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On superpositions of automata

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We say that an automaton **A** *realises* an automaton **B** if **B** can be given as an A-homomorphic image of an A-subautomaton of **A**. If there exists a one-to-one homomorphism having the above property then it is said that **B** *can be embetted A-isomorphically* into **A**.

Let **A** be a finite automaton and denote by $C(\mathbf{A})$ the class of all finite superpositions of automata having fewer states than **A**. For any natural number l, let $C_l(\mathbf{A})$ be the class of all automata from $C(\mathbf{A})$ whose factors have not more states than l.

For any finite automaton **A** and natural number l one can raise the following questions:

(a) Whether there exists an $\mathbf{A}_1 \in C_l(\mathbf{A})$ such that $\mathbf{A}_1$ is A-isomorphic to **A**.

(b) Whether **A** can be embetted A-isomorphically into a superposition from $C_l(\mathbf{A})$.

(c) Whether **A** can be realized by an automaton in $C_l(\mathbf{A})$.

Using results published by M. Yoeli [6], we can solve (a). Moreover, by specializing Theorem 4.3.2. stated by F. Gécseg [2], problem (b) can also be solved. In both cases we can give an effective procedure.

In this paper, using a result mentioned by F. Gécseg and some results achieved by R. J. Nelson [5] and H. P. Zeiger [8], we present an algorithm to decide for any automaton **A** whether it can be realized by an automaton **B** from $C(\mathbf{A})$. Moreover, if such **B** exists then it can be given by a procedure presented in this paper.

Before studying these questions, we introduce some notions and notations.

In the sequel by an automaton we always mean a finite automaton.

Take two automata $\mathbf{A}_1=\mathbf{A}_1(X_1, A_1, Y_1, \delta_1, \lambda_1)$ and $\mathbf{A}_2=\mathbf{A}_2(X_2, A_2, Y_2, \delta_2, \lambda_2)$ with $Y_1 \subseteq X_2$. It is said that the automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ with $X=X_1$, $A=A_1 \times A_2$ and $Y=Y_2$ is the *superposition* of $\mathbf{A}_1$ by $\mathbf{A}_2$ (in notation: $\mathbf{A}=\mathbf{A}_1 * \mathbf{A}_2$) if for any $x \in X$ and $(a_1, a_2) \in A$,

$$\delta((a_1, a_2), x)=(\delta_1(a_1, x), \delta_2(a_2, \lambda_1(a_1, x)))$$

and

$$\lambda((a_1, a_2), x) = \lambda_2(a_2, \lambda_1(a_1, x))$$

hold.

The concept of superposition can be generalized in a natural way for any finite system of automata $\mathbf{A}_i=\mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(i=1, 2, \ldots, n)$ with $Y_j \subseteq X_{j+1}$ $(j=1, 2, \ldots, n-1)$.

Let k be a natural number and $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ be an automaton. Then by $\mathbf{A}^k$ we mean the automaton $\mathbf{B}=\mathbf{B}(X, B, Y', \delta', \lambda')$ with

$$B = \underbrace{A\times A\times \ldots \times A}_{k\text{-times}} \text{ and } Y' = \underbrace{Y\times Y\times \ldots \times Y}_{k-\text{times}}$$

such that for any $x\in X$ and $(a_1, a_2, \ldots, a_k)\in B$, we have

$$\delta'\big((a_1, \ldots, a_k), x\big) = \big(\delta(a_1, x), \ldots, \delta(a_k, x)\big)$$

and

$$\lambda'\big((a_1, \ldots, a_k), x\big) = \big(\lambda(a_1, x), \ldots, \lambda(a_k, x)\big).$$

Let $\mathbf{A}_i=\mathbf{A}_i(X, A_i, Y, \delta_i, \lambda_i)$ $(i=1, \ldots, n)$ be a system of automata such that for any $i, j\in\langle 1, \ldots, n\rangle$, $A_i\cap A_j=\emptyset$ if $i\neq j$. Then the automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ is called the *direct sum* of $\mathbf{A}_i(i=1, \ldots, n)$ if $A=\bigcup_{i=1}^{k} A_i$ and for any $x\in X$ and $a\in A$,

$$\delta(a, x) = \delta_i(a, x) \quad (a\in A_i)$$

and

$$\lambda(a, x) = \lambda_i(a, x) \quad (a\in A_i)$$

hold.

Take an arbitrary automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$. An $x\in X$ is called *reset signal* if there exists an $a\in A$ such that $\delta(b, x)=a$ for any $b\in A$. We say that this a belongs to x. An input signal $x\in X$ is said to be *permutation signal* if $\eta_x\colon a\to\delta(a, x)$ $(a\in\in A)$ is a permutation of A. Generally, for an automaton $\mathbf{A}$ with input set X, X_R denotes the set of all reset signals and X_P is the set of all permutation signals. An automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ is *reset, permutation* and *permutation-reset* automaton if respectively $X=X_R$, $X=X_P$ and $X=X_R\cup X_P$.

For any set H let $F(H)$ denote the free semigroup freely generated H. Furthermore, let ap be the last letter in the word $\delta(a, p)$ $\big(a\in A, p\in F(x)\big)$. Let $\mathbf{A}$ be an automaton and B a subset of the state set A of $\mathbf{A}$. Then for any input word p, we set $B^p = =\langle c|c=bp|b\in B\rangle$. Moreover we say that a system $\Gamma=\langle B_1, \ldots, B_n\rangle$ of subsets of A is *cover* of $\mathbf{A}$ if $\bigcup_{i=1}^{n} B_i=A$, $B_i\neq B_j$ implies $i\neq j$ and for any $B_i\in\Gamma$ and $x\in X$ there exists a $B_j\in\Gamma$ such that $B_i^x\subseteqq B_j$. For any $B_i\in\Gamma$ take a 1—1 mapping Φ_{B_i} of $\langle 1, 2, \ldots, \bar{\bar{B}}_i\rangle$ onto B_i. We say that a pair $(\mathbf{A}_1, \mathbf{A}_2)$ of automata is an *SR*-system of $\mathbf{A}$ belonging to Γ if the following conditions are satisfied:

$$\mathbf{A}_1 = \mathbf{A}_1(X, \Gamma, \Gamma\times X, \delta_1, \lambda_1), \quad \mathbf{A}_2 = \mathbf{A}_2(\Gamma\times X, \langle 1, \ldots, l\rangle, Y, \delta_2, \lambda_2),$$

where $l=\max\limits_{B_i\in\Gamma} \bar{\bar{B}}_i$; furthermore, for any $x\in X$, $B_i\in\Gamma$ and $k\in\langle 1, \ldots, l\rangle$,

$$B_i^x \subseteqq \delta_1(B_i, x),$$

$$\lambda_1(B_i, x) = (B_i, x),$$

$$\delta_2\big(k, (B_i, x)\big) = \begin{cases} \Phi^{-1}_{\delta_1(B_i, x)}\big(\delta(\Phi_{B_i}(k), x)\big) & \text{if } k \leqq \bar{\bar{B}}_i, \\ \text{arbitrary } m\in\langle 1, 2, \ldots, l\rangle\text{-otherwise,} \end{cases}$$

$$\lambda_2\big(k, (B_i, x)\big) = \begin{cases} \lambda\big(\Phi_{B_i}(k), x\big) & \text{if } k \leqq \bar{\bar{B}}_i, \\ \text{arbitrary } y\in Y\text{-otherwise.} \end{cases}$$

It has been proved (see [5]) that for any such pair $\mathbf{A}_1, \mathbf{A}_2$ the superposition $\mathbf{A}_1 * \mathbf{A}_2$ realises $\mathbf{A}$.

A system $(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ of automata is called an *SR-system of* $\mathbf{A}$ *with rank* k if $A_1 * \ldots * \mathbf{A}_n$ realizes $\mathbf{A}$, at least one $\mathbf{A}_i$ $(1 \leqq i \leqq n)$ has k states and none of $\mathbf{A}_1, \ldots, \mathbf{A}_n$ has more than k states.

Finally, it is said that $\mathbf{A}$ *can be mapped MA-homomorphically (MA-isomorphically) onto* $\mathbf{B}$ if the automaton without output belonging to $\mathbf{A}$ can be mapped A-homomorphically (A-isomorphically) onto the automaton without output belonging to $\mathbf{B}$.

Now we are ready to present our algorithm.

Let $\mathbf{A} = \mathbf{A}(X, A, Y, \delta, \lambda)$ be an arbitrary automaton. We shall investigate whether $\mathbf{A}$ has an SR-system of rank less than $\bar{\bar{A}}$.

We distinguish the following cases:

(I) If $\bar{\bar{A}} \leqq 2$ then $\mathbf{A}$ has no SR-system of rank less than $\bar{\bar{A}}$.

(II) Let $X = X_R$ and $\bar{\bar{A}} > 2$. Then every system $\Gamma^{(2)} = \langle B_1^{(2)}, B_2^{(2)} \rangle$ with $B_1^{(2)} \cup B_2^{(2)} = A$ and $1 \leqq \bar{\bar{B}}_1^{(2)}, \bar{\bar{B}}_2^{(2)} < \bar{\bar{A}}$ is a cover of $\mathbf{A}$. Giving an SR-system $(\mathbf{A}_1^{(2)}, \mathbf{A}_2^{(2)})$ of $\mathbf{A}$ belonging to Γ, we get the desired construction.

(III) Let $X = X_p$, $\bar{\bar{A}} > 2$ and assume that $\mathbf{A}$ can be given as a direct sum of two automata $\mathbf{B}$ with state set $B = \langle b_1, \ldots, b_n \rangle$ and $\mathbf{C}$ with state set C such that $\bar{\bar{B}} \leqq \bar{\bar{C}}$. In this case $\Gamma^{(3)} = \langle \langle b_1 \rangle, \langle b_2 \rangle, \ldots, \langle b_n \rangle, C \rangle$ is a cover of $\mathbf{A}$. Therefore, since $\bar{\bar{B}} \leqq \bar{\bar{C}}$ and $\bar{\bar{A}} > 2$ thus every SR-system $(\mathbf{A}_1^{(3)}, \mathbf{A}_2^{(3)})$ of $\mathbf{A}$ belonging to $\Gamma^{(3)}$ is suitable for our purpose.

(IV) Assume that $X = X_p$, $\bar{\bar{A}} > 2$ and $\mathbf{A}$ cannot be given as a direct sum of any two automata. Consider all proper subsets C_j of A having at least two elements and for any C_j give a cover $\Gamma_j = \langle C_j^p | p \in F(X) \rangle$. For any such Γ_j, let us consider an SR-system $(\mathbf{B}_j, \mathbf{A}_j)$ of $\mathbf{A}$ belonging to Γ_j. If one of these SR-systems has rank less than $\bar{\bar{A}}$ then it is a suitable SR-system of $\mathbf{A}$. If none of them has rank less than $\bar{\bar{A}}$ then take all pairs $(\mathbf{B}_j, \mathbf{A}_j)$ such that the number of states of $\mathbf{B}_j * \mathbf{A}_j$ is less than $\bar{\bar{A}}!$. (In this case this is only a formal requirement since the number of states of any $\mathbf{B}_j * \mathbf{A}_j$ is less than $\bar{\bar{A}}!$). For any subset C_{ij} of the state set of such $\mathbf{B}_j$ having at least two elements, let us construct a cover $\Gamma_{ij} = \langle C_{ij}^p | p \in F(X) \rangle$ of $\mathbf{B}_j$ and an SR-system $(\mathbf{B}_{ij}, \mathbf{A}_{ij})$ belonging to this cover. If one of these triples $(\mathbf{B}_{ij}, \mathbf{A}_{ij}, \mathbf{A}_j)$ is of rank less than $\bar{\bar{A}}$ then we get a desired SR-system of $\mathbf{A}$. If there exists no such system let us consider all systems $(\mathbf{B}_{ij}, \mathbf{A}_{ij}, \mathbf{A}_j)$ for which the number of states of $\mathbf{B}_{ij} * \mathbf{A}_{ij} * \mathbf{A}_j$ is less than $\bar{\bar{A}}!$. Now repeating the above process, we get the following cases:

(IV. A) We get an SR-system $(\mathbf{A}_1^{(4)}, \ldots, \mathbf{A}_n^{(4)})$ of $\mathbf{A}$ with rank less than $\bar{\bar{A}}$.

(IV. B) For all sequences $(\mathbf{B}, \mathbf{A}_1, \ldots, \mathbf{A}_n)$, $\bar{\bar{B}} \geqq \bar{\bar{A}}$ and the number of states of $\mathbf{B} * \mathbf{A}_1 * \ldots * \mathbf{A}_n$ is not less than $\bar{\bar{A}}!$. In this case $\mathbf{A}$ cannot be realized by a superposition of automata having fewer states than $\mathbf{A}$.

(V) Assume that $X = X_R \cup X_P$, $X_R \neq \emptyset$, $X_P \neq \emptyset$ and $\bar{\bar{A}} > 2$. If the X-subautomaton of $\mathbf{A}$ having input set X_P can be given as a direct sum then let us apply to this X-subautomaton the procedure presented in (III); in the opposite case let us apply to it the procedure given in (IV). In case (IV. B) the automaton $\mathbf{A}$ cannot be realized by a superposition of automata having fewer states than $\mathbf{A}$. If we get (IV. A) then one can apply (III) or, using (VII), we get a desired SR-system $(\mathbf{A}_1^{(5)}, \ldots, \mathbf{A}_n^{(5)})$ of $\mathbf{A}$.

(VI) Let $X \setminus (X_R \cup X_P) \neq \emptyset$, $\bar{\bar{A}} > 2$ and consider the construction given by H. P. Zeiger in [8]: For any $x \in X \setminus X_P$, let $a(x)$ denote the state of $\mathbf{A}$ such that $\delta(a', x) \neq$

$\neq a(x)$ where $a' \in A$ is arbitrary. Consider the cover $\Gamma^{(6)} = \langle B | B = A \setminus \langle a \rangle, a \in A \rangle$ and take the automaton $\mathbf{A}_1^{(6)} = \mathbf{A}_1^{(6)}(X, \Gamma^{(6)}, \Gamma^{(6)} \times X, \delta_1, \lambda_1)$ such that for any $x \in X$ and $B \in \Gamma^{(6)}$,

$$\delta_1(B, x) = \begin{cases} B^x & \text{if} \quad x \in X_P, \\ A \setminus \langle a(x) \rangle\text{-otherwise,} \end{cases}$$

and

$$\lambda_1(B, x) = (B, x).$$

Now choosing a suitable automaton $\mathbf{A}_2^{(6)}$, we get an SR-system $(\mathbf{A}_1^{(6)}, \mathbf{A}_2^{(6)})$ of $\mathbf{A}$ such that the number of states of $\mathbf{A}_2^{(6)}$ is less than $\mathbf{A}$, $\mathbf{A}_1^{(6)}$ is permutation-reset; moreover, if $X_P \neq \emptyset$ then the X-subautomata of $\mathbf{A}$ and $\mathbf{A}_1^{(6)}$ having input set X_P are A-isomorphic (see [5]).

Thus we get the following subprocedures.

(VI. A) If $A_1^{(6)}$ is a reset automaton then apply (II) to it. In this case $(\mathbf{A}_1^{(2)}, \mathbf{A}_2^{(2)}, \mathbf{A}_2^{(6)})$ is a required system.

(VI. B) If $\mathbf{A}_1^{(6)}$ has a permutation signal then apply (V) to it. If $\mathbf{A}_1^{(6)}$ has no SR-system with rank less than $\bar{A}$ then neither has $\mathbf{A}$. In the opposite case $(\mathbf{A}_1^{(5)}, \mathbf{A}_2^{(5)}, \ldots, \mathbf{A}_n^{(5)}, \mathbf{A}_2^{(6)})$ is an SR-system of $\mathbf{A}$ with rank less than $\bar{A}$.

(VII) Assume that $X \setminus X_R \neq \emptyset$, $X_R \neq \emptyset$ and the superposition $\mathbf{A}_1 * \mathbf{A}_2 * \ldots * \mathbf{A}_n$ of the automata $\mathbf{A}_i = A_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(\bar{A}_i < \bar{A};\ i = 1, \ldots, n)$ realises the X-subautomaton $\mathbf{B}$ with input set $X \setminus X_R$ of the automaton $\mathbf{A}$. Let ψ be an A-homomorphism of an A-subautomaton of the superposition $\mathbf{A}_1 * \mathbf{A}_2 * \ldots * \mathbf{A}_n$ onto $\mathbf{B}$. For any $x \in X_R$ take an element $(a_1(x), \ldots, a_n(x))$ of $A_1 \times A_2 \times \ldots \times A_n$ such that $\psi((a_1(x), \ldots, a_n(x)))$ is an element of A belonging to x. Construct the automaton $\mathbf{A}_i^{(7)} = \mathbf{A}_i^{(7)}(X_i', A_i, Y_i', \delta_i', \lambda_i')$ $(i = 1, \ldots, n)$ with $X_1' = X$ and $Y_n' = Y$ such that for any $j(=2, \ldots, n)$ and $k(=1, \ldots, n-1)$, $X_j' = A_1 \times A_2 \times \ldots \times A_{j-1} \times X$ and $Y_k' = A_1 \times \ldots \times A_k \times X$; furthermore, for any $i(=1, \ldots, n)$, $x_i \in X_i'$, and $a_i \in A_i$,

$$\delta_i'(a_i, x_i) =$$

$$= \begin{cases} \delta_i(a_i, x_i) & \text{if} \quad i = 1 \text{ and } x_i \notin X_R, \\ a_i(x_i) & \text{if} \quad i = 1 \text{ and } x_i \in X_R, \\ \delta_i(a_i, \lambda_{i-1}(a_{i-1}, \ldots, \lambda_1(a_1, x), \ldots)) & \text{if} \quad i > 1,\ x_i = (a_1, a_2, \ldots, a_{i-1}, x),\ x \notin X_R, \\ a_i(x) & \text{if} \quad i > 1,\ x_i = (a_1, a_2, \ldots, a_{i-1}, x),\ x \in X_R, \end{cases}$$

$$\lambda_i'(a_i, x_i) =$$

$$= \begin{cases} (a_i, x_i) & \text{if} \quad i = 1, \\ (a_1, a_2, \ldots, a_i, x) & \text{if} \quad 1 < i < n \text{ and } x_i = (a_1, \ldots, a_{i-1}, x), \\ \lambda(\psi(a_1, a_2, \ldots, a_n), x) & \text{if} \quad i = n,\ x_i = (a_1, \ldots, a_{n-1}, x) \text{ and } \psi((a_1, a_2, \ldots, a_n)) \text{ is} \\ & \qquad \text{defined, arbitrary } y \in Y\text{-otherwise.} \end{cases}$$

The system $(\mathbf{A}_1^{(7)}, \ldots, \mathbf{A}_n^{(7)})$ given above is an SR-system of $\mathbf{A}$ with rank less than $\bar{A}$.

We now show that the process given above is right. Superpositions of automata with one-element state sets have one-element state sets, too. Moreover, the state set is never void. Therefore (I) is obviously valid.

It can be seen directly from the definition that (II) and (III) are valid.

After proving (IV) and (VII), the validity of (V) follows obviously, and (VI) is valid by the results published in [8].

In order to deal with the construction given in (VII) take the partial mapping $\psi': A_1 \times A_2 \times \ldots \times A_n \to A$ given as follows: For any $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \ldots \times A_n$, let

$$\psi'((a_1, \ldots, a_n)) = \begin{cases} \psi((a_1, \ldots, a_n)) & \text{if} \quad \psi((a_1, \ldots, a_n)) \text{ is defined,} \\ \text{undefined-otherwise.} \end{cases}$$

It can be proved easily that ψ' is an A-homomorphism of a suitable A-subautomaton of $\mathbf{A}_1^{(7)} * \ldots * \mathbf{A}_n^{(7)}$ onto $\mathbf{A}$, i.e., the superposition $\mathbf{A}_1^{(7)} * \mathbf{A}_2^{(7)} * \ldots * \mathbf{A}_n^{(7)}$ realizes $\mathbf{A}$. This shows the applicability of (VII).

It remains to show that (IV) is valid. To do this consider the following two results.

Theorem 1. Let $\mathbf{A}$ be an automaton with n states. Then for any natural number k, every connected A-subautomaton of the A-direct power $\mathbf{A}^k$ of $\mathbf{A}$ is MA-isomprphic to a suitable A-subautomaton of the A-direct power $\mathbf{A}^n$.

Theorem 2. (R. J. Nelson [5]). Every permutation automaton is strongly connected or can be given as a direct sum of strongly connected permutation automata.

We now prove two lemmas. Applying them, we get Theorem 3 which shows the validity of (IV).

Lemma 1. Let n and l be arbitrary natural numbers such that $1 < l < n$. Furthermore, let $\mathbf{A}$ be a connected permutation automaton with n states having an SR-system $(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ of rank less than or equal to l.

Assume that an SR-system $(\mathbf{B}, \mathbf{C})$ of $\mathbf{A}$ has the following properties.

(a) $\mathbf{B} * \mathbf{C}$ is an MA-homomorphic image of a connected A-subautomaton of $\mathbf{A}^n$,

(b) $(\mathbf{A}_1, \ldots, \mathbf{A}_i)$ $(1 < i \leq m)$ is an SR-system of $\mathbf{B}$.

Then, using (IV), one can find an SR-system $(\mathbf{B}_1, \mathbf{C}_1)$ of $\mathbf{B}$ and a natural number t such that

(c) $\mathbf{B}_1 * \mathbf{C}_1 * \mathbf{C}$ is MA-homomorphic image of an A-subautomaton of $\mathbf{A}^{n \cdot t}$,

(d) $\mathbf{A}_1 * \ldots * \mathbf{A}_{i-1}$ realises $\mathbf{B}_1$,

(e) $\mathbf{C}_1$ has a number of states not exceeding l.

Proof. Using Theorem 2, it can be proved easily that every connected A-subautomaton of $\mathbf{A}^n$ is strongly connected permutation automaton. Therefore, the same is true for $\mathbf{B} * \mathbf{C}$, too. Thus $\mathbf{B}$ (as the first component of $\mathbf{B} * \mathbf{C}$) should be strongly connected permutation automaton. From this it follows, by an easy computation, that $\mathbf{A}_1 * \ldots * \mathbf{A}_{i-1}$ has a strongly connected A-subautomaton $\mathbf{D}$ such that $\mathbf{D} * \mathbf{A}_i$ realizes $\mathbf{B}$.

Let us denote by $F(X)$ the input semigroups of $\mathbf{A}$ and $\mathbf{D}$. Moreover, let D and A_i be the state sets of $\mathbf{D}$ and $\mathbf{A}_i$, respectively. Take an A-homomorphism ψ of a suitable A-subautomaton of $\mathbf{D} * \mathbf{A}_i$ onto $\mathbf{B}$. For any $d \in D$, define the set

$$\Delta(d) = \langle \psi((d, a_i)) \mid a_i \in A_i \rangle. \tag{1}$$

Since $\mathbf{B}$ is strongly connected thus $\Gamma = \langle (\Delta(d))^p \mid p \in F(x) \rangle$ is a cover of $\mathbf{B}$ for any $d \in D$.

Accomplishing a step of (IV), we get an SR-system $(\mathbf{B}_1, \mathbf{C}_1)$ of $\mathbf{B}$ belonging to Γ.

On the other hand, since the number of states of $\mathbf{A}_i$ does not exceed l and, by definition (1), $\overline{\overline{\Delta(d)}} \leqq l (d \in D)$ thus $\mathbf{C}_1$ has not more states than l. Therefore, (e) is valid.

Define a partitition Π on D as follows: $d_1 \equiv d_2(\Pi)$ if and only if $\Delta(d_1)=\Delta(d_2)(d_1, d_2 \in D)$. Then, by (1), Π is congruent. Therefore, $\mathbf{B}_1$ is an MA-homomorphic image of $\mathbf{D}$, i.e., (d) is valid.

Now in order to prove our Lemma it is enough to show that, choosing a suitable natural number t, (c) is also true. Since $\mathbf{B}$ is a permutation automaton thus $\overline{\overline{(\Delta(d))p}} = \overline{\overline{\Delta(d)}}$ holds for arbitrary $d \in D$ and $p \in F(X)$. Therefore, it is easy to prove that for any $d \in D$ and $p \in F(X)$,

$$(\Delta(d))^p = \Delta(dp). \tag{2}$$

By this equality (2), we can use the notation $\Delta(d)(d \in D)$ for the elements of Γ.

For any $\Delta(d) \in \Gamma$, let $\Phi_{\Delta(d)}$ be the one-to-one mapping of $\langle 1, 2, \ldots, \overline{\overline{\Delta(d)}} \rangle$ onto $\Delta(d)$ determined by $\mathbf{C}_1$. Moreover, let ψ' be the MA-homomorphism of a suitable connected A-subautomaton of $\mathbf{A}^n$ onto $\mathbf{B} * \mathbf{C}$. Since this subautomaton is strongly connected permutation automaton (see Theorem 2) thus the number of elements of arbitrary class of the partitition induced by ψ' is the same natural number t_1. Denote by C the state set of $\mathbf{C}$ and let $t = t_1 \cdot \overline{\overline{\Delta(d)}} \cdot \overline{\overline{C}} (d \in D)$.

For arbitrary state $(\Delta(d), c_1, c)$ of $\mathbf{B}_1 * \mathbf{C}_1 * \mathbf{C}$, let

$$\Omega(\Delta(d), c_1, c) = \Big\langle (a_1, a_2, \ldots a_{n \cdot t}) \Big| \bigcup_{i=0}^{t-1} \langle \psi'((a_{i \cdot n+1}, \ldots, a_{(i+1) \cdot n})) \rangle =$$

$$= \Delta(d) \times C, \psi'((a_1, \ldots, a_n)) = (\Phi_{\Delta(d)}(c_1), c) \Big\rangle. \tag{3}$$

We show that for any pair $(\Delta(d), c_1, c), (\Delta(d'), c_1', c')$,

$$(\Delta(d), c_1, c) \neq (\Delta(d'), c_1', c') \Rightarrow \Omega(\Delta(d), c_1, c) \cap \Omega(\Delta(d'), c_1', c') = \emptyset. \tag{4}$$

Assume that $\Delta(d) \neq \Delta(d')$. Then it can also be assumed that there exists a $b \in \Delta(d)$ with $b \notin \Delta(d')$. Take a state $(a_1'', \ldots, a_n'')$ from $\mathbf{A}^n$ such that $\psi'((a_1'', \ldots, a_n'')) \in \in \langle b \rangle \times C$. Then, by (3), every element $(a_1, \ldots, a_{n \cdot t})$ of $\Omega(\Delta(d), c_1, c)$ has a part $(a_{i \cdot n+1}, \ldots, a_{(i+1) \cdot n})$ $(0 \leqq i \leqq t-1)$ which is equal to $(a_1'', \ldots, a_n'')$, and for any element $(a_1', \ldots, a_{n \cdot t}')$ of $\Omega(\Delta(d'), c_1', c')$ we have $(a_{j \cdot n+1}', \ldots, a_{(j+1) \cdot n}') \neq (a_1'', \ldots, a_n'')$ $(j = 0, 1, \ldots, t-1)$. Therefore (4) is true.

Let $\Delta(d) = \Delta(d')$ and assume that $(c_1, c) \neq (c_1', c')$. Then by (3) for any pair $(a_1, a_2, \ldots, a_{n \cdot t}) \in \Omega(\Delta(d), c_1, c)$, $(a_1', a_2', \ldots, a_{n \cdot t}') \in \Omega(\Delta(d'), c_1', c')$ we have that $(a_1, \ldots, a_n) \neq (a_1', \ldots, a_n')$. This completes the proof of (4).

Let us show that for any state $(a_1, a_2, \ldots, a_{n \cdot t})$ of the A-direct power $\mathbf{A}^{n \cdot t}$ defined by (3) and for any input word $p \in F(X)$

$$(a_1, a_2, \ldots, a_{n \cdot t}) \in \Omega(\Delta(d), c_1, c) \Rightarrow (a_1, \ldots, a_{n \cdot t}) \cdot p \in \Omega((\Delta(d), c_1, c) \cdot p). \tag{5}$$

Since $\mathbf{B}$ and $\mathbf{B} * \mathbf{C}$ are permutation automata thus

$$(\forall (d, p))(d \in D, p \in F(X))(\overline{\overline{(\Delta(d))^p}} = \overline{\overline{\Delta(d)}}, \overline{\overline{(\Delta(d) \times C)^p}} = \overline{\overline{\Delta(d) \times C}}),$$

i.e. $(\Delta(d)\times C)^p=(\Delta(d))^p\times C$. Thus for arbitrary element $(a_1, a_2, \ldots, a_{n\cdot t})$ of $\Omega(\Delta(d), c_1, c)$ we have

$$\bigcup_{i=0}^{t-1}\langle\psi'((a_{i\cdot n+1}, \ldots, a_{(i+1)\cdot n})\cdot p)\rangle=(\Delta(d))^p\times C. \tag{6}$$

From $(a_1, \ldots, a_{n\cdot t})\in\Omega(\Delta(d), c_1, c)$ and (3)

$$\psi'((a_1, a_2, \ldots, a_n)\cdot p)=(\Phi_{\Delta(d)\cdot p}(c_1'), c') \tag{7}$$

where c_1' and c' are the second and third components of $(\Delta(d), c_1, c)\cdot p$. Hence used the definition (3) implies the (6) and (7) the (5) is valid, too.

From (4) and (5) we have that an A-subautomaton of A-direct power $\mathbf{A}^{n\cdot t}$ can be mapped MA-homomorphically onto $\mathbf{B}_1*\mathbf{C}_1*\mathbf{C}$. The classes of this homomorphism are represented by definition (3). This completes the proof of Lemma 1.

The following holds.

Lemma 2. Let $(\mathbf{B}, \mathbf{C})$ be an SR-system of a connected permutation automaton $\mathbf{A}$ and assume that $\mathbf{C}$ has fewer states than $\mathbf{A}$. Then it can be found an SR-system $(\mathbf{B}'\ \mathbf{C}')$ of $\mathbf{A}$ such that

(a) $\mathbf{B}'$ is MA-isomorphic to a strongly connected A-subautomaton of $\mathbf{B}$,

(b) using (IV) $\mathbf{C}'$ can be constructed as the second component of an SR-system of $\mathbf{A}$,

(c) $\mathbf{C}'$ has not more states than $\mathbf{C}$,

(d) $\mathbf{B}'*\mathbf{C}'$ is strongly connected,

(e) $\mathbf{B}*\mathbf{C}$ realises $\mathbf{B}'*\mathbf{C}'$.

Proof. Let ψ be an A-homomorphism of an A-subautomaton $\mathbf{M}$ of $\mathbf{B}*\mathbf{C}$ onto $\mathbf{A}$ and take a fixed state (b_0, c_0) of $\mathbf{M}$. Since $\mathbf{A}$ is strongly connected thus it can be assumed that $\mathbf{M}$ is also strongly connected.

Let $\mathbf{B}=\mathbf{B}(X, B, Y, \delta_B, \lambda_B)$ and take

$$\Delta(b_0)=\langle\psi(b_0, c)\,|\,c\in C\rangle, \quad\text{and}\quad \Delta(b)=(\Delta(b_0))^p \tag{8}$$

where $b=b_0p$ $(b\in B, p\in F(X)$ and C is the state set of $\mathbf{C})$.

Since $\mathbf{M}$ is strongly connected thus $\Delta(b_0)$ is non-empty. Therefore, $\Gamma=\langle(\Delta(b_0))^p|p\in F(X)\rangle$ is a cover of $\mathbf{A}$.

Denote by $(\mathbf{B}_1, \mathbf{C}')$ an SR-system of $\mathbf{A}$ belonging to Γ. By (8) and the construction of Γ, it can be seen that $\mathbf{C}'$ satisfies conditions (b) and (c) of Lemma 2.

Now let us define the automaton $\mathbf{B}'=\mathbf{B}'(X, B', \Gamma\times X, \delta_{B'}, \lambda_{B'})$ in the following way: $B'=\langle b|b=b_0p, p\in F(X)\rangle$ and for any $x\in X$ and $b\in B$, $\delta_{B'}(b, x)=\delta_B(b, x)$ and $\lambda_{B'}(b, x)=(\Delta(b), x)$.

By our construction, it is clear that $(\mathbf{B}', \mathbf{C}')$ is an SR-system of $\mathbf{A}$; furthermore, conditions (a) and (d) of Lemma 2 is satisfied.

Again, since $\mathbf{A}$ is a permutation automaton thus

$$(\Delta(b))^p=\Delta(bp) \tag{9}$$

for any $b\in B'$ and $p\in F(X)$.

For any $(b, k)\in B'\times\langle 1, 2, \ldots, \overline{\overline{\Delta(b)}}\rangle$, take

$$\Omega(b, k)=\langle(b, c)\,|\,c\in C, \psi((b, c))=\Phi_{\Delta(b)}(k)\rangle \tag{10}$$

where $\Phi_{\Delta(b)}$ is the one-to-one mapping of $\langle 1, 2, \ldots, \overline{\overline{\Delta(b)}}\rangle$ onto $\Delta(b)$ determined by $\mathbf{C}'$. By (9), the set $\Omega(b, k)$ given by (10) is defined for any (b, k) from $B' \times \langle 1, 2, \ldots, \ldots, \max_{b \in B'} \overline{\overline{\Delta(b)}}\rangle$. On the other hand, since the mappings $\Phi_{\Delta(b)}: \langle 1, 2, \ldots, \Delta(b)\rangle \to \Delta(b)$ defined by $\mathbf{C}'$ are 1—1 thus the sets $\Omega(b, k)(b \in B', k \in \langle 1, \ldots, \overline{\overline{\Delta(b)}}\rangle)$ forms a partition of a given subset of $B \times C$. Taking into consideration that ψ is a homomorphism this partition can be induced by a homomorphism ψ' onto $\mathbf{B}' * \mathbf{C}'$ because of (9). Therefore, $\mathbf{B} * \mathbf{C}$ realizes $\mathbf{B}' * \mathbf{C}'$ which ends the proof of Lemma 2.

It can be proved that if **A** is an permutation automaton with n states then none of the strongly connected A-subautomata of $\mathbf{A}^n$ has more states than $n!$ Thus the validity of (IV) follows from.

Theorem 3. Let n and l be natural numbers with $1 < l < n$. Moreover, assume that the connected permutation automaton **A** with n states has an SR-system $(\mathbf{A}_1, \ldots, \ldots, \mathbf{A}_m)$ of rank l. Then, using (IV), we get an SR-system $(\mathbf{B}_1, \ldots, \mathbf{B}_m)$ of **A** with rank not exceeding l such that $\mathbf{A}^n$ has an A-subautomaton which can be mapped MA-homomorphically onto $\mathbf{B}_1 * \ldots * \mathbf{B}_m$.

Proof. Let $\mathbf{B}_{m+1}$ an automaton with one state having the same input set as **A**; moreover, under any input signal x, $\mathbf{B}_{m+1}$ produces the same output signal x.

Let $\mathbf{B} = \mathbf{A}$, $\mathbf{C} = \mathbf{B}_{m+1}$ and $i = m$. It is clear hat for any $(\mathbf{B}, \mathbf{C})$ and natural i, the conditions of Lemma 1 are satisfied. By Lemma 2, it can be assumed that for the pair $(\mathbf{D}_0, \mathbf{B}_m)$ $(\mathbf{D}_0 = \mathbf{B}_1, \mathbf{B}_m = \mathbf{C}_1)$ given at the first step of (IV), $\mathbf{D}_0 * \mathbf{B}_m$ is strongly connected, i.e., $\mathbf{A}^{n \cdot t}$ has a strongly connected A-subautomaton which can be mapped MA-homomorphically onto $\mathbf{D}_0 * \mathbf{B}_m$. Since $\mathbf{B} = \mathbf{A}$ thus $(\mathbf{D}_0, \mathbf{B}_m)$ is an SR-system of **A**; i.e., we can disregard $\mathbf{B}_{m+1}$.

Using Theorem 1, there is an A-subautomaton of $\mathbf{A}^n$ which can be mapped MA-homomorphically onto. $\mathbf{D}_0 * \mathbf{B}_m$. Thus the system $\mathbf{B} = \mathbf{D}_0$, $\mathbf{C} = \mathbf{B}_m$, $i = m - 1$ satisfies the conditions of Lemma 1.

By Lemma 2, it can be assumed that for any pair $(\mathbf{D}_1, \mathbf{B}_{m-1})$ obtained at the second step of (IV), $\mathbf{D}_1 * \mathbf{B}_{m-1}$ is strongly connected. Again, using Lemma 2, it can also be shown that $\mathbf{D}_1 * \mathbf{B}_{m-1} * \mathbf{B}_m$ is strongly connected. This, by Theorem 1, implies that $\mathbf{A}^n$ has a strongly connected A-subautomaton which can be mapped MA-homomorphically onto $\mathbf{D}_1 * \mathbf{B}_{m-1} * \mathbf{B}_m$. Therefore, the system $\mathbf{B} = \mathbf{D}_1$, $\mathbf{C} = \mathbf{B}_{m-1} * \mathbf{B}_m$, $i = m - 2$ satisfies the conditions of Lemma 2. Repeating this process, we get an SR-system $(\mathbf{B}_1, \ldots, \mathbf{B}_m)$ of **A** such that

(a) $\mathbf{A}_1$ realizes $\mathbf{B}_1$ and the number of states of $\mathbf{B}_1$ and $\mathbf{B}_i$ $(i = 2, \ldots, m)$ do not exceed l,

(b) $\mathbf{A}^n$ has an A-subautomaton which can be mapped MA-homomorphically onto $\mathbf{B}_1 * \ldots * \mathbf{B}_m$,

(c) the system $(\mathbf{B}_1, \ldots, \mathbf{B}_m)$ (except one-state components) can be given by applications of (IV).

This completes the proof of Theorem 3 and at the same time we proved that our process is right.

We now show the validity of.

Theorem 4 (see [2]). There exists an automaton **A** with four states such that **A** can be realized by a superposition of three automata having fewer states than **A** but no superposition of two automata having fewer states than **A** realizes **A**.

Proof. Let $\mathbf{A}=\mathbf{A}(X, A, A\times X, \delta, \lambda)$ be the automaton with $X=\langle x_1, x_2\rangle$ given by the transition table below

δ	x_1	x_2
a_1	a_2	a_2
a_2	a_3	a_3
a_3	a_4	a_1
a_4	a_1	a_4

The $\lambda: A\times X\to A\times X$ output function induces the identical mapping.

It can be proved easily that any cover of $\mathbf{A}$ has at least four elements. Therefore, using a result by M. Yoeli [7], $\mathbf{A}$ cannot be realized as a superposition of two automata having fewer states than $\mathbf{A}$.

Now take an SR-system $(\mathbf{B}_1, \mathbf{A}_3)$ belonging to the cover $\Gamma_0=\langle\langle a_1, a_2\rangle^p|p\in F(X)\rangle$ of $\mathbf{A}$. Furthermore, let $(\mathbf{A}_1, \mathbf{A}_2)$ be an SR-system belonging to the cover $\Gamma_1=\langle\langle\langle a_1, a_2\rangle, \langle a_3, a_4\rangle\rangle^p|p\in F(X)\rangle$ of $\mathbf{B}_1$. By the constructions of Γ_0 and Γ_1, it can be proved easily that $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ have fewer states than four. This ends the proof of Theorem 4.

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