Local and global reversibility of finite inhomogeneous cellular automaton*

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Cellular automata are highly parallel working systems, so they have high importance in computational applications (for example sorting [4], matrix operations, etc.). It seems difficult to apply the classical infinite, homogeneous cellular automata to these purposes [1], [2]. For this reason the classical definitions are modified in this work. In point 1. we introduce the notion of finite, inhomogeneous cellular automaton. The reason of first modification (using by many authors, e.g. [7]) is clear: only finite automaton is realisable in practice. Further the second modification (the inhomogeneity) makes the cellular automaton more flexible [11], without excluding the homogeneity in hardware [3].

In the theory of cellular automata there is a very important and interesting question, that how appear the characteristics of local maps in the global map, and conversely. This is the basic conception of present work too, having in the centre the problem of reversibility. This subject has been investigated by many authors (in particular by T. Toffoli [8], [9]), but always in the global sense. In this context the reversibility is equivalent to the bijectivity of global map.

To the contrary, we mean the reversibility in local sense: a cellular automaton we shall call reversible, if its local maps may be changed so, that the new global map is the inverse of the original one.

The bijectivity of global map forms necessary condition for our "strong reversibility". Therefore in point 2. a connection will be proved between the local maps and the number of eden-configurations, from which derives a necessary condition for bijectivity (it is the generalization of results in [5]).

In point 3. a necessary and sufficient condition is presented to the reversibility. With this criterion we can decide the reversibility of a given cellular automaton, and construct its reverse.

The point 4. contains concrete investigations in case of one-dimensional cellular automaton, with the result: only very simple reversible cellular automata exist in this special case.

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1. Basic definitions

(i) *Inhomogeneous cellular automaton* is a \((C, A, N, \Phi)\) four-tuple, where
- \(C = \{c_1, \ldots, c_m\}\) is the finite set of cells,
- \(A = \{0, 1, \ldots, s-1\}\) is the set of cell-states,
- \(N: c_i \mapsto (c_{i_1}, \ldots, c_{i_n})\) is the *neighbourhood function*, which assigns to each cell its neighbours. (The specification of neighbours may be different cell by cell, i.e. the cellular automaton has totally arbitrary topology.)
- \(\Phi: c_i \mapsto f_i\) is the *function-system*, which assigns to each cell an \(f_i: A^n \to A\) local map. (The local maps also may be different cell by cell.)

(ii) *Configuration* is a map \(\alpha: C \to A\), we denote it always with Greek letters.

(iii) *Neighbourhood* of cell \(c_i\) in a given configuration is the \(n_i\)-tuple of states of its neighbours.

(iv) The *global map* of a cellular automaton is a map \(F: \mathcal{A} \to \mathcal{A}\) where \(\mathcal{A}\) is the set of all configurations, and \(F(\alpha) = \beta\) if for all \(i, f_i(a_{i_1}, \ldots, a_{i_n}) = \beta(c_i)\) (where \((a_{i_1}, \ldots, a_{i_n})\) is the neighbourhood of \(c_i\) in \(\alpha\)).

In further we use the abbreviation CA instead of cellular automaton.

2. Relation between the local maps and the number of eden-configurations

We consider a CA \((C, A, N, \Phi)\) with the global map \(F\).

The following definition is well-known from the literature:

*Definition.* A configuration \(\alpha\) will be called *garden-of-eden configuration* (in short *eden-configuration*), if there is no \(\beta\), for which \(F(\alpha) = \beta\).

We have an obvious equivalence:

\[ F \text{ is bijective } \iff \text{ there is no eden-configuration}. \]

Let be \(c\) a cell with \(n\) neighbours, and \(f\) its local map. Suppose, that there are \(p_a\) different neighbourhoods of \(c\), where the new cell-state given by \(f\) is \(a\). The number of all possible neighbourhoods is \(s^n\), consequently \(\sum_{a \in A} p_a = s^n\).

*Definition.* We say, that the local map \(f\) is balanced, if \(\forall a: p_a = p\), where obviously \(p = s^n/s = s^{n-1}\).

When \(f\) is unbalanced, the measure of this may be characterized with the quantity

\[ q = \sum_{a \in A \atop p_a < p} (p - p_a), \]
and we say: \(f\) is \(q\)-unbalanced.

*Theorem.* Let be \((C, A, N, \Phi)\) an arbitrary CA, \(c\) a cell in it, and \(f\) its local map. If \(f\) is \(q\)-unbalanced, then the CA has at least \(q \cdot s^{m-n}\) eden-configurations (\(m\) is the number of cells, \(s\) is the number of cell-sates).

*Proof.* It is clear, that there are \(s^{m-n}\) different configurations, where the neighbourhood of \(c\) is a given \((a_1, \ldots, a_s)\). So there are exactly \(p_a \cdot s^{m-n}\) configurations, where the new cell-state of \(c\) is \(a\). At the same time the number of all configurations, where the state of \(c\) is \(a\), is \(s^{m-1} = p \cdot s^{m-n}\). Consequently if \(p_a < p\), then among these \(p \cdot s^{m-n}\) configurations there are \((p - p_a) \cdot s^{m-n}\) eden-configurations.

We find the same situation by all state \(a\) having the property \(p_a < p\), consequently the CA has at least \(\sum_{p_a < p} (p - p_a) \cdot s^{m-n}\) eden-configurations. \(\square\)
Corollaries. (i) If in a CA for any \( i \) the local map of \( c_i \) is \( q_i \)-unbalanced, then the CA has at least \( \max_{1 \leq i \leq m} (q_i \cdot s^{m-n}) \) eden-configurations.

(ii) To the bijectivity of global map is necessary condition, that all local maps are balanced.

Similar results are published in works [5], [6] on classical infinite, homogeneous CA.

### 3. The problem of reversibility

**Algorithm for decision of reversibility, and construction of the reverse**

**Definition.** A CA \( (C, A, N, \Phi) \) with a global map \( F \) is reversible, if there exists another function-system \( \Phi' \) such, that the CA \( (C, A, N, \Phi') \) generates the global map \( F^{-1} \).

The first problem in this subject: to decide from a given CA, whether it is reversible. On this purpose we introduce a general algorithm, which is suitable for constructing the reverse, too.

Let be \( (C, A, N, \Phi) \) a CA, \( c_i \) a cell in it. Let's denote with \( N_1 \) the neighbours of \( c_i \), and with \( N_2 \) the neighbours of neighbours (with a bit incorrect notation \( N_1 = N(c_i), N_2 = N(N(c_i)) \)). It is clear, that the state of \( N_1 \) at time \( t+1 \) is determined by the state of \( N_2 \) at time \( t \). If we know the local functions in \( N_1 \), we may describe this transition with a table called in following as *inverse-constructing-table* (ICT in short). In case of one-dimensional, two-state CA it is illustrated on figure 1.

If the cell \( c_i \) has an \( f_i' \) reverse local function, then this function gives back from any \( N_1 \)-state of column \( t+1 \) of ICT the state of \( c_i \) in column \( t \). Consequently the existence of \( f_i' \) has the following necessary condition: if two \( N_1 \)-states in column \( t+1 \) of ICT are equal, then the corresponding \( c_i \)-states in column \( t \) also should be equal. Furthermore this condition is sufficient to the existence of \( f_i' \) reverse function, because we may construct it by the ICT.

![The construction of ICT in case of one-dimensional two-state CA.](image)
So the following is obtained:

**Proposition.** A CA \((C, A, N, \Phi)\) is reversible \iff for each cell \(c_i\), its \(1\text{CT}\) satisfies: if two \(N_1\)-states in column \(t+1\) agree, then the corresponding \(c_t\)-states in column \(t\) must agree too.

If this condition is satisfied, then we can construct the reverse function-system.

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### 4 The reversibility of one-dimensional two-state cellular automaton

The preceding algorithm decides only about a given \(\Phi\) whether it is reversible, but does not help to find concrete reversible function-systems. It is clear, that there exist trivial ones, for example the identical function-system (where each cell keeps its state, independently of neighbours), or the shift function-system, (where each cell receives the state of the same neighbour).

Nontrivial reversible function-systems have high importance in practice, but to construct them is very difficult. In further we give a necessary condition to the reversibility of one-dimensional two-state CA, from which we shall see, that in one-dimension only very special function-systems are reversible, consequently it is easy to construct them.

So in following the CA \((C, A_0, N_0, \Phi)\) will be investigated, where\(C=\{c_1, \ldots, c_m\}\), \(m \geq 5\) is supposed (this assumption makes easier the investigation),

\(A_0 = \{0, 1\}\),

\(N_0: c_i \rightarrow (c_{i-1}, c_i, c_{i+1})\), the indexes are interpreted cyclically (i.e. \(c_1\) and \(c_m\) are neighbours). Thus we have a circle-topology.

\(\Phi\) is arbitrary.

We need the following general definition:

**Definition.** In a CA \((C, A, N, \Phi)\) the cell \(c_i\) depends on its neighbour \(c_j\), if there are two neighbourhoods of \(c_i\) such, that they differ only in state of \(c_j\), and the corresponding new states of \(c_i\) are different.

Using this notion we take a remark to the definition of \((C, A_0, N_0, \Phi)\): if \(\Phi\) is such, that \(c_1\) and \(c_m\) are independent each of other, then the circle-topology we may replace with a section-topology. So our definition contains the section-topology too.

Two lemmas will be proved in further. In proofs we shall use often the fact, that for reversibility is necessary condition that all local maps are balanced. (It results from the second corollary in point 2.) Moreover we shall use the notation \(\overline{a}\), which denotes the opposite of cell-state \(a\).

**Lemma 1.** Suppose that \(\Phi\) is reversible, and its reverse is \(\Phi'\). In this case if \(c_{i-1}\) depends on \(c_{i-2}\) by the function-system \(\Phi\), then \(c_i\) is independent of \(c_{i-1}\) by \(\Phi'\).

**Proof.** Suppose, that \(c_{i-1}\) depends on \(c_{i-2}\), i.e. there are \(a, b\) such, that \(f_{i-2}(0, a, b) = x\), and \(f_{i-2}(1, a, b) = \overline{x}\).

Now let's consider the function \(f_{i+1}!\). We have two different cases:

(i) \(\exists y: \forall c, d: f_{i+1}(b, c, d) = y\).

The function \(f_{i+1}\) is balanced, therefore \(\forall c, d: f_{i+1}(b, c, d) = \overline{y}\), that is to say, \(c_{i+1}\) depends only on \(c_i\). Thus by the reverse \(c_i\) depends only on \(c_{i+1}\).

(ii) \(\exists c, d\) and \(\exists c', d': f_{i+1}(b, c, d) = y\) and \(f_{i+1}(b, c', d') = \overline{y}\).
Let $f_i(a, b, c) = p$, $f_i(a, b, c') = q$. So the ICT of cell $c_i$ contains the following part:

\[
\begin{array}{c|ccc|cc}
& a & b & c & d & x & y \\
\hline
0 & a & b & c & d & x & p & y \\
1 & a & b & c & d & \bar{x} & p & y \\
0 & a & b & c' & d' & x & q & \bar{y} \\
1 & a & b & c' & d' & \bar{x} & q & \bar{y} \\
\end{array}
\]

The four binary triples in column $t+1$ are different, and the reverse function $f'_i$ constructed by the table assigns to each triple the same state $b$. But $f'_i$ is balanced, so it assigns to the other four triple the state $\bar{b}$. By this the table of $f'_i$ is known. We can see from it, that $c_i$ is independent of $c_{i-1}$.

The second lemma needs the following definition:

**Definition.** Let $c_i, \ldots, c_j$ be a section of cells. We say, that it is isolated, if $c_i$ is independent of $c_{i-1}$, and $c_j$ of $c_{j+1}$.

**Lemma 2.** Suppose that $\Phi$ is reversible, and its reverse is $\Phi'$. In this case if the section $c_i, \ldots, c_j$ is isolated by $\Phi$, then it is isolated by $\Phi'$ too.

**Proof.** Two configurations will be called equivalent (with respect to the section $c_i, \ldots, c_j$), if their sections corresponding to the $c_i, \ldots, c_j$ are equal. So a classification is obtained on the set $\mathcal{A}$.

It is easy to prove the following chain: $c_i, \ldots, c_j$ is isolated by $\Phi$ if and only if $F(c_i, \ldots, c_j)$ is F-compatible (i.e. $\forall a, \beta: a \sim \beta \Rightarrow F(a) \sim F(\beta)$) if it is $F^{-1}$-compatible too (because $F$ is one-to-one) $\Rightarrow c_i, \ldots, c_j$ is isolated by $\Phi$. □

**Definition.** A function-system we call a shift function-system, if each cell depends only on its left (or only on its right) neighbour.

**Theorem.** If the CA $(C, A_0, N_0, \Phi)$ is reversible, then there exists one of the following two cases:

(i) Each cell stands in an isolated section containing maximum three cells.

(ii) $\Phi$ is a shift function-system.

**Proof.** (i) Suppose that there are $c_i$ and $c_j$ such, that $c_i$ is independent of its left neighbour, and $c_j$ is independent of its right neighbour. The cellular automaton has circle-topology, consequently the section $c_i, \ldots, c_j$ always exists. Furthermore this section is isolated, and — having applied the lemma 2. — it is isolated by $\Phi'$ too.

Now let's consider an arbitrary cell $c_k$. According to the lemma 1. either $c_{k-1}$ is independent of $c_{k-2}$, or by the reverse $c_{k}$ is independent of $c_{k-1}$. In the first case the section $c_{k-1}, \ldots, c_j$, in the second case the section $c_k, \ldots, c_j$ is isolated. Applying the geometrical inverse of lemma 1. we get: either $c_i, \ldots, c_{k+1}$ or $c_i, \ldots, c_{k}$ is isolated. The common part of two isolated sections is isolated too, so we have: $c_k$ stands in an isolated section containing maximum three cells.

(ii) Suppose the negation of the previous case, that is each cell depends (for example) on the left neighbour. We shall prove, that in this case each cell is independent of the right neighbour: suppose, that for any $k$ $c_{k+1}$ depends on $c_{k+2}$. At the same time $c_{k-1}$ depends on $c_{k-2}$, and from the lemma 1. we get, that $c_k$ is an isolated cell. This fact contradicts to the original assumption.

So each cell has only two real neighbours: the left cell and itself. We may classify
the balanced local maps for two neighbours in three types:

<table>
<thead>
<tr>
<th>I. 0 0</th>
<th>a</th>
<th>II. 0 0</th>
<th>a</th>
<th>III. 0 0</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1</td>
<td>a</td>
<td>0 1</td>
<td>b</td>
<td>0 1</td>
<td>b</td>
</tr>
<tr>
<td>1 0</td>
<td>b</td>
<td>1 0</td>
<td>a</td>
<td>1 0</td>
<td>b</td>
</tr>
<tr>
<td>1 0</td>
<td>b</td>
<td>1 0</td>
<td>b</td>
<td>1 1</td>
<td>a</td>
</tr>
</tbody>
</table>

In our case each cell depends on the left neighbour, so the type II. is out of the question. If all functions have the type III., then \( \forall a: F(a) = F(\bar{a}) \), thus the global map is not one-to-one.

If there are functions type I. and type III. at the same time, then there exists a cell \( C_i \) such, that \( f_i \) has the type I., and \( f_{i+1} \) has the type III. Therefore the ICT of \( C_i \) contains the following part:

\[
\begin{array}{cccc}
  t & t+1 \\
  a & b & c & d & e & x & y & z \\
  a & b & \bar{c} & d & e & x & y & z
\end{array}
\]

These two lines exclude the reversibility.

So we get: all local maps have the type I., i.e. \( \Phi \) is a shift function-system.

**Corollaries.**
1. If \((C, A_0, N_0, \Phi)\) has section topology, then each reversible \( \Phi \) has the type (i).
2. If \((C, A_0, N_0, \Phi)\) is homogeneous, then we have only the six trivial reversible function-systems: the identical one, and its contrary (where each cell alters its state independently of neighbours), the left and right shift function-systems, and their contrary.

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References


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