\begin{aligned} (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \dots \vee (a_{1n} \wedge x_n) &= b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \dots \vee (a_{mn} \wedge x_n) &= b_m \end{aligned} \tag{2}$$

with coefficients $A = (a_{ij}) \in L^{I \times J}$, unknowns $X = (x_j) \in L^{J \times \{1\}}$ and constants $B = (b_i) \in L^{I \times \{1\}}$. We assume $|I| = m \in \mathbf{N}$, $|J| = n \in \mathbf{N}$.

The matrix $X^0 = (x_j^0) \in L^{J \times \{1\}}$ is a *point solution* of (2) if and only if $A \cdot X^0 = B$ holds. An $n$-tuple $(X_1, \dots, X_n)$ with $X_i \subseteq L$ is an *interval solution* of (2) if each $n$-tuple $(x_1, \dots, x_n)$ with $x_i \in X_i$ is a point solution of (2) and the $n$-tuple $(X_1, \dots, X_n)$ is maximal with respect to this property.

The system (2) is *solvable* if it possesses at least one solution, otherwise it is not solvable or *unsolvable*.

The computing method for obtaining the interval solutions of (2) consists of the following.

For each $i \in I$ we denote by $S_i$, $E_i$, $G_i$ the sets

$$S_i = \{j \in J\colon a_{ij} < b_i\}, \quad E_i = \{j \in J\colon a_{ij} = b_i\}, \quad G_i = \{j \in J\colon a_{ij} > b_i\}$$

According to Proposition 1 we can form the set $V_i$ of the interval solutions of the $i^{\text{th}}$ equality in (2):

$$V_i \equiv \{X^{i, r_i}\colon X^{i, r_i} = (X_1^{i, r_i}, \dots, X_n^{i, r_i}), \quad r_i \leqq |E_i \cup G_i|\}.$$

Let $r_i \leqq |E_i \cup G_i|$ be fixed for each $i \in I$. We denote by $X = (X_j)$, $j \in J$, an interval solution of (2) in which each $X_j$ is the following nonempty intersection:

$$X_j = \bigcap_{i \in I} X_j^{i, r_i} \neq \emptyset \tag{3}$$

Let $V=\{X\colon X=(X_j),\ j\in J\}$ be the set of all interval solutions of (2). The elements of $V$ are determined by (3). The cardinality of the set $V$ is finite.

Using the above symbols we propose the following algorithm for computing the interval solutions of the system (2):

**Algorithm 1**

Step 1. For each $i\in I$ obtain the sets $S_i$, $E_i$, $G_i$, $V_i$.
Step 2. According to the expression (3) obtain the set $V$.

**Proposition 2.** The time complexity function of Algorithm 1 is exponential in the number of $m$ equations.

*Proof.* It follows from the fact that the cardinality of the set $V$ does not exceed the bound $\prod_{i=1}^{m} |E_i\cup G_i|\leqq n^m$.

In many cases we do not need all of the interval solutions of (2). For example, in fuzzy automata theory it is interesting whether the system is solvable or not and if the system is solvable — to compute one of its solutions. For this purpose we shall consider some of the properties of the system (2).

Two systems over L are called *equivalent* if each solution of the first one is a solution of the second and vice versa.

**Proposition 3.** If the system (2′) is obtained from the system (2) after a permutation of the equations then the systems (2) and (2′) are equivalent.

Let the system $A\cdot X=B$ be given. We denote by $(A:B)$ the matrix

$$(A\colon B)=\begin{pmatrix} a_{11}\dots a_{1n}\colon\ b_1 \\ \dots\dots\dots\dots \\ a_{m1}\dots a_{mn}\colon\ b_m \end{pmatrix}$$

We shall denote by (2′) the system $A'\cdot X=B'$:

$$\begin{matrix} (a'_{11}\wedge x_1)\vee\dots\vee(a'_{1n}\wedge x_n)=b'_1 \\ \dots\dots\dots\dots\dots\dots \\ (a'_{m1}\wedge x_1)\vee\dots\vee(a'_{mn}\wedge x_n)=b'_m \end{matrix} \tag{2'}$$

obtained from (2) after a permutation of the equations in such a way that $b'_1\geqq b'_2\geqq\dots\geqq b'_m$.

The systems (2) and (2′) are equivalent according to Proposition 3.

Let $A^*$ be the matrix, obtained from the matrix $A'$ by the rule

$$a^*_{ij}=\begin{cases} 0 & \text{if } a'_{ij}<b'_i, \\ b'_i & \text{if } a'_{ij}=b'_i, \\ 1 & \text{if } a'_{ij}>b'_i, \end{cases} \tag{4}$$

**Proposition 4.** The systems $A\cdot X=B$ and $A^*\cdot X=B'$ are equivalent.

*Proof.* According to Proposition 3 the systems $A\cdot X=B$ and $A'\cdot X=B'$ are equivalent. We shall prove that $A'\cdot X=B'\Leftrightarrow A^*\cdot X=B'$. If $A'\cdot X=B'$ then for each $i \in I$

$$\exists j \in J\colon (a'_{ij} \geqq x_j = b'_i) \Leftrightarrow$$

$$\exists j \in J\colon (a'_{ij} \geqq b'_i \wedge x_j = b'_i) \vee (a'_{ij} = b'_i \wedge x_j \geqq b'_i) \Leftrightarrow$$

$$\exists j \in J\colon (a^*_{ij} = 1 \wedge x_j = b'_i) \vee (a^*_{ij} = b'_i \wedge x_j \geqq b'_i),$$

i.e. for each $i \in I$ there exists a $j \in J$ such that $a^*_{ij} \wedge x_j = b'_i$ and hence $A^* \cdot X = B'$.

Let $\underline{B} = \{b_1, \ldots, b_m\}$ be the set of the distinct elements in the matrix $B$ of the system (2), resp. (2'). Having in mind the expression (4), it is clear that the elements $a^*_{ij}$ of $A^*$ and $b'_i$ of $B'$ belong to the set $\underline{B} \cup \{0, 1\}$.

**Proposition 5.** Let $X = (X_j)$ be an interval solution of the system $A^* \cdot X = B'$, where the components $X_j$, $j \in J$, are determined by (3). Each component $X_j$ is among the following intervals: $L$, $[0, b_{p1}]$, $[b_{p2}, b_{p3}]$, $[b_{p4}, 1]$, where $b_{p1}, b_{p2}, b_{p3}, b_{p4} \in \underline{B}$.

The proof follows from Proposition 1 and the expression (3).

**Corollary 1.** Each interval solution of (2) has all its components among the following intervals: $L$, $[0, b_{p1}]$, $[b_{p2}, b_{p3}]$, $[b_{p4}, 1]$, where $b_{p1}, b_{p2}, b_{p3}, b_{p4} \in \underline{B}$.

Let $B^n_m$ be the set of all $n$-fold variations with repetitions on the elements of the set $\underline{B}$.

**Corollary 2.** The system (2) is solvable if and only if there exists an $X^0 \in B^n_m$ such that $A \cdot X^0 = B$ holds.

*Proof.* If there exists an $X^0 \in B^n_m$ with $A \cdot X^0 = B$ then the system (2) is solvable. Conversely, if the system (2) is solvable, then each component $X_j$ of an interval solution has the interval form determined by Corollary 1. Hence we can choose each component $x^0_j$ of a point solution of (2) to be equal to an element of $\underline{B}$, i.e. $X^0 \in B^n_m$.

Having in mind Corollary 2 we propose the following algorithm for computing a point solution of the system (2), or for establishing its solvability.

**Algorithm 2**

Step 1. Find the set $\underline{B}$.
Step 2. Compute the set $B^n_m$.
Step 3. For each $X^0 \in B^n_m$ check whether it is a point solution of the system (2).
Step 4. List all point solutions determined in Step 3.
Step 5. If there exists no $X^0 \in B^n_m$ with $A \cdot X^0 = B$ then the system is unsolvable. Otherwise it is solvable and a set of point solutions is given in step 4.

**Proposition 6.** The time complexity function of Algorithm 2 is exponential in the number of $n$ variables.

*Proof.* We can check whether $X^0 \in B^n_m$ is a solution of (2) in a polynomial time, but in a search problem manner. The set $B^n_m$ is finite and $|B^n_m| = |\underline{B}|^n \leqq m^n$. Hence the Algorithm 2 is finite with exponential in the number of $n$ variables time complexity function.

## § 4. A polynomial time algorithm

We propose a polynomial time algorithm for computing a point solution of the system (2) if it is solvable or for listing the numbers of the contradictory equations if the system is unsolvable.

In order to simplify the problem we introduce a symbol-matrix $\underline{A}$ with symbol-coefficients obtained from those of $A^*$ if for each $a^*_{ij}$ we put the corresponding type letter $S$, $E$ or $G$ (without index):

$$a_{ij} = \begin{cases} S & \text{if} \quad a^*_{ij} = 0, \\ E & \text{if} \quad a^*_{ij} = b'_i, \\ G & \text{if} \quad a^*_{ij} = 1. \end{cases} \tag{5}$$

The set of the solutions of the system (2) remains unchanged after this reduction step (5).

Let the system (2) be given and $X=(X_j)$ denote an interval solution of (2). Let the system (2′) and the matrix $\underline{A}$ be obtained. We assume $j\in J$ to be fixed in $\underline{A}$. In the following we denote by $r$ the smallest number of the row with $E$-type coefficient in its $j^{\text{th}}$ column and by $k$ the greatest number of the row with $G$-type coefficient in its $j^{\text{th}}$ column in $\underline{A}$.

In order to find a point solution of (2) we are interested in finding a point $x_j\in X_j$ with $a_{ij}\wedge x_j\leqq b_i$ for each $i\in I$. Especially we mark the $i^{\text{th}}$ equation in a marker vector $IND$ if $a_{ij}\wedge x_j=b_i$ holds.

Having in mind the above notions we obtain the following

**Proposition 7.** Let the system $A\cdot X=B$ be given.

i) if the $j^{\text{th}}$ column in $\underline{A}$ contains a $G$-type coefficient then $x_j=b'_k$ implies $a'_{ij}\wedge x_j=b'_i$ for $i=k$ and for each $i>k$ with $a'_{ij}=b'_i$;

ii) if the $j^{\text{th}}$ column in $\underline{A}$ does not contain any $G$-type coefficient but it contains an $E$-type coefficient then $x_j=b'_r$ implies $a'_{ij}\wedge x_j=b'_i$ for $i=r$ and for each $i>r$ with $a'_{ij}=b'_i$;

iii) if the $j^{\text{th}}$ column in $\underline{A}$ does not contain neither $G$-type nor $E$-type coefficients then $a'_{ij}\wedge x_j<b'_i$ for each $x_j\in L$.

*Proof.* i) if $x_j=b'_k$ and $i=k$ then $a'_{ij}\wedge x_j=a'_{kj}\wedge b'_k=b'_k$ since $a'_{kj}>b'_k$; if $x_j=b'_k$, $i>k$ and $a'_{ij}=b'_i$ then $a'_{ij}=b'_i\leqq b'_k$ according to the order in (2′) implies $a'_{ij}\wedge x_j= =b'_i\wedge b'_k=b'_i$;

ii) if $x_j=b'_r$ and $i=r$ then $a'_{ij}\wedge x_j=a'_{rj}\wedge b'_r=b'_r\wedge b'_r=b'_r$; if $x_j=b'_r$, $i>r$ and $a'_{ij}=b'_i$ then $a'_{ij}=b'_i\leqq b'_r$ according to the order in (2′) implies $a'_{ij}\wedge x_j=b'_i\wedge b'_r=b'_i$;

iii) if the $j^{\text{th}}$ column in $\underline{A}$ contains only $S$-type coefficients then $a'_{ij}\wedge x_j\leqq a'_{ij}<b'_i$ for each $i\in I$ and arbitrary $x_j\in L$.

On this base we propose the following algorithm:

**Algorithm 3**

*Step 1.* Enter the matrix $(A:B)$.
*Step 2.* Form the matrix $\underline{A}$.
*Step 3.* Erase the marker vector $IND$.
*Step 4.* $j=0$.
*Step 5.* $j=j+1$.

*Step 6.* If $j>n$ go to 10.

*Step 7.* If the $j^{\text{th}}$ column in $\underline{A}$ does not contain any $G$-type coefficient then go to 8. Otherwise $x_j=b'_k$. Put a mark in $IND$ for $i=k$ and for each $i>k$ with $a'_{ij}=b'_i$. Put a mark in $IND$ for each $i<k$ if $a'_{ij}\geqq b'_i=b'_k$. Go to Step 5.

*Step 8.* If the $j^{\text{th}}$ column does not contain any $E$-type coefficient then go to step 9. Otherwise $x_j=b'_r$, put marks in $IND$ for $i=r$ and for each $i>r$ with $a'_{ij}=b'_i$. Go to Step 5.

*Step 9.* $x_j=1$. Go to Step 5.

*Step 10.* If there exists at least one unmarked row in $IND$ then the system is unsolvable and the unmarked equations are in contradiction with the marked ones. The marked equations form a compatible system. If all rows in $IND$ are marked then the system is compatible and the components of the point solution $X=(x_j)$ are determined in Steps 7, 8, 9.

**Theorem.** The following problems are algorithmically decidable in polynomial time for the system (2):

i) whether the system is solvable or not;

ii) computing a point solution if the system is solvable;

iii) obtaining the numbers of the contradictory equations if the system is unsolvable.

The proof follows from Algorithm 3.

The program realisation of Algorithm 3 is available at the Center of Applied Mathematics in the Higher Institute for Mechanical and Electrical Engineering.

We shall consider two examples as a simple illustration of Algorithm 3.

**Example 1.** Solve the system

$$(0{,}3\wedge x_1)\vee(0{,}5\wedge x_2)\vee(0{,}4\wedge x_3)\vee(0{,}7\wedge x_4) = 0{,}2$$
$$(0{,}8\wedge x_1)\vee(0{,}2\wedge x_2)\vee(0{,}7\wedge x_3)\vee(0{,}5\wedge x_4) = 0{,}5$$
$$(0{,}2\wedge x_1)\vee(0{,}7\wedge x_2)\vee(0{,}5\wedge x_3)\vee(0{,}3\wedge x_4) = 0{,}3$$

The (′)-system is

$$(0{,}8\wedge x_1)\vee(0{,}2\wedge x_2)\vee(0{,}7\wedge x_3)\vee(0{,}5\wedge x_4) = 0{,}5$$
$$(0{,}2\wedge x_1)\vee(0{,}7\wedge x_2)\vee(0{,}5\wedge x_3)\vee(0{,}3\wedge x_4) = 0{,}3$$
$$(0{,}3\wedge x_1)\vee(0{,}5\wedge x_2)\vee(0{,}4\wedge x_3)\vee(0{,}7\wedge x_4) = 0{,}2$$

The matrix $(\underline{A}:B')$ and the marker vector $IND$ are

$$(A\colon B') = \begin{pmatrix} G & S & G & E\colon 0{,}5 \\ S & G & G & E\colon 0{,}3 \\ G & G & G & G\colon 0{,}2 \end{pmatrix} \quad IND = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$

The system is unsolvable. The contradictory equations have 0 in $IND$.

**Example 2.** Compute a point solution of the system

$$(0{,}2\wedge x_1)\vee\ 0{,}5\wedge x_2)\vee(0{,}7\wedge x_3) = 0{,}4$$
$$(0{,}8\wedge x_1)\vee(0{,}2\wedge x_2)\vee(0{,}1\wedge x_3) = 0{,}2$$

The matrix $(\underline{A}:B')$ and the marker vector $IND$ are

$$(\underline{A}:\ B') = \begin{pmatrix} S & G & G:\ 0{,}4 \\ G & E & S:\ 0{,}2 \end{pmatrix} \quad IND = \begin{pmatrix} * \\ * \end{pmatrix}$$

The column vector $X=(0{,}2\ 0{,}4\ 0{,}4)^t$ is a point solution of this system.

I would like to express gratitude to prof. V. Trnkova and Dr. S. Ivanov for the valuable discussions and the interest in my work.

*Abstract.* A polynomial time algorithm for computing a point solution of a system of linear equations over a bounded chain is given.

## References

[1] Cuninghame-Green, R. A., Minimax Algebra, Lecture Notes in Economics and Mathematical Systems, Vol. 166, Springer-Verlag, Berlin, 1979.
[2] Eilenberg, S., Automata, Languages and Machines, Vol. A, Acad. Press, New York, 1974.
[3] Garey, M. R., Johnson, D. S., Computers and Intractability, A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[4] Kuich, W., Formal Power Series, Cycle-Free Automata and Algebraic Systems, Techn. Univ. Wien, Wien, 1982.
[5] MacLane, S., Birkhoff, G., Algebra, Macmillan Publ. Co., New York, 1979.
[6] Peeva, K. G., Linear Systems and Ordered Semirings, Proc. of the IX Summer School, Varna, 1983.
[7] Santos, E., On Reduction of Maxi-min Machines, J. Math. Anal. Appl., 40, 60—78 (1972).
[8] Starke, P., Abstracte Automaten, Berlin, 1969.
[9] Topencharov, V. V., Peeva, K. G., Equivalence, Reduction and Minimization of Finite Fuzzy Automata, J. Math. Anal. Appl., 84, 270—281 (1981).
[10] Zimmermann, U., Linear and Combinatorial Optimization in Ordered Algebraic Structures, Amsterdam etc., North-Holl. 1981.

*Received Oct. 11, 1984*

SOFIA 1000
P.O. BOX 384
CENTER OF APPLIED MATHEMATICS