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On minimal and maximal clones

L. Szabó*[†]

1 Introduction

A composition closed set of finitary operations on a fixed universe A containing all projections is a clone. For example the set $\mathbf{J}$ of all projections and the set $\mathbf{O}$ of all operations on A are clones. The clones, ordered by inclusion, form an algebraic lattice $\mathbf{L}$ with least element $\mathbf{J}$ and greatest element $\mathbf{O}$. For $|A| = 2$, $\mathbf{L}$ is the well-known countable Post lattice [5], but already for $|A| > 2$ there are $2^{\aleph_0}$ clones. For A finite $\mathbf{L}$ has finitely many coatoms, called maximal clones , and they are fully known ([7],[8]). On the other hand $\mathbf{L}$ has finitely many atoms, called minimal clones, and are fully known only for $|A| \leq 3$ ([3], [5]). It is also known (see e.g. [6]) that the meet of all maximal clones is $\mathbf{J}$, and the join of all minimal clones is $\mathbf{O}$.

The aim of the present paper is to show that in general there are three maximal clones with meet $\mathbf{J}$ and there are three minimal clones with join $\mathbf{O}$; moreover, for a prime element universe, two maximal clones, resp., two minimal clones have the above properties.

2 Preliminaries

Let A be a fixed universe with $|A| \geq 2$. For any positive integer n let $\mathbf{O}^{(n)}$ denote the set of all n-ary operations on A (i.e. maps $A^n \to A$) and let $\mathbf{O} = \bigcup_{n=1}^{\infty} \mathbf{O}^{(n)}$. For $1 \leq i \leq n$ let e_i^n denote the n-ary i-th projection (trivial operation). Further let $\mathbf{J} = \{e_i^n | 1 \leq i \leq n < \infty\}$. The operations in $\mathbf{O} \setminus \mathbf{J}$ are called *nontrivial operations.* By a *clone* we mean a subset of $\mathbf{O}$ which is closed under superpositions and contains all projections. The set of clones ordered by inclusion form a lattice $\mathbf{L}$ in which every meet is the set-theoretical intersection. For $F \subseteq \mathbf{O}$ denote by $[F]$ the clone generated by F, and instead of $[\{f\}]$ we write $[f]$.

A minimal clone, resp., a maximal clone is an atom, resp., a dual atom of $\mathbf{L}$. It is well-known that $\mathbf{L}$ is an atomic and dually atomic algebraic lattice, and has finitely many minimal clones and maximal clones. Furthermore, the intersection of all maximal clones is $\mathbf{J}$, and the minimal clones generate $\mathbf{O}$ (see e.g. [6]). The maximal clones are fully known and was given by I. G. Rosenberg ([7], [8]). For $|A| = 2$, $\mathbf{L}$ is the well-known Post lattice [5]. Considering the Post lattice we immediately see that for two element set there are three maximal clones with intersection $\mathbf{J}$ and the intersection of two maximal clones cannot be $\mathbf{J}$. Moreover, there are three minimal clones with join $\mathbf{O}$ and the join of two minimal clones cannot be $\mathbf{O}$.

*Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary

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A subset $F \subseteq \mathbf{O}$ as well as the algebra (A, F) is *primal* or *complete* if the clone generated by F (i.e. the set of all term functions of (A, F)) is equal to $\mathbf{O}$; F as well as the algebra (A, F) is *functionally complete* if F together with all constant operations is primal.

A ternary operation f on A is a *majority function* if for all $x, y \in A$ we have $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$; f is a *Mal'tsev function* if $f(x,y,y) = f(y,y,x) = x$ for all $x, y \in A$. An n-ary operation t on A is said to be an i-th *semi-projection* $(n \geq 3,\ 1 \leq i \leq n)$ if for all $x_1, \ldots, x_n \in A$ we have $t(x_1, \ldots, x_n) = x_i$ whenever at least two elements among $x_1, \ldots, x_n$ are equal. We are going to formulate Rosenberg's Theorem ([7], [8]) which is the main tool in proving our results. First, however, we need some further definitions.

Let $n, h \geq 1$. An n-ary operation $f \in \mathbf{O}^{(n)}$ is said to *preserve* the h-ary relation $\rho \subseteq A^h$ if ρ is a subalgebra of the h-th direct power of the algebra $(A; f)$. Then the set of operations preserving ρ forms a clone, which is denoted by Polρ. We say that a relation ρ is a *compatible relation* of the algebra (A, F) if $F \subseteq$ Polρ. A binary relation is called *nontrivial* if it is distinct from the identity relation and from the full relation.

An h-ary relation ρ on A is called *central* if $\rho \neq A^h$ and there exists a non-void proper subset C of A (called the *center* of ρ) such that

(a) $(a_1, \ldots, a_h) \in \rho$ whenever at least one $a_i \in C (1 \leq i \leq h)$;

(b) ρ is *totally symmetric*, i.e. $(a_1, \ldots, a_h) \in \rho$ implies $(a_{1\pi}, \ldots, a_{h\pi}) \in \rho$ for every permutation ϕ of the indices $1, \ldots, h$;

(c) ρ is *totally reflexive*, i.e. $(a_1, \ldots, a_h) \in \rho$ if $a_i = a_j$ for some $i \neq j (1 \leq i, j \leq h)$.

Let $h \geq 3$. A family $T = \{\Theta_1, \ldots, \Theta_m\}$ $(m \geq 1)$ of equivalence relations on A is called *h-regular* if each Θ_i $(1 \leq i \leq m)$ has exactly h blocks and $\Theta_T = \Theta_1 \cap \ldots \cap \Theta_m$ has exactly h^m blocks (i.e. the intersection $\bigcap_{i=1}^m B_i$ of arbitrary blocks B_i of $\Theta_i (i = 1, \ldots, m)$ is nonempty). The relation determined by T is

$$\lambda_T = \{(a_1, \ldots, a_h) \in A^h : a_1, \ldots, a_h \text{ are not pairwise incongruent modulo } \Theta_i \text{ for all } i (1 \leq i \leq m)\}.$$

Note that h-regular relations are both totally reflexive and totally symmetric.

Now we are in a position to state Rosenberg's Theorem:

Theorem A (I. G. Rosenberg [7],[8]). *A subset of* $\mathbf{O}$ *is a maximal clone if and only if it is of the form* Polρ *for a relaton* ρ *of one of the following six types:*

1. *a bounded partial order;*
2. *a binary relation* $\{(a, a\pi) | a \in A\}$ *where* π *is a permutation of* A *with* $|A|/p$ *cycles of the same length* p *(p is a prime number);*
3. *a quaternary relation* $\{(a_1, a_2, a_3, a_4) \in A^4 | a_1 + a_2 = a_3 + a_4\}$ *where* $(A; +)$ *is an elementary abelian p-group (p is a prime number);*
4. *a nontrivial equivalence relation;*
5. *a central relation;*
6. *a relation determined by an h-regular family of equivalence relations.*

Moreover, a finite algebra $\mathbf{A} = (A, F)$ *is primal if and only if* $F \subseteq \mathrm{Pol}\rho$ *for no relation* ρ *of any of the above six types.*

3 Results

From now on A is supposed to be the set $\{0, \dots, k-1\}$ with $k > 2$.

Theorem 3.1 *There exist three maximal clones such that their intersection is* $\mathbf{J}$*. Moreover, if* k *is a prime number then there are two maximal clones such that their intersection is* $\mathbf{J}$*.*

Proof. For any $a \in A$ define a binary relation ρ_a on A as follows:

$$\rho_a = \{(x, y) | x = a \text{ or } y = a \text{ or } x = y\}.$$

Observe that ρ_a is a central relation with center $\{a\}$. Choose two fixed point free permutation σ and τ on A of prime orders such that $\{\sigma, \tau\}$ generates a transitive permutation group on A. If k is a prime number then we can choose σ and τ with $\sigma = \tau$. Then, by Theorem A, $\mathrm{Pol}\rho_a$ $(a \in A)$, $\mathrm{Pol}\sigma$ and $\mathrm{Pol}\tau$ are maximal clones. Put $F = \mathrm{Pol}\rho_0 \cap \mathrm{Pol}\sigma \cap \mathrm{Pol}\tau$. We show that $F = \mathbf{J}$.

Consider the algebra $\mathbf{A} = (A; F)$. Then ρ_0 is a compatible relation, σ and τ are automorphisms of $\mathbf{A}$. Therefore, by the choice of σ and τ, Aut $\mathbf{A}$ is transitive, which implies that every operation of $\mathbf{A}$ is surjective. Moreover, if $\pi \in$ Aut $\mathbf{A}$ then

$$\rho_{0\pi} = \{(x\pi, y\pi) | (x, y) \in \rho_0\}$$

is also a compatible relation of $\mathbf{A}$. Therefore, by the transitivity of Aut $\mathbf{A}$, we have that ρ_a is a compatible relation of $\mathbf{A}$ for every $a \in A$. From this it follows that for every distinct $a, b \in A$

$$\rho_{ab} = \rho_a \cap \rho_b = \{(a, b), (b, a)\} \cup \{(x, x) | x \in A\}$$

is also a compatible relation of $\mathbf{A}$.

It is well-known that if a surjective operation preserves a central relation then it preserves its center (see e.g. [9]). Thus we have that every operation in F is idempotent. Suppose that $\mathbf{A}$ has a nontrivial operation. Then it has either a nontivial binary operation or a majority function or a Mal'tsev function or a nontrivial semi-projection among its term functions (see e.g. [4]).

First consider the case when $\mathbf{A}$ has a nontrivial binary term function f. Let $a, b \in A$ be arbitrary distinct elements. Then from $(a, b), (b, b) \in \rho_{ab}$ we have that $(f(a, b), b) = (f(a, b), f(b, b)) \in \rho_{ab}$, implying that $f(a, b) = a$ or $f(a, b) = b$. Suppose that $f(a, b) = a$ and choose an arbitrary element $c \in A$ with $c \neq a, b$. Then $(a, a), (b, c) \in \rho_{bc}$ implies that $(f(a, c), a) = (f(a, c), f(a, b)) \in \rho_{bc}$ and $f(a, c) = a$. This fact together with the transitivity of Aut $\mathbf{A}$ shows that f is the first projection, a contradiction. If $f(a, b) = b$ then a similar argument yields that f is the second projection.

Now let d be a majority term function of $\mathbf{A}$, and let $a, b, c \in A$ be pairwise different elements. Then $d(a, b, c)$ is different from two of the elements a, b, c, say from a and b. Then $(a, a), (b, a), (c, c) \in \rho_b$ implies that $(d(a, b, c), a) = (d(a, b, c), d(a, a, c)) \in \rho_b$, a contradiction.

If t is a Mal'tsev function among the term functions of $\mathbf{A}$, then for any two distinct elements $a, b \neq 0$, $(a, 0), (0, 0), (0, b) \in \rho_0$ implies that $(a, b) = (t(a, 0, 0), (t(0, 0, b)) \in \rho_0$, a contradiction.

Finally, let l be a nontrivial n-ary first semi-projection among the term functions of **A**. Since l is not the first projection, there are $a, a_2, \dots, a_n \in A$ such that $l(a, a_2, \dots, a_n) = b \neq a$. Choose $c \in A$ with $c \neq a, b$. Then $(a, c), (a_2, a), \dots, (a_n, a) \in \rho_a$ implies that $(b, c) = (l(a, a_2, \dots, a_n), l(c, a, \dots, a)) \in \rho_a$, a contradiction. This completes the proof.

Theorem 3.2 *There exist three minimal clones such that their join is* **O**. *Moreover, if k is a prime number then there are two minimal clones such that their join is* **O**.

Proof. First consider the case when k is a prime number and let σ be the permutation $(0\ 1\ \dots\ k-1)$ on A. Clearly, $[\sigma]$ is a minimal clone. Define a ternary operation f on A as follows:

$$f(x, y, z) = \max(\min(x, y), \min(x, z), \min(y, z))$$

for all $x, y, z \in A$. Then f is a majority function and $[f]$ is a minimal clone (see e.g. [6]). We show that f together with σ generates **O**. Put $F = \{f, \sigma\}$.

Taking into consideration Theorem A, we have to show that $F \subseteq \mathrm{Pol}\rho$ for no relation of any of the types (1)-(6). Since σ generates a transitive permutation group, it is easy to show that it cannot preserve a relation of type (1) and (5). Moreover, making use of the fact that k is a prime number, one can show easily that σ do not preserve a relation of type (4). Furthermore, f being a majority function - as it is well-known (see e.g. [4]) - do not preserve a relation of type (3) and (6). Finally suppose that ρ is a relation of type (2) determined by a permutation π with $F \subseteq \mathrm{Pol}\rho$. Then π is an automorphism of the algebra $\mathbf{A} = (A; F)$. Since π and σ commute we have that π is a power of σ, and then σ is also a power of π (k is prime). Hence σ is an automorphism of **A**. Therefore, we have that $1 = f(0, 1, 2) = f((k-1)\sigma, 0\sigma, 1\sigma) = f(k-1, 0, 1)\sigma = 1\sigma = 2$, a contradiction. This completes the proof when k is a prime number.

Now suppose that k is not a prime, and let p be a prime number such that $k/2 < p < k$. Consider the permutations $\sigma = (0\ 1\ \dots\ p-1)$ and $\tau = (k-p\ \ k-(p-1)\ \dots\ k-1)$ on A. Clearly, $[\sigma]$ and $[\tau]$ are minimal clones. Define a ternary operation d on A as follows:

$$d(x, y, z) = \begin{cases} x, & \text{if } x = y, \\ z, & \text{otherwise.} \end{cases}$$

Then d is the well-known dual discriminator, which generates a minimal clone (see e.g. [2]). We show that σ and τ together with d generate **O**. Put $F = \{d, \sigma, \tau\}$.

Again, by Theorem A, we have to show that $F \subseteq \mathrm{Pol}\rho$ for no relation of any of the types (1)-(6). Suppose that $F \subseteq \mathrm{Pol}\rho$ for a relation of one of the type (1)-(6). It is known that $\{d\}$ is a functionally complete set (see e. g. [1]). Therefore $d \notin \mathrm{Pol}\rho$ if $\mathrm{Pol}\rho$ contains all constant operations. Hence ρ is of type (2) or a unary central relation. Since $[\sigma]$ and $[\tau]$ generate a transitive permutation group, they do not preserve a unary central relation.

Finally suppose that ρ is a relation of type (2) determined by a permutation π. Observe that if π is of order q then π is the product of k/q cycles of the same length q. Moreover, since k is not a prime number, we have $q \leq k/2$. Then π commutes with σ and τ. Let $0\pi = i$. If $i > p-1$ then for all $j \in \{0, 1, \dots, p-1\}$ we have $j\pi = 0\sigma^j\pi = 0\pi\sigma^j = i\sigma^j = i$, showing that π is not injective, a contradiction. Hence $i < p$, and for all $j \in \{0, 1, \dots, p-1\}$ we have $j\pi = 0\sigma^j\pi = 0\pi\sigma^j = i\sigma^j = 0\sigma^i\sigma^j = 0\sigma^j\sigma^i = j\sigma^i$ showing that π contains the cycle σ^i of length p. Therefore we have $p = q \leq k/2$, a contradiction. This completes the proof.

Problem 1 *Find all natural numbers k for which there exist two maximal clones on the set* $\{0, \ldots, k-1\}$ *such that their intersection is* **J**.

Problem 2 *Find all natural numbers k for which there exist two minimal clones on the set* $\{0, \ldots, k-1\}$ *such that their join is* **O**.

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