On Sperner families in which no 3 sets have an empty intersection

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1. Introduction

Let \( \mathcal{G}(r, k) \) denote the set of all Sperner families \( \mathcal{F} \) (i.e. \( X \subset Y \) for all different \( X, Y \in \mathcal{F} \) on \( R = [1, r] \) (the interval of the first \( r \) natural numbers with \( r \equiv 3 \)) satisfying \( \bigcup_{i=1}^{k} X_i \subset R \) for all \( X_i \in \mathcal{F} \) \( (i = 1, \ldots, k) \) where \( \subset \) is used in the strong sense. Furthermore we use the following notations:

\[
\mathcal{G}^1(r, k) = \{ \mathcal{F}: \mathcal{F} \in \mathcal{G}(r, k), \bigcup_{X \in \mathcal{F}} X = R \},
\]

\[
\mathcal{G}^0(r, k) = \{ \mathcal{F}: \mathcal{F} \in \mathcal{G}(r, k), \bigcup_{X \in \mathcal{F}} X \subset R \},
\]

\[
n(r, k) = \max_{\mathcal{F} \in \mathcal{G}^0} |\mathcal{F}|, \quad n^1(r, k) = \max_{\mathcal{F} \in \mathcal{G}^1} |\mathcal{F}| \quad \text{and} \quad n^0(r, k) = \max_{\mathcal{F} \in \mathcal{G}^0} |\mathcal{F}|.
\]

We notice that \( \mathcal{G}^1(r, k) = \emptyset \) holds for \( k \geq r \).

\( n(r, 2) \) was determined by E. C. Milner [6] (for the dual case) and later by A. Brace and D. E. Daykin [1], and \( n(r, k) \) with \( k \geq 4 \) was determined by the author [3].

For \( n(r, 3) \) the following two configurations are known:

\[
n(r, 3) = \left( \frac{r-1}{2} \right) + 1 \quad (1)
\]

and

\[
n(r, 3) = \left( \frac{r-1}{2} \right) \quad (2)
\]

P. Frankl [2] proved (1) for large enough even \( r \) (e.g. for \( r \geq 1000 \)) and (2) for large enough odd \( r \) (e.g. for \( r \geq 300 \)). The author [3] showed (1) for \( r = 7 \) and even...
For all odd \( r \) with the exception of the following 12 values: 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37 and 43.

In the present paper we prove
(1) for \( r=4, 6, 114 \) and even \( r \geq 120 \) and
(2) for \( r=11, 17, 23, 29, 35, 43 \).

We observe that exchanging all \( X \in \mathcal{F} \) by \( R \setminus X \) we get analogous results for Sperner families in which no 3 sets have an empty intersection.

We shall sharpen Theorem 5 of [3] in the case \( k=3 \). There we divided a maximal family \( \mathcal{F} \subseteq \mathcal{F}(r, 3) \) to two families \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), and showed

\[
|\mathcal{F}_0| = \left( \begin{bmatrix} r-1 \\ \frac{r-2}{2} \end{bmatrix} \right) \quad \text{and} \quad |\mathcal{F}_1| = \left( \begin{bmatrix} r-1 \\ \frac{r-1}{3} \end{bmatrix} - 1 \right).
\]

In fact \( |\mathcal{F}_1| \) depends on \( |\mathcal{F}_0| \). For \( k=3 \) and even \( r \), \( |\mathcal{F}_0| = \left( \begin{bmatrix} r-1 \\ \frac{r-2}{2} \end{bmatrix} \right) \) implies \( |\mathcal{F}_1| = 1 \).

In section 2 we shall present our main results and give a new type estimation of families of sets, which will be used in section 3 to prove a theorem analogous to Theorem 5 [3]. Finally, in section 4 we shall prove our main result.

2. Main results

Throughout this paper let \( a = \left\lfloor \frac{r-2}{2} \right\rfloor \) and \( b = \left\lfloor \frac{r-1}{3} \right\rfloor \).

Theorem 1. \( 1^o \) \( n(r, 3) = \left( \begin{bmatrix} r-1 \\ \frac{r-1}{2} \end{bmatrix} + 1 \right) \) for \( r=4, 6, 114 \) and even \( r \geq 120 \),

\( 2^o \) \( n(r, 3) = \left( \begin{bmatrix} r-1 \\ \frac{r-1}{2} \end{bmatrix} \right) \) for \( r=11, 17, 23, 29, 35, 43 \).

Let \( r \geq 4 \). Then \( n(r, 3), n^1(r, 3) \) and \( n^0(r, 3) \) exist and it holds \( n(r, 3) = \max (n^1(r, 3), n^0(r, 3)) \).

For \( \mathcal{F} \subseteq \mathcal{F}(r, 3) \) there is an element \( v \in R \) such that \( \mathcal{F} \) is a Sperner family on \( R \setminus \{v\} \), and it follows by Sperner's theorem [7]:

Lemma 1. \( n^0(r, 3) = \left( \begin{bmatrix} r-1 \\ \frac{r-1}{2} \end{bmatrix} \right) \).

We shall use the following lemma shown in more general form in [3] (Lemma 2).
Lemma 2. Let $F \in \mathcal{P}(r, 3)$ such that $|F| = n^1(r, 3)$ and $\max_{X \in F} |X|$ is minimal. Then $|X| \geq a$ holds for all $X \in F$.

Lemma 3. Let $s \geq \frac{r}{2}$ be an integer and let $F_s$ denote an arbitrary family of different $s$-element subsets of $R$. Finally, let $F^*_s$ denote the largest family of $(2s)$-element subsets of $R$ such that for every $X \in F^*_s$ there is at least one pair $(Y, Z)$ of subsets of $F_s$ satisfying $Y \cup Z = X$. Then

$$|F^*_s| \geq \binom{r-s}{s} \cdot |F_s| - \binom{r}{2s}.$$

Proof. Let us consider the following families:

$$\overline{F}_s = \{X : X \subseteq R, |X| = s, X \notin F_s\},$$

$$\overline{F}^*_s = \{X : X \subseteq R, |X| = 2s, X \notin F^*_s\}.$$

Then for any $X \in \overline{F}^*_s$ there is no pair $(Y, Z)$ of sets of $F_s$ with $Y \cup Z = X$. For every such $X \in \overline{F}^*_s$ there exist exactly $\frac{1}{2} \binom{2s}{s} = \binom{2s-1}{s}$ unordered pairs $(Y, Z)$ with $|Y| = |Z| = s$ and $Y \cup Z = X$. All these sets are mutually disjoint, i.e., at least $\binom{2s-1}{s}$ $s$-element subsets belong to $\overline{F}_s$ for every $X \in \overline{F}^*_s$.

On the other hand for every $s$-element set $Y$ of $R$ there exist exactly $\binom{r-s}{s}$ disjoint $s$-element sets $Z$. Hence

$$|\overline{F}^*_s| \binom{2s-1}{s-1} \geq |\overline{F}_s| \binom{r-s}{s}.$$

Using $|\overline{F}^*_s| = \binom{r}{2s} - |F^*_s|$ and $|\overline{F}_s| = \binom{r}{s} - |F_s|$ we obtain the inequality of Lemma 3. \(\square\)

3. An upper bound for $n^1(r, 3)$

Let $F \in \mathcal{P}(r, 3)$ such that $|F| = n^1(r, 3)$ and $\max_{X \in F} |X|$ is minimal. By Lemma 2, we have $|X| \geq a$ for all $X \in F$. The numbers $p_i = |\{X : X \in F, |X| = i\}|$ ($i = 0, \ldots, r$) are called parameters of the family $F$. $\mathcal{F}$ denotes the canonical Sperner family (see A. J. W. Hilton [4]).

Now we decompose $F$ to the subfamilies $D, E$ and $H$ defined as follows.

- $D$ is a subfamily of $F$ with $\mathcal{F}D = \{X : X \in \mathcal{F}, r \notin X\}$.
- $E = \{X : X \in F \setminus D, |X| \geq r - 2a - 1\}$.
- $H = \{X : X \in F \setminus D, |X| \geq r - 2a\}$.
1. It has been proved by A. J. W. Hilton [4] that all \( X \in \mathcal{F} \) with \( |X| > b \) belong to \( \mathcal{D} \). \( \mathcal{D} \) is a Sperner family on \( R \setminus \{r\} \). Using \( \binom{r-1}{|X|} = \binom{r-1}{a-1} \) for \( |X| \geq a - 1 < \frac{r-1}{2} \), by Lubell’s inequality [5] we obtain

\[
\sum_{X \in \mathcal{D}} \frac{1}{\binom{r-1}{|X|}} = \sum_{X \in \mathcal{D}} \frac{1}{\binom{r-1}{a}} + \sum_{X \in \mathcal{D}^*} \frac{1}{\binom{r-1}{|X|}} \leq 1,
\]

\[
\frac{p_a}{(r-1)} + \frac{(|\mathcal{D}| - p_a)}{(r-1)} \leq 1,
\]

and

\[
|\mathcal{D}| = |\mathcal{D}| = \frac{a}{r-a} \binom{r-1}{a} + \frac{r-2a}{r-a} p_a.
\]

2. \( \mathcal{J} = \{X: X \cup \{r\} \in \mathcal{J}(D \cup \mathcal{S}), r \notin X\} \) is a Sperner family of cardinality \(|\mathcal{S}|\) on \( R \setminus \{r\} \) and \(|X| \geq r - 2a - 2\) holds for all \( X \in \mathcal{J} \).

By Lubell’s inequality [5] we obtain

\[
\sum_{X \in \mathcal{J}} \frac{1}{\binom{r-1}{|X|}} = 1, \quad \frac{|\mathcal{J}|}{\binom{r-1}{r-2a-2}} \leq 1 \quad \text{and} \quad |\mathcal{S}| = |\mathcal{J}| \leq \binom{r-1}{r-2a-2}.
\]

3. Let \( \mathcal{F}^{**} = \{X: R \setminus X \in \mathcal{F}^{**}\} \). Then \( \mathcal{D} \cup \mathcal{H} \cup \mathcal{F}^{**} \) is a Sperner family. We notice that \(|X| \geq r - 2a\) holds for all \( X \in \mathcal{D} \cup \mathcal{H} \) and \(|X| = r - 2a\) holds for all \( X \in \mathcal{F}^{**} \).

Clearly, \( \mathcal{D} \cup \mathcal{H} \) and \( \mathcal{F}^{**} \) are Sperner families themselves. We have only to show that there is no pair \((Y, Z)\) with \( Y \in \mathcal{F}^{**} \) and \( Z \in \mathcal{D} \cup \mathcal{H} \) satisfying \( Y \subseteq Z \). Let us assume the contrary. Then there are two sets \( Y_1, Y_2 \in \mathcal{D} \) with \( Y_1 \cup Y_2 = R \setminus Y \).

The sets \( Y_1, Y_2, Z \in \mathcal{F} \) it follows \( Y_1 \cup Y_2 \cup Z = (R \setminus Y) \cup Z \supseteq (R \setminus Y) \cup Y = R \), which is impossible for \( \mathcal{F} \in \mathcal{G}(r, 3) \).

\( \mathcal{J}' = \{X: X \cup \{r\} \in \mathcal{J}(D \cup \mathcal{H} \cup \mathcal{F}^{**}), r \notin X\} \) is a Sperner family on \( R \setminus \{r\} \). If \( q_i, q_i', q_i'' \) are the parameters of the families \( \mathcal{J}' \), \( \mathcal{H} \) and \( \mathcal{F}^{**} \), respectively, then \( q_i = q_{i+1}' + q_{i+1}'' \) holds. By Lubell’s inequality [5], using \( \binom{r-1}{|X|} \leq \binom{r-1}{b} \) for \( |X| \leq b < \frac{r-1}{2} \), we get

\[
\sum_{X \in \mathcal{J}'} \frac{1}{\binom{r-1}{|X|}} = 1, \quad \sum_{X \in \mathcal{J}'} \frac{1}{\binom{r-1}{|X|-1}} + \sum_{X \in \mathcal{F}^{**} \cup \mathcal{H}} \frac{1}{\binom{r-1}{r-2a-1}} \leq 1
\]

and

\[
\frac{|\mathcal{J}'|}{\binom{r-1}{b-1}} + \frac{|\mathcal{F}^{**} \cup \mathcal{H}|}{\binom{r-1}{r-2a-1}} \leq 1.
\]

By Lemma 3 using \( |\mathcal{F}| = n^2(r, 3) \) and the estimations for \( \mathcal{D}, \mathcal{S} \) and \( \mathcal{H} \) we obtain

\[\text{as } \min_{X \in \mathcal{D} \cup \mathcal{H}} |X| = r - 2a - 1 \text{ would imply } \mathcal{H} = \emptyset \text{ and, together with 1. and 2., the estimation given in Theorem 2.}\]
Theorem 2.

\[ n^1(r, 3) \leq \max_{p_a} \left( \frac{a}{r-a} \left( \frac{r-1}{a} \right) + \frac{r-2a}{r-a} p_a + \left( \frac{r-1}{r-2a-2} \right) + \frac{2(r-a)}{r-2a} \left( \frac{1}{\binom{r-1}{a}} \right) \right) \]

4. Proof of Theorem 1

Clearly, \( n(r, 3) = \max \left\{ n^1(r, 3), \binom{r-1}{a} \right\} \) holds by Lemma 1.

1º. Let \( r \) be even. Then all \( a \)-element subsets of \( R \setminus \{ r \} \) and the set \( \{ r \} \) form a family \( \mathcal{F} \in \mathcal{G}(r, 3) \) having the cardinality \( \binom{r-1}{a} + 1 \). So we have only to show that the right side of the inequality of Theorem 2 has the value \( \binom{r-1}{a} + 1 \), too.

For \( r = 4 \) it is easy to see that \( n^1(4, 3) = 4 \) holds.

Now let \( r = 6, 114 \) or \( r \geq 120 \).

The function \( f(p_a) \), of which we consider the maximum in Theorem 2, is a linear function in \( p_a \). We have to take the maximum over the interval \( \left[ 0, \frac{r-1}{a} \right] \), as an immediate consequence of A. J. W. Hilton's result [4] which we used in the definition of \( \mathcal{D} \). We have \( f \left( \binom{r-1}{a} \right) = \binom{r-1}{a} + 1 \). We have only to show that the factor of \( p_a \) in \( f(p_a) \) is positive (or equal to 0), i.e., using \( r - 2a = 2 \),

\[ \frac{2}{r-a} - (r-a) \frac{\binom{r-1}{b-1}}{\binom{r-1}{a}} > 0. \]  

(3) is equivalent to

\[ M(r) = \frac{2(r-b)(r-b-1)...(r-a+1)}{(r-a)a(a-1)...b} \geq 1. \]  

(4)

Furthermore,

\[ \frac{M(6t+10)}{M(6t+4)} = \frac{2^{10} \binom{t+7}{4} t^{3/2} t^{5/4} \binom{t+1}{2} \binom{t+7}{2}^{1/2}}{3^6 t^{t+2} \binom{t+5}{3} t^{\frac{5}{3}} \binom{t+4}{3}^{1/2}} = g(t) \]

is monotonically increasing, because \( \frac{t+x}{t+y} \) is monotonically increasing for fixed \( x \) and \( y \) with \( x < y \).
For \( t \equiv 20 \) we obtain \( g(t) \equiv g(20) = \frac{127766373}{99866624} > 1 \). By induction it follows that \( M(6t+4) > 1 \) for \( t \equiv 20 \).

Moreover we have

\[
\frac{M(6t+2)}{M(6t+4)} = \frac{9}{8} \frac{t+1}{t+\frac{3}{4}} \frac{t+\frac{3}{4}}{t+\frac{2}{3}} > \frac{9}{8} > 1
\]

and

\[
\frac{M(6t)}{M(6t+4)} = \frac{3^4}{2^8} \frac{t+1}{t+\frac{3}{4}} \frac{t+\frac{3}{4}}{t+\frac{1}{2}} \frac{t+\frac{1}{2}}{t-\frac{1}{2}} > \frac{81}{64} > 1,
\]

which proves \( M(2t) > 1 \) for \( t \equiv 60 \).

Finally we complete our proof by \( \frac{M(114)}{M(124)} = \frac{59025914157}{53793208352} > 1 \).

2°. In [3] the author proved the following estimation for \( |\mathcal{S} \cup \mathcal{H}|: |\mathcal{S} \cup \mathcal{H}| \equiv \frac{(r-1)}{(b-1)} \). Using our estimation for \( |\mathcal{S}| \) we obtain \( |\mathcal{S}| \equiv \left(\frac{r-1}{a}\right) \frac{a}{r-a} + \frac{r-2a}{r-a} p_a + \frac{(r-1)}{(a+1)} \). Both, this estimation and the bound given in Theorem 2 are valid for each \( |\mathcal{S}| \). It suffices to show that for every \( p_a \) one of our upper bounds is less than \( \left(\frac{r-1}{a+1}\right) \), because in this case \( r \) is odd, i.e. \( \left[\frac{r-1}{2}\right] = a+1 \). We distinguish the following cases.

1. \( p_a < \frac{a+3}{3} \left(\frac{r-1}{a+1}\right) - \left(\frac{r-1}{a-1}\right) - \left(\frac{r-1}{b-1}\right) \). Then \( |\mathcal{S}| < \left(\frac{r-1}{a+1}\right) \) follows from our last estimation.

2. \( p_a \equiv \frac{a+3}{3} \left(\frac{r-1}{a+1}\right) - \left(\frac{r-1}{a-1}\right) - \left(\frac{r-1}{b-1}\right) \equiv \frac{2}{3} \frac{a+3}{a+1} \left(\frac{r-1}{a}\right) - \frac{a+3}{3} \left(\frac{r-1}{b-1}\right) \).

Then we use the estimation of Theorem 2. First we prove that the factor of \( p_a \) in \( f(p_a) \) is negative, i.e.

\[
\frac{r-2a}{r-a} - \frac{2(r-a)}{r-2a} \left(\frac{r-1}{a}\right) < 0. \tag{5}
\]

(5) is equivalent to

\[
N(r) = \frac{9(r-b)(r-b-1)\ldots(a+4)}{2(a+3)a(a-1)\ldots b} < 1.
\]

We have that

\[
\frac{N(6t+5)}{N(6t-1)} = \frac{2^{10}}{3^6} \frac{t+\frac{3}{4}}{t+\frac{4}{3}} \frac{t+\frac{1}{2}}{t+\frac{4}{3}} \frac{t+\frac{1}{4}}{t+\frac{2}{3}} \frac{t-\frac{1}{2}}{t-\frac{1}{3}} = g'(t)
\]
is monotonically increasing by our remark above.

For \(2 \leq t \leq 5\) we obtain \(g'(t) \leq g'(5) = \frac{6072}{6137} < 1\). From \(N(11) = \frac{3}{7} = 0.428571\), \(N(6t-1) < 1\) follows by induction for \(2 \leq t \leq 6\). Finally, we get \(N(43) = \frac{10179}{59432} < 1\). \(f(p_s)\) takes the maximum in the described interval at \(p_s = \frac{a+3}{3} \left(\frac{r-1}{a+1} - \frac{r-1}{a} - \frac{r-1}{b-1}\right)\), consequently. We will complete our proof by showing the following inequality.

\[
\frac{3}{2} \frac{2a+3}{a+1} \frac{(r-1)}{a} - \frac{a+3}{3} \left(\frac{r-1}{b-1}\right) - \frac{3}{a+3} \left(\frac{r-1}{a} - \frac{b-1}{a}\right) + \frac{a}{a+3} \left(\frac{r-1}{a}\right) + (r-1) + \frac{2}{3} (a+3) \left(\frac{r-1}{b-1}\right) < \left(\frac{r-1}{a+1}\right).
\]

This inequality is equivalent to

\[
w(r) = \left(\frac{r-1}{b-1}\right) \left[1 + \frac{2(a+3)^2}{9(a+1)} \left(1 - \frac{b-1}{a+1} \left(\frac{r-1}{a}\right)\right)\right] - (r-1) > 0.
\]

\(w(11) = 112 > 0\).

Furthermore we prove the inequality \(w'(r) = \frac{(a+3)(a+1)}{2a+13} \left(\frac{r-1}{b-1}\right) \leq \frac{1}{2}\) for \(r = \{17, 23, 29, 35, 43\}\) by referring to the following table:

<table>
<thead>
<tr>
<th>(r)</th>
<th>17</th>
<th>23</th>
<th>29</th>
<th>35</th>
<th>43</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w'(r))</td>
<td>140</td>
<td>154</td>
<td>442</td>
<td>9044</td>
<td></td>
</tr>
<tr>
<td></td>
<td>297</td>
<td>323</td>
<td>1035</td>
<td>19981</td>
<td></td>
</tr>
</tbody>
</table>

Using this estimation of \(w'(r)\) we get first

\[
w(r) = \left(\frac{r-1}{b-1}\right) \left[1 + \frac{2(a+3)^2}{9(a+1)} \left(1 - \frac{2a+13}{2(a+3)}\right)\right] - (r-1)
\]

\[
= \left(\frac{r-1}{b-1}\right) \frac{2(a-6)}{9(a+1)} - (r-1),
\]
then \( \psi(17) \geq \frac{311}{9} > 0 \). \( r \geq 17 \) implies \( \frac{a - 6}{a + 1} \geq \frac{1}{8} \) and for \( 2 \leq i \leq b - 1 \) we have \( \frac{r - b - 1 + i}{i} > 3 \). Hence for \( r \in \{23, 29, 35, 43\} \):

\[
\psi(r) \equiv (r - 1) \frac{2(a - 6)}{9(a + 1)} \prod_{i=2}^{b-1} \frac{r - b - 1 + i}{i} -(r - 1)
\]

\[
\equiv (r - 1) \frac{2}{9} \frac{1}{8} 3^{b-2} -(r - 1)
\]

\[
\equiv (r - 1) \frac{1}{4} 3^{3} -(r - 1)
\]

> 0 follows.

5. Concluding remark

The author conjectures that (1) holds for the remaining even \( r \) and (2) holds for the remaining odd \( r \), i.e. 13, 19, 25, 31 and 37.

References


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