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Representation of automaton mappings in finite length

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In [3] we introduced a family of products, and in some cases it was decided for two such products whether one of them is a real generalization of the other one with respect to the homomorphic representation of automata. In this paper we investigate similar problems concerning representations of automaton mappings in finite length.

To make this paper self-contained we recall the notions and notations of automata theory used in our later discussions.

By a *finite automaton* we mean a system $\mathbf{A}=(X, A, Y, \delta, \lambda)$, where X, A and Y are finite (nonvoid) sets, called input, state and output sets, respectively. δ denotes the transition function and λ is the output function of $\mathbf{A}$.

Let $F(X)$ denote the free monoid generated by the input set X. The transition function δ can be extended to $A\times F(X)$ in the following way: for any $p=p'x\in F(X)$ and $a\in A$, $\delta(a,p)=\delta(\delta(a,p'),x)$. In the sequel we use the more convenient notation $ap_{\mathbf{A}}$ for $\delta(a,p)$. If there is no danger of confusion we omit the index $\mathbf{A}$.

Take an $a\in A$. We define a mapping $f_{\mathbf{A},a}$: $F(X)\to F(Y)$ in the following way: for any $p=x_1x_2...x_n\in F(X)$, let $f_{\mathbf{A},a}(p)=y_1y_2...y_n$ where $y_1=\lambda(a,x_1)$, $y_2=\lambda(ax_1,x_2)$, $\ldots$, $y_n=\lambda(ax_1...x_{n-1},x_n)$. This $f_{\mathbf{A},a}$ is called the *mapping induced by* $\mathbf{A}$ *in the state* a. For convenience, further on we give an automaton in the form $\mathbf{A}=(X, A, a_0, Y, \delta, \lambda)$ if we are interested in $f_{\mathbf{A},a_0}$, and use the notation $f_{\mathbf{A}}$ for $f_{\mathbf{A},a_0}$. In this case it is said that $\mathbf{A}$ is an *initial automaton* with the *initial state* a_0.

A mapping f: $F(X)\to F(Y)$ ($|X|, |Y|<\aleph_0$) is called an *automaton mapping* if there exists a (not necessarily finite) automaton $\mathbf{A}=(X, A, a, Y, \delta, \lambda)$ such that $f=f_{\mathbf{A}}$. Moreover, let n be a natural number. We say that $\mathbf{A}$ *induces* f *in length* n if $f(p)=f_{\mathbf{A}}(p)$ for all $p\in F_n(X)$, where $F_n(X)$ denotes the set of all input words of $\mathbf{A}$ with length nonexceeding n.

If we omit the output set and output function of an automaton $\mathbf{A}=(X, A, Y, \delta, \lambda)$ then we get the *semiautomaton* belonging to $\mathbf{A}$. Thus, a semiautomaton has the form $\mathbf{A}=(X, A, \delta)$. Let n be a natural number, and for an initial semiautomaton $\mathbf{A}=(X, A, a, \delta)$ set $A^{(n)}=\{ap\mid p\in F_n(X)\}$. Take two semiautomata $\mathbf{A}=(X, A, a, \delta)$

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and $\mathbf{B}=(X, B, b, \delta')$. Then a mapping τ of $A^{(n)}$ onto $B^{(n)}$ is called an *n-homomorphism* of $\mathbf{A}$ *onto* $\mathbf{B}$ if $\tau(ap)=bp$ holds for any $p\in F_n(X)$.

One can easily prove the following:

Lemma 1. Take an automaton $\mathbf{B}=(X, B, b, Y, \delta', \lambda')$ and let $\mathbf{A}'=(X, A, a, \delta)$ be a semiautomaton. Assume that for a natural number n, there exists an n-homomorphism of $\mathbf{A}'$ onto $\mathbf{B}'=(X, B, b, \delta')$. Then there is a mapping $\lambda\colon A\times X\to Y$ such that $\mathbf{A}=(X, A, a, Y, \delta, \lambda)$ induces $f_{\mathbf{B}}$ in length $n+1$.

In the sequel by an automaton (semiautomaton) we always mean a finite automaton (semiautomaton).

Let $\mathbf{A}_i=(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(i=1, \dots, n)$ be arbitrary automata, X and Y finite (nonvoid) sets. Moreover, take two mappings

$$\varphi\colon A_1\times\dots\times A_n\times X\to X_1\times\dots\times X_n$$

and

$$\varphi'\colon A_1\times\dots\times A_n\times X\to Y.$$

Then it is said that the automaton $\mathbf{A}=(X, A, Y, \delta, \lambda)$ with $A=A_1\times\dots\times A_n$ is the *(general) product* of $\mathbf{A}_1, \dots, \mathbf{A}_n$ with respect to X, Y, φ and φ' if

$$\delta((a_1, \dots, a_n), x) = (\delta_1(a_1, x_1), \dots, \delta_n(a_n, x_n))$$

and

$$\lambda((a_1, \dots, a_n), x) = \varphi'(a_1, \dots, a_n, x)$$

hold for any $(a_1, \dots, a_n)\in A$ and $x\in X$, where $(x_1, \dots, x_n)=\varphi(a_1, \dots, a_n, x)$ (cf. [4]). For this product we shall use the short notation $\mathbf{A}=\prod_{i=1}^{n}\mathbf{A}_i\,[X, Y, \varphi, \varphi']$. The general product of semiautomata can be defined analogously; as it is determined by the input set X and the feedback function completely, we can write $\mathbf{A}=\prod_{i=1}^{n}\mathbf{A}_i[X, \varphi]$ in this case. If $X=X_1\times\dots\times X_n$ and $\varphi(a, x)=x$ $(a\in A,\ x\in X)$ then we speak of a *direct product*. Moreover, if $\varphi(a, x)$ is independent of a for any $a\in A$ then $\mathbf{A}$ is called a *quasi-direct product*.

Let α be a mapping of the set N of all natural numbers into itself such that $\alpha(i)\geqq i$ for all $i\in N$. A product $\mathbf{A}=\prod_{i=1}^{n}\mathbf{A}_i[X, Y, \varphi, \varphi']$ is an *α-product* if φ can be given in the form

$$\varphi(a_1, \dots, a_n, x) = (\varphi_1(a_1, \dots, a_n, x), \dots, \varphi_n(a_1, \dots, a_n, x))$$

such that φ_i $(1\leqq i\leqq n)$ is independent of states having indices greater than or equal to $\alpha(i)$. We denote by α_j $(\colon N\to N; j=0, 1, \dots)$ the mapping for which $\alpha_j(i)=i+j$ $(i\in N)$. It can be proved (cf. [1]) that the α_0-product is the same as the loop-free composition introduced in [5].

Take a natural number n. An automaton $\mathbf{A}=(X, A, a, Y, \delta, \lambda)$ is called n-free if $ap\neq aq$ for all $p, q\in F_n(X)$ with $p\neq q$.

The following result is obvious.

Lemma 2. Take two semiautomata $\mathbf{A}=(X, A, a, \delta)$ and $\mathbf{B}=(X, B, b, \delta')$. If $\mathbf{A}$ is n-free then there exists an n-homomorphism of $\mathbf{A}$ onto $\mathbf{B}$.

We say that the α_i-product is *metrically equivalent* to the α_j-product (general product) if for any natural number n and system Σ of automata, a mapping f can be induced in length n by an α_i-product of automata from Σ if and only if f can be induced in length n by an α_j-product (general product) of automata from Σ.

Now we are ready to prove the following

Theorem. For all $i=0, 1, \ldots,$ the α_i-product is metrically equivalent to the general product.

Proof. Let Σ be a system of automata. Take a (general) product $\mathbf{A}=(X, A, a_0, Y, \delta, \lambda)=\prod_{j=1}^{k} \mathbf{A}_j[X, Y, \varphi, \varphi']$ $(\mathbf{A}_j \in \Sigma)$. By Lemma 1, it is enough to show that there exists an α_0-product $\mathbf{B}=(X, B, b, Y, \delta', \lambda')$ of automata from Σ such that for any natural number n, the semiautomaton $\mathbf{B}=(X, B, b, \delta')$ can be mapped n-homomorphically onto $\mathbf{A}'=(X, A, a_0, \delta)$. Thus in the sequel we may confine ourselves to semiautomata, i.e., we can assume that Σ consists of semiautomata.

For a semiautomaton $\mathbf{C}^*=(X, C, \delta_{\mathbf{C}}^*)$ we say that a state $c \in C$ is *ambiguous* if there are $x, x' \in X$ such that $\delta_{\mathbf{C}}^*(c, x) \neq \delta_{\mathbf{C}}^*(c, x')$. Let n be a fixed natural number, and let $u<n$ be the greatest number for which there exist a $\mathbf{C}=(Z, C, \delta_{\mathbf{C}}) \in \Sigma$, $c \in C$ and $p \in F(Z)$ with $|p|=u$ such that cp is ambiguous. ($|p|$ denotes the length of p.) Assume that there exists such a u. Then for all $t \leqq u$ there are states $c_t \in C$ and words $p_t \in F(Z)$ with $|p_t|=t$ such that $c_t p_t$ are ambiguous. (Indeed, c_t and p_t can respectively be chosen as cq_t and q_t', where q_t is the prefix of p with $|q_t|=u-t$ and q_t' is the suffix of p with $|q_t'|=t$.)

First we construct a $(u+1)$-free semiautomaton $\mathbf{D}=(W, D, d, \delta_{\mathbf{D}})$ as an α_0-product of semiautomata from the one-element set $\{\mathbf{C}\}$, where $W=\{w_1, w_2\}$. For each $c_t \in C$ $(t=0, 1, \ldots, u)$ choose two inputs $z_t, z_t' \in Z$ such that $\delta_{\mathbf{C}}(c_t p_t, z_t) \neq \delta_{\mathbf{C}}(c_t p_t, z_t')$. Form the α_0-product $\mathbf{D}_1=(W, D_1, d_1, \delta^{(1)})=\mathbf{C}[W, \varphi^{(1)}]$, where $d_1=c_u$ and for all $d \in D_1$ and $w_r \in W$

$$\varphi^{(1)}(d, w_r)=\begin{cases} z_u & \text{if } r=1, \\ z_u' & \text{if } r=2. \end{cases}$$

It is obvious that $\mathbf{D}_1$ is a 1-free semiautomaton. Now assume that for all $m \leqq s (\leqq u)$ we have constructed an m-free α_0-power $\mathbf{D}_m=(W, D_m, d_m, \delta^{(m)})$ of $\mathbf{C}$. Furthermore, suppose that $p_s=\bar{z}_1 \ldots \bar{z}_s$ and let l be a natural number such that $2^l \geqq 2^{s+1}+s+1$. Take the l-th direct power $C'=(Z', C', c, \delta_{\mathbf{C}}')$ of $\mathbf{C}$, where $c=(c_s, \ldots, c_s)$. Moreover, let $\bar{Z}$ be the subset of Z' consisting of all elements $\bar{z}$ whose each component is either z_s or z_s', and $cp_s\bar{z} \neq cp_s'$ for any prefix p_s' of p. Denote by $\mathbf{D}_{s+1}=(W, D_{s+1}, d_{s+1}, \delta^{(s+1)})$ the α_0-product $(\mathbf{D}_s \times \mathbf{C}')[W, \varphi^{(s+1)}]$, where $d_{s+1}=(d_s, c)$ and for any $p, q \in F(W)$, $w \in W$ and $c', c'' \in C'$,

(i) $\varphi_1^{(s+1)}(d_s p, c', w)=w$,

(ii) $\varphi_2^{(s+1)}(d_s p, c', w)=\bar{z}_{v+1}$ if $|p|=v<s$,

(iii) if $|p|=|q|=s$ then $\varphi_2^{(s+1)}(d_s p, c', w) \in \bar{Z}$ such that

$$\varphi_2^{(s+1)}(d_s p, c', w) \neq \varphi_2^{(s+1)}(d_s q, c'', w') \quad \text{if} \quad (d_s p, w) \neq (d_s q, w').$$

$(\varphi_2^{(s+1)}(d_s p, c', w)$ with $|p|=s$ can be chosen in this way, since $|\bar{Z}| \geqq 2^{s+1}$.)

(iv) in all other cases $\varphi^{(s+1)}$ is defined arbitrarily such that the resulting product is an α_0-product.

We prove that $\mathbf{D}_{s+1}$ is an $(s+1)$-free semiautomaton. Take two words $p, q \in F(W)$ with $p \neq q$. Now let us distinguish the following three cases:

1) $|p|, |q| \leqq s$. Then $d_s p \neq d_s q$ since $\mathbf{D}_s$ is an s-free semiautomaton. Therefore, $d_{s+1} p = (d_s, c) p \neq (d_s, c) q = d_{s+1} q$.

2) $|p| = v \leqq s$ and $|q| = s+1$. Let us assume that $q = q'w$ $(w \in W)$. Then by the definition of $\varphi^{(s+1)}$, $(d_s, c)p = (d_s p, c\bar{z}_1 \dots \bar{z}_v)$ and $(d_s, c)q = (d_s q, cp_s \varphi_2^{(s+1)}(d_s q', cp_s, w))$. Again, by the definition of $\varphi^{(s+1)}$, $c\bar{z}_1 \dots \bar{z}_v \neq c_s p_s \varphi_2^{(s+1)}(d_s q', cp_s, w)$.

3) $|p| = |q| = s+1, p = p'w$ and $q = q'w'$ $(w, w' \in W)$. Now, by the definition of $\varphi_2^{(s+1)}$, since $\mathbf{D}_s$ is an s-free semiautomaton, thus $\varphi_2^{(s+1)}(d_s p', cp_s, w) \neq \varphi_2^{(s+1)}(d_s q', cp_s, w')$. Therefore, $(d_s, c)p = (d_s p, cp_s \varphi_2^{(s+1)}(d_s p', cp_s, w)) \neq (d_s q, cp_s \varphi_2^{(s+1)}(d_s q', cp_s, w')) = (d_s, c) q$.

Thus we have shown that for all $s \leqq u+1$, $\mathbf{D}_s$ is an s-free semiautomaton. Then $\mathbf{D}$ can be chosen as $\mathbf{D}_{s+1}$.

We now construct a $(u+1)$-free semiautomaton $\mathbf{E} = (X, E, e_0, \delta_{\mathbf{E}})$ as a quasi-direct product of semiautomata from the one-element set $\{\mathbf{D}\}$. Let t be a natural number such that $2^t \geqq |X|$. Moreover, take a one-to-one mapping ψ of X into W^t. We shall prove that $\mathbf{E} = (X, E, e_0, \delta_{\mathbf{E}}) = \mathbf{D}^t[X, \psi]$ with $e_0 = (d, \dots, d)$ is a $(u+1)$-free semiautomaton. (The feed-back function ψ of $\mathbf{E}$ can be given in this form, since for quasi-direct products the feed-back function is independent of states.) Take two words $p, q \in F_{u+1}(X)$ with $p \neq q$. Assume that $p = x_1 \dots x_r$ and $q = x_1' \dots x_s'$. Then there exists an i $(1 \leqq i \leqq t)$ such that $\psi_i(x_1) \dots \psi_i(x_r) \neq \psi_i(x_1') \dots \psi_i(x_s')$. (Note that ψ is given in the form $\psi = (\psi_1, \dots, \psi_t)$.) Therefore, $d\psi_i(x_1) \dots \psi_i(x_r) \neq d\psi_i(x_1') \dots \psi_i(x_s')$ since $\mathbf{D}$ is a $(u+1)$-free semiautomaton. Thus we have got that $e_0 p \neq e_0 q$, showing that $\mathbf{E}$ is a $(u+1)$-free semiautomaton.

Let us now consider the following two cases:

I) $u+1 = n$. In this case, by Lemma 2, $\mathbf{A}'$ is an n-homomorphic image of $\mathbf{E}$.

II) $u+1 < n$. Then take the direct product $\mathbf{G} = (X', G, g_0, \delta_{\mathbf{G}}) = \Pi(\mathbf{A}_j \mid j = 1, \dots, k)$, where $G = A$ and $g_0 = a_0$. Now form the α_0-product $\mathbf{H} = (X, H, h, \delta_{\mathbf{H}}) = (\mathbf{E} \times \mathbf{G})[X, \gamma]$, where $h = (e_0, a_0)$, and for all $x \in X$, $p \in F(X)$, $e \in E$ and $g \in G$,

$$\gamma(e_0 p, g, x) = (x, \varphi(a_0 p_{\mathbf{A}}, x)) \quad \text{if} \quad |p| \leqq u+1$$

and $\gamma(e, g, x) = (x, x')$, where x' is an arbitrary element of X' if e cannot be given in the form $e_0 p$ with $p \in F_{u+1}(X)$.

Since for a given $p \in F_{u+1}(X)$ there exists no $q \in F_{u+1}(X)$ such that $p \neq q$ and $e_0 p = e_0 q$, thus γ is well defined.

Let us take a mapping $\tau \colon H^{(n)} \to A^{(n)}$ in the following way: $\tau((e, a)) = a$ $((e, a) \in H^{(n)})$. (Here $A^{(n)}$ is considered in $\mathbf{A}$.) We show that τ is an n-homomorphism of $\mathbf{H}$ onto $\mathbf{A}$. Take an arbitrary word $p \in F_n(X)$ with $|p| = l$. We proceed by induction on the length l of p. For $|p| = 0$, $\tau((e_0, a_0)p) = a_0 p_{\mathbf{A}}$ is obviously valid. Assume that our statement has been proved for all words with length t $(< n)$. Now let $p = p'x$ $(x \in X)$ such that $|p| = j+1$ $(\leqq t+1)$. If $|p'| \leqq u+1$ then

$$(e_0, a_0)p = (e_0 p, a_0 p'_{\mathbf{A}} \varphi(a_0 p'_{\mathbf{A}}, x)) = (e_0 p, a_0 p_{\mathbf{A}}),$$

i.e., $\tau((e_0, a_0)p) = \tau((e_0 p, a_0 p_{\mathbf{A}})) = a_0 p_{\mathbf{A}} = \tau((e_0, a_0)) p_{\mathbf{A}}$.

Now consider the case $n>|p'|>u+1$. Then $(e_0, a_0)p=(e_0, a_0)p'(x, \gamma(e, a, x))=$ $=(ex, a\gamma(e, a, x))$, where $(e, a)=(e_0, a_0)p'$. Observe that $ax_{\mathbf{A}}=ax'_{\mathbf{A}}$ for any $x, x'\in X$, since otherwise there exist an $\mathbf{A}_j$ $(1\leqq j\leqq k)$, $a_j\in A_j$ and $p_j\in F(X_j)$ with $n>|p_j|>u+1$ such that a_jp_j is ambiguous, contradicting our assumption that u is the greatest number having this property. Thus, taking into consideration the induction hypothesis $a=a_0p'_{\mathbf{A}}$, we get $(e_0, a_0)p=(e_0p, a_0p'_{\mathbf{A}}\gamma(e, a, x))=(e_0p, a_0p'_{\mathbf{A}}\varphi(a_0p'_{\mathbf{A}}, x))=(e_0p, a_0p_{\mathbf{A}})$, proving that $\tau((e_0, a_0)p)=a_0p_{\mathbf{A}}=\tau(e_0, a_0)p_{\mathbf{A}}$. Therefore, we have shown that τ is an n-homomorphism of $\mathbf{H}$ onto $\mathbf{A}$.

If there is no ambiguous state in any semiautomaton from Σ then $\mathbf{A}$ is isomorphic to a quasi-direct product of $\mathbf{A}_1, \ldots, \mathbf{A}_k$.

Since the direct product and quasi-direct product are special cases of the α_0-product, and the α_0-product of α_0-products is also an α_0-product thus $\mathbf{H}$ can be given as an α_0-product of semiautomata from Σ. This ends the proof of the Theorem.

A system Σ of automata is *metrically complete* with respect to the α_i-product (general product) if for any natural number n and automaton mapping f: $F(X)\to F(Y)$ $(|X|, |Y|<\aleph_0)$ there exists an α_i-product (general product) of automata from Σ inducing f in length n. In [2] it was shown that there exists an algorithm to decide for a finite system Σ of automata whether Σ is metrically complete with respect to the α_0-product. Using this result, from our above Theorem we get the following

Corollary. There exists an algorithm to decide for a finite system Σ of automata whether Σ is metrically complete with respect to the general product or any α_i-product $(i=0, 1, \ldots)$.

Представление автоматных отображений в конечном длине

В статье [3] было введено понятие α_i-произведения автоматов $(i=0, 1, \ldots)$. Пусть Σ — произвольное множество конечных автоматов и n — некоторое натуральное число. В настоящей работе доказывается, что автоматное отображение f можно индуцировать в длине n некоторым α_i-произведением автоматов из Σ тогда и только тогда f индуцируется в длине n некоторым произведением автоматов из Σ.

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