A note on the interconnection structure of cellular networks

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Using the concept of the structure automaton it is proved that every cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.

1. Introduction

The interconnection structures of infinite cellular automata, called also tessellation automata [3], are usually taken to be networks based on direct sum of infinite cyclic groups. Such networks have a great degree of uniformity [4]. Realizations of finite and infinite cellular automata by various types of uniform networks were described in [2]. It was shown there that such realizations may be described by the use of the theory of groups. The structure of cellular automata realized by nonuniform cellular networks has not been investigated because of the lack of the proper description method for such networks. In this paper a step in this direction is presented using the concept of the structure automaton. It is proved that every cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.

2. Preliminaries

Definition. A cellular network is a system \( \mathcal{N} = (C, S, \delta, f) \) where \( C \) — is a countable set of cells, \( S \) — is a finite set of cell-states, \( \delta : S^k \rightarrow S \) — is a cell transition function, \( f : C \rightarrow C^k \) — is the neighbourhood function.

Definition. The cellular automaton (CA) realized by a cellular network \( \mathcal{N} \) is a pair \( \mathcal{A} = (\mathcal{N}, F) \) where \( S^C = \{ h : C \rightarrow S \} \) — is the set of CA configurations \( F : S^C \rightarrow S^C \) is the global map defined by \( F(h(c)) = \delta \cdot h^k \cdot f(c) \) where \( h^k(c_1, c_2, \ldots, c_k) = (h(c_1), h(c_2), \ldots, h(c_k)) \).

Let \( \mathcal{N}_1 = (C_1, S_1, \delta_1, f_1) \) and \( \mathcal{N}_2 = (C_2, S_2, \delta_2, f_2) \) be two cellular networks with \( f_1 : C_1 \rightarrow C_1^k \) and \( f_2 : C_2 \rightarrow C_2^k \). A network \( \mathcal{N}_1 \) is a realization of the network \( \mathcal{N}_2 \) when there exists a pair of functions \( (\varphi, \psi) \), \( \varphi : C_1 \rightarrow C_2 \), \( \varphi(C_1) = C_2 \), \( \psi : S_1 \rightarrow S_2 \) such that \( \forall \varphi^k(f_1(c_1)) = f_2(\varphi(c_1)) \) and \( \forall \psi \cdot \delta_1(s_1, s_2, \ldots, s_k) = \delta_2(\psi(s_1), \psi(s_2), \ldots, \psi(s_k)) \). These equalities mean that \( (\varphi, \psi) \) is a homomorphism of \( \mathcal{N}_1 \) onto \( \mathcal{N}_2 \).

In [2] the following theorem was proved.
Theorem 1. If a cellular network $\mathcal{N}_1$ realizes the cellular network $\mathcal{N}_2$ then the cellular automaton $\mathcal{A}(\mathcal{N}_1)$ simulates the cellular automaton $\mathcal{A}(\mathcal{N}_2)$ i.e. a function $H$ from $S_1^c$ onto $S_2^c$ exists such that

$$\forall s^c \in S_1^c \quad H(F_1(s^c)) = F_2(H(s^c)).$$

Simulations of CA realized by various types of cellular networks having a high degree of uniformity were described in [2]. It was shown there that the simulation of CA realized by such networks is essentially a problem of the group homomorphism and, in some cases, a problem of permutation groups generators.

No attempts were reported on the simulation of CA realized by nonuniform networks. Particularly interesting question is whether there is an algebraic structure for the description similarly as the theory of groups in the case of uniform networks. The answer for this question is given here. It states that simulations of all CA may be described by the use of the theory of semigroups.

3. Results

Let $\mathcal{N}=(C, S, \delta, f)$ be a cellular network defined as above. For notational convenience we label cell inputs 1, 2, ..., $k$ of the cells in $\mathcal{N}$ by $x_1, x_2, ..., x_k$.

Definition. A structure automaton of the cellular networks is a triple $\mathcal{N}_A = \langle X, C, \omega \rangle$ where $X$ is the input alphabet of cell input labels, $C$ is a countable set of cells of $\mathcal{N}$, $\omega : X \times C \rightarrow C$ is a transition function defined by $\forall \forall \omega(x_i, c_k) = c_i \iff$ the $i$-th component of the neighbourhood function value $f(c_i)$ is equal to $c_k$.

It is easy to see that every cellular network may be described by some structure automaton. Classical results [1] obtained in the theory of automata may be now applied to the description of cellular networks. For example we can generalize the classification of networks as follows: (For the notions below [1] may be consulted).

1. **Connected networks** described by connected structure automata.
2. **Strongly connected networks** described by strongly connected structure automata.
5. **Arrays** [2] described by perfect automata.

The enumeration above is done according to the generality of specific class of networks. The first two classes are important from the point of information flow in CA. In the cellular network described by the connected structure automaton there are some parts from (or to) which information flows in only one direction, and there are no such parts in the cellular network described by the strongly connected structure automaton.

Definition. A cellular network $\mathcal{N}$ is of semigroup-type if there is one-to-one correspondence $\alpha$ between the set of cells $C$ and certain semigroup $J$ with operation
* such that for some subset \( L = \{l_1, l_2, \ldots, l_k\} \subset J \),
\[
\forall \alpha \in \mathcal{C} \quad \forall \alpha^k(f(c)) = (l_1 \ast \alpha(c), l_2 \ast \alpha(c), \ldots, l_k \ast \alpha(c)).
\]

**Theorem 2.** If \( \mathcal{N} \) is a cellular network described by the strongly connected structure automaton \( \mathcal{N}_A \) then there exist a cellular network \( \mathcal{M} \) of semigroup-type such that the CA \( \mathcal{A}^{\ast} \) simulates the CA \( \mathcal{A}(\mathcal{N}) \).

**Proof.** By Theorem 1, it is sufficient to consider cellular network realizations. Let \( \mathcal{N}_A = (X, C, \omega) \). With every input symbol \( x_i \) we can associate a transformation \( \omega(x_i, c) \) for all \( c \in C \). Let \( J \) be the transformation semigroup generated by all \( \omega(x_i) \) for \( x_i \in X \).

We define the following structure automaton \( \mathcal{M}_J = (\Omega, J, m) \), where \( \Omega = \{\omega(x_i) \mid x_i \in X\} \) — input alphabet, \( m(\omega(x_i), j) = \omega(x_i) \cdot j \) — transition function defined as a composition of mappings in \( J \).

Let \( H: J \to C \) be a function defined by \( \forall j \in J \quad H(j) = j(c_0) \) for some fixed \( c_0 \in C \). \( H \) is onto \( C \) because the semigroup \( J \) is transitive. We shall prove that \( H \) is a homomorphism of the structure automaton \( \mathcal{M}_J(\Omega, J, m) \) onto the structure automaton \( \mathcal{N}_A = (X, C, \omega) \). We have

\[
\forall \omega x_i \in \Omega \quad H(m(\omega x_i, j)) = H(\omega x_i \cdot j) = \omega x_i \cdot j(c_0) = \omega(x_i, H(j)).
\]

From Theorem 1 it follows that the CA realized by the semigroup-type network \( \mathcal{M} \) described by the structure automaton \( \mathcal{M}_J \) simulates the CA realized by the network \( \mathcal{N} \). □

Now, we will extend Theorem 2 to cellular networks described by connected structure automata. In this case the transformation semigroup \( J \) is not transitive.

An extension of the semigroup \( J \) will be defined in two steps. First, when there is no identity, we add an identity \( e \) to the semigroup \( J \) obtaining the semigroup \( J \cup e = J_e \). Let \( C_g \subset C \) be the set (possibly with the smallest cardinality) such that \( J_e(C_g) = C \).

In the second step, let the elements of \( C_g \) be numbered \( c_1, c_2, \ldots, c_i, \ldots \) For each element \( c_i \in C_g \) a set of vectors is constructed

\[
[J_e, I_i] = \{(0, 0, \ldots, 0, j_e, 0, \ldots)\}
\]

where \( j_e \) is on the \( i \)-th position and 0 is an element such that \( 0 \cdot 0 = 0 \cdot j_e = j_e \cdot 0 = 0 \).

Let \( [J_e] = \bigcup_i [J_{e,i}] \). It is easy to see that the set \( [J_e] \) together with componentwise multiplication forms a semigroup.

Let \( H_e \) be a function \( H_e: J_e \to C \) such that for each \( c_i \in C_g \), \( H_e([J_{e,i}]) \) is defined as

\[
H|[0, 0, \ldots, 0, j_e, 0, \ldots]) = j(c_i).
\]

\( H \) is onto \( C \) because of the definition of the set \( C_g \) and vector semigroup \( [J_e] \).

From these constructions we finally have

**Theorem 3.** Any cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.
4. Conclusion

It was proved that semigroupe-type cellular networks are universal in the sense that any cellular automaton may be simulated by a network of such type. This result may compared with the simulation power of the group-type and Abelian group-type networks [2]. Note, that in the case of connected networks with infinite number of cells the semigroup for simulation may be not finitely generated, which gives a new level of complexity in the theory of cellular automata. Further investigation is needed in two directions. First, on the computational capability of the cellular automata realized by semigroup-type networks comparing to tesselation automata. Second, in the finite case, on the algebraic characterization of the structure of finite cellular automata using finite structure automata.

References


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