On cofinal and definite automata

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1. Introduction

Cofinal or directable automata were introduced in [1] and further investigated in [2, 7, 8, 9]. Cofinal automata are automata whose states can be directed to a single state by a suitable input word. We will call a cofinal automaton definite if there is an integer n such that all input words of length greater than or equal n direct the state set to a single state. Perles et al. [10] investigated definite events and definite automata. In particular they used shift registers, a special type of definite automata, in their discussion of the synthesis problem. Moreover, Stoklosa [12, 13] investigated these automata from an algebraic point of view. In section 2 of this paper we will prove a graph theoretic property of shift registers, namely that the transition diagram of a shift register contains a hamiltonian circle. In section 3 we apply this result in order to investigate the determination whether an arbitrary automaton is cofinal or not. In section 4 we determine the structure of all strongly definite automata with the aid of shift registers. Finally, in section 5, we characterize the general structure of definite automata. Let us give precise definitions first.

Definition 1.1. An *automaton* (more exactly, an *X-automaton*) A, denoted by $A=(S, X)$, consists of the following data: (i) S is a nonempty finite *set of states*. (ii) X is a nonempty finite *set of inputs*. (iii) There exists a function M_A of $S \times X^*$ into S, called a *state transition function*, such that $M_A(s, pq)=M_A(M_A(s, p), q)$ and $M_A(s, e)=s$ for all $s \in S$ and all $p, q \in X^*$, where X^* is the free monoid over X and e is its identity.

Note that in the following sp^A will often be used to denote $M_A(s, p)$.

Definition 1.2. An automaton $A=(S, X)$ is said to be *cofinal* (or *directable* in [1, 2]) if there exists $p \in X^*$ such that $Sp^A=\{sp^A | s \in S\}$ is a singleton.

Definition 1.3. An automaton $A=(S, X)$ is called a *definite* automaton if there exists an integer $n \geqq 0$ such that $|Sp^A|=1$ holds for all $p \in X^*$ with $|p| \geqq n$. If A is a definite automaton, then the least integer n such that the above condition holds is called the *degree of A* and denoted by $d(A)$.

A definite automaton is cofinal. The class of definite automata A with $d(A)=0$ is exactly the class of all one-state automata. Furthermore, if $d(A)=n \geqq 1$ for a definite automaton A, then there exists a $q \in X^*$ with $|q|=n-1$ and $|Sq^A|>1$. A definite automaton $A=(S, X)$ is called a *strongly definite* automaton if it is *strongly connected*, i.e., if for all $s, s' \in S$ there exists $p \in X^*$ such that $sp^A=s'$ holds. If $|X|=1$ for a strongly definite automaton, then $|S|=1$ holds too.

Definition 1.4. Let n be a nonnegative integer, X a finite set and X^n the set of all words over X of length n. Then the automaton $A(n)=(X^n, X)$ whose state transition function is defined by $(yp)x^{A(n)}=px$ for all $(y, p, x) \in X \times X^{n-1} \times X$ if $n \geqq 1$ and $ex^{A(n)}=e$ for all $x \in X$ if $n=0$ is called an *n-stage shift register without feedback* (or briefly an nSR).

Obviously, n-stage shift registers are strongly definite automata.

2. A graph theoretic property of nSR's

The purpose of this section is to prove the following theorem. (For the notion of a hamiltonian circle in a directed graph see [5].)

Theorem 2.1. There exists a hamiltonian circle in the state transition diagram of an nSR.

Note that the state transition diagram of an nSR is the directed graph whose vertices are states and where there is a directed edge from p to q, labelled by x, iff $px^{A(n)}=q$ for $(p, x, q) \in X^n \times X \times X^n$. If $n=0$, the theorem holds trivially. Therefore we assume $n \geqq 1$ for the rest of this section. Before proving the theorem, we need the following definition.

Definition 2.1. Let $r \geqq 1$. A sequence $p_1, p_2, \ldots, p_r$ of distinct elements of X^n with $p_i x_i^{A(n)}=p_{i+1}$ with $x_i \in X$ for $1 \leqq i \leqq r-1$ is called a *chain of length* $r-1$ and denoted by $p_1 \xrightarrow[x_1]{} p_2 \xrightarrow[x_2]{} p_3 \xrightarrow[x_3]{} \cdots \xrightarrow[x_{r-2}]{} p_{r-1} \xrightarrow[x_{r-1}]{} p_r$ (or briefly $p_1 \to p_2 \to p_3 \to \ldots \to p_{r-1} \to p_r$).

Now we first provide some lemmata.

Lemma 2.1. Let $p_1 \to p_2 \to \ldots \to p_r$ with $p_i=y_i q_i$, $(y_i, q_i) \in X \times X^{n-1}$, for $1 \leqq i \leqq r$, be a chain of length $r-1$. Then there exists a $p \in X^n$ such that $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ iff there exists an $x \in X$ such that $q_r x \notin \{p_1, p_2, \ldots, p_{r-1}, p_r\}$.

The proof is easy and thus omitted.

Lemma 2.2. Let $p_1 \to p_2 \to \ldots \to p_r$. If there is no $p \in X^n$ such that $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ holds, then there exists some $x \in X$ such that $p_r \xrightarrow[x]{} p_1$, i.e., we have a circle $\langle p_1, p_2, \ldots, p_{r-1}, p_r \rangle$ in the state transition diagram of $A(n)$.

Proof. Let $p_i=y_i q_i$ with $(y_i, q_i) \in X \times X^{n-1}$ for $1 \leqq i \leqq r$ and $p_i \xrightarrow[x_i]{} p_{i+1}$ for $1 \leqq i \leqq r-1$. By Lemma 2.1, we have $q_r x \in \{p_1, p_2, \ldots, p_r\}$ for all $x \in X$. This means that $q_r X=\{q_r x | x \in X\} \subseteqq \{p_1, p_2, \ldots, p_r\}$. Let $Xq_r=\{xq_r | x \in X\}$. It is obvious that $|q_r X|=|Xq_r|=|X|$ holds. Now assume $p_1=y_1 q_1 \notin q_r X$. This implies $q_r X \subseteqq \{p_2, \ldots, p_r\}$. Furthermore we have $p_i=y_i q_i \in q_r X$ iff $p_{i-1}=y_{i-1} q_{i-1} \in Xq_r$ for all i with $2 \leqq i \leqq r$. Therefore the set $\{p_1, p_2, \ldots, p_{r-1}\}$ contains $|q_r X|$ elements of Xq_r. Together with $p_r \in Xq_r$ we obtain $|Xq_r| \geqq |q_r X|+1$ in contradiction to the fact $|Xq_r|=|q_r X|$. Hence $p_1 \in q_r X$. Since $p_r = y_r q_r$, there exists some $x \in X$ such that $p_r \xrightarrow[x]{} p_1$.

Lemma 2.3. Let $p_1 \to p_2 \to \ldots \to p_r$ and $\{p_1, p_2, \ldots, p_r\} \neq X^n$. Then there exists a $p_1' \to p_2' \to \ldots \to p_r' \to p_{r+1}'$ such that $\{p_1, p_2, \ldots, p_r\} \subsetneqq \{p_1', p_2', \ldots, p_r', p_{r+1}'\}$.

Proof. If we have $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ for some $p \in X^n$, there is nothing to do. Now, assume that there does not exist such $p \in X^n$. By Lemma 2.2, we have a circle $\langle p_1, p_2, \ldots, p_r \rangle$ in the state transition diagram of $A(n)$. Let $p \in X^n - \{p_1, p_2, \ldots, p_r\}$. Then it is easy to see that $pp_1^{A(n)} = p_1$. From this it can easily be shown that there exist some $p', p'' \in X^*, x \in X$ and i with $1 \leqq i \leqq r$ such that $p_1 = p'xp''$, $pp'^{A(n)} \in X^n - \{p_1, p_2, \ldots, p_r\}$ and $(pp'^{A(n)})x^{A(n)} = p_i$. It is obvious that in this case we have $pp'^{A(n)} \to p_i \to p_{i+1} \to \ldots \to p_1 \to p_2 \to \ldots \to p_{i-2} \to p_{i-1}$ and $\{p_1, p_2, \ldots, p_r\} \subsetneqq \{pp'^{A(n)}, p_i, p_{i+1}, \ldots, p_1, p_2, \ldots, p_{i-1}\}$. This completes the proof of the lemma.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $p_1 \to p_2 \to \ldots \to p_r$ be one of the longest chains in the transition diagram of $A(n)$. Then, by Lemma 2.3, we have $X^n = \{p_1, p_2, \ldots, p_r\}$. Moreover, by Lemma 2.2, $\{p_1, p_2, \ldots, p_r\}$ forms a circle $\langle p_1, p_2, \ldots, p_{r-1}, p_r \rangle$, i.e., the state transition diagram of $A(n)$ has a hamiltonian circle.

Remark 2.1. Note that the previous results provide an algorithm for obtaining a hamiltonian circle in the state transition diagram of an nSR.

3. Cofinal automata and cofinal congruences

We will now apply the foregoing theorem to investigate the determination whether an arbitrary automaton is cofinal or not and to give a characterization of the minimal cofinal congruence of an arbitrary automaton. In this section all automata are assumed to be automata over a fixed alphabet X. Let us first give

Definition 3.1. Let n be a positive integer and $\mathscr{A}(n) = \{A = (S, X) \,|\, |S| = n$ and A is cofinal$\}$. Then by $\delta(n)$ we denote the value $\max\limits_{A \in \mathscr{A}(n)} \min \{|p| \,|\, p \in X^*$ and $|Sp^A| = 1\}$.

In [1, 11], $\delta(n)$ is investigated. Černy et al. [1] conjectured that $\delta(n) = (n-1)^2$. However at present only $(n-1)^2 \leqq \delta(n) \leqq 0(n^3)$ is known. The following result is obvious.

Proposition 3.1. Let $A = (S, X)$ be an automaton such that $|S| = n$. Then A is cofinal iff there exists a $p \in X^{\delta(n)}$ such that $|Sp^A| = 1$. ($X^{\delta(n)}$ is the set of all words over X with length $\delta(n)$.)

To test whether or not an automaton $A = (S, X)$ with n states is cofinal, we have to check whether or not Sp^A is a singleton for each $p \in X^{\delta(n)}$. Another more economical method would be to merge all $p \in X^{\delta(n)}$ in a single word w and to check the property "cofinal" with this word w. We first introduce some notions. Let $u, w \in X^*$. u is called a *subword* of w iff $w = u'uu''$ for some $u', u'' \in X^*$. Now let $w \in X^*$ such that every $u \in X^{\delta(n)}$ is a subword of w. Then w is called a *merged word of* $X^{\delta(n)}$ [3]. Obviously we have:

Proposition 3.2. Let $A=(S, X)$ be an automaton with $|S|=n$. Then A is cofinal iff $|Sw^A|=1$, where w is a merged word of $X^{\delta(n)}$.

It can easily be seen that the length of a merged word of $X^{\delta(n)}$ is greater than or equal to $|X|^{\delta(n)}+\delta(n)-1$. Moreover, with the aid of Theorem 2.1, we can show:

Lemma 3.1. There exists a merged word w of $X^{\delta(n)}$ such that $|w|=|X|^{\delta(n)}+$ $+\delta(n)-1$.

Proof. By Theorem 2.1, the state transition diagram of $A(\delta(n))=(X^{\delta(n)}, X)$ has a hamiltonian circle $\langle p_1, p_2, \ldots, p_r\rangle$ with $r=|X|^{\delta(n)}$. Let $p_1 \xrightarrow[x_1]{} p_2 \xrightarrow[x_2]{} p_3 \xrightarrow[x_3]{} \ldots$ $\ldots \xrightarrow[x_{r-2}]{} p_{r-1} \xrightarrow[x_{r-1}]{} p_r$ and put $w=p_1x_1x_2\ldots x_{r-2}x_{r-1}$. This proves the lemma.

Now we can state the following:

Theorem 3.1. There exists a $w\in X^*$ satisfying the following conditions:

(i) $|w|=|X|^{\delta(n)}+\delta(n)-1$.
(ii) For each automaton $A=(S, X)$ with $|S|=n$, A is cofinal iff $|Sw^A|=1$.

Remark 3.1. In [3], Dömösi discussed a general method to obtain the shortest merged word w of L, where L is a finite subset of X^*.

We will now use Lemma 3.1 to characterize the minimal cofinal congruence of an arbitrary automaton. To this end, we first recall the following notions. Let $A=(S, X)$ be an automaton. An equivalence relation ϱ on S is called *congruence on* A if $(s, s')\in\varrho$ implies $(sx^A, s'x^A)\in\varrho$ for all $s, s'\in S$ and $x\in X$. Let ϱ, ϱ' be congruences on A. Then $\varrho\wedge\varrho'$ and $\varrho\vee\varrho'$, the *product* and *sum* of ϱ and ϱ', are defined as usual (see e.g. [6]). $R(A)$, the set of all congruences on A, forms a lattice w.r.t. $\wedge$ and $\vee$. We now define:

Definition 3.2. Let $A=(S, X)$ be an automaton. A congruence ϱ on A is said to be *cofinal* if for all $s, s'\in S$ there exists a $p\in X^*$ such that $(sp^A, s'p^A)\in\varrho$ holds.

Let π_ϱ denote the partition of S induced by ϱ and $\pi_\varrho(s)$ the block of π_ϱ containing $s\in S$. We have:

Lemma 3.2. Let $A=(S, X)$ be an automaton and ϱ a congruence on A. Then ϱ is cofinal iff there exist a $p\in X^*$ and an $s_0\in S$ with $Sp^A\subseteq\pi_\varrho(s_0)$.

Proof. The "if part" is obvious. Conversely, let ϱ be cofinal and T a maximal subset of S such that there exist a $p\in X^*$ and an $s_0\in S$ with $Tp^A\subseteq\pi_\varrho(s_0)$. Assume $T\neq S$ and let $s\in S-T$. Then we have $(sp^A, s_0)\notin\varrho$. Since ϱ is cofinal, there exists a $p'\in X^*$ such that $(sp^Ap'^A, s_0p'^A)\in\varrho$. Since ϱ is a congruence, we have $(T\cup\{s\})(pp')^A\subseteq\pi_\varrho(s_0p'^A)$. This contradicts the minimality of T, hence $S=T$.

By $R_{cf}(A)$ we denote the set of all cofinal congruences on A. Let $\varrho, \varrho'\in R_{cf}(A)$. By Lemma 3.2, there exist $p, p'\in X^*$ such that $(sp^A, s'p^A)\in\varrho$ and $(sp'^A, s'p'^A)\in\varrho'$ for all $s, s'\in S$. This implies $(s(pp')^A, s'(pp')^A)\in\varrho\wedge\varrho'$ for all $s, s'\in S$. Therefore, $\varrho\wedge\varrho'\in R_{cf}(A)$. $\varrho\vee\varrho'\in R_{cf}(A)$ can be shown in a similar way. Thus $R_{cf}(A)$ forms a sublattice of $R(A)$. We now give

Definition 3.3. Let $A=(S, X)$ be an automaton. The minimal element of $R_{cf}(A)$, denoted by ϱ_{cf}, is called the *minimal cofinal congruence on* A.

Now we will characterize ϱ_{cf}.

Theorem 3.2. Let $A=(S, X)$ be an automaton with $|S|=n$ and ϱ a congruence on A. Let w be a merged word of $X^{\delta(n)}$. Then $\varrho=\varrho_{cf}$ iff ϱ is the minimal congruence on A such that $Sw^A\subseteqq\pi_\varrho(s_0)$ for some $s_0\in S$.

Proof. The assertion follows from Proposition 3.2 and the fact that ϱ is cofinal iff the quotient automaton A/ϱ is cofinal.

Remark 3.2. We can develop further properties of cofinal congruences and their quotient automata along the line of [4], where similar notions for commutative congruences were introduced.

4. The structure of strongly definite automata

In this section we consider homomorphic images of nSR's in order to characterize strongly definite automata. We have:

Theorem 4.1. Let $A=(S, X)$ be an automaton and let n be a positive integer. Then A is a strongly definite automaton with $d(A)\leqq n$ iff A is a homomorphic image of $A(m)=(X^m, X)$ for all integers m with $m\geqq n$.

Proof. It is easy to see that $A(m)$ is a strongly definite automaton of degree m. Let A be a homomorphic image of $A(m)$. Then A is a strongly definite automaton with $d(A)\leqq d(A(m))=m$. This completes the proof of the "if" part. Now let A be a strongly definite automaton with degree $d(A)\leqq n$ and $m\geqq n$. Let h be the following mapping of X^m into $S: h(p)=Sp^A$ for all $p\in X^m$. Since $d(A)\leqq m$, this mapping is well defined. Note that a singleton Sp^A is considered as an element of S. We prove that h is surjective. Let $s\in S$ and $p'\in X^m$. Since A is strongly connected, there exists a $q\in X^*$ such that $(Sp'^A)q^A=s$. Let $p'q=p''p$ with $p\in X^m$. Then we have $s=S(p'q)^A=S(p''p)^A=(Sp''^A)p^A=Sp^A$. Finally, we prove that h is a homomorphism of $A(m)$ onto A. Let $p=x'p'$ with $x'\in X, p'\in X^{m-1}$ and $x\in X$. Then we have $h(px^{A(m)})=h((x'p')x^{A(m)})=h(p'x)=S(p'x)^A=Sx'^A(p'x)^A=$ $=S(x'p')^Ax^A=(Sp^A)x^A=h(p)x^A$. This completes the proof of the "only if" part.

Remark 4.1. We can prove that the homomorphism h in the above proof is the unique homomorphism of $A(m)$ onto A. In general, if there exists a homomorphism of a strongly cofinal automaton onto another automaton, it is uniquely determined. For this, see [8].

The following corollary is obvious. Note that the inequality $|S|\geqq d(A)+1$ follows directly from Theorem 7 of [10].

Corollary 4.1. Let A be a strongly definite automaton. Then we have $|X|^{d(A)}\geqq|S|\geqq d(A)+1$. Moreover, $|X|^{d(A)}=|S|$ iff A is isomorphic to $A(d(A))$.

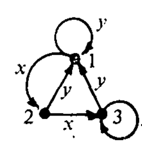


Fig. 1

Example 4.1. Let A be given by the diagram of Fig. 1. If A is a strongly definite automaton, then $2^{d(A)}\geqq 3\geqq d(A)+1$, hence $d(A)=2$. On the other hand, we have $\{1, 2, 3\}(xx)^A=3, \{1, 2, 3\}(xy)^A=$ $=1, \{1, 2, 3\}(yx)^A=2$ and $\{1, 2, 3\}(yy)^A=1$. This shows that A is really a strongly definite automaton with degree 2. Furthermore, A is not isomorphic to $A(2)$. Finally, the homomorphism h of $A(2)$ onto A is given as follows: $h(xx)=3$, $h(xy)=1$, $h(yx)=2$ and $h(yy)=1$.

Remark 4.2. In Theorem 2.1 we proved that the state transition diagram of a shift register has a hamiltonian circle. Moreover, in Theorem 4.1 we proved that the set of all homomorphic images of shift registers coincide with the set of all strongly definite automata. It seems to be interesting to consider the following problem: Under what conditions may the state diagram of a strongly definite automaton have a hamiltonian circle?

5. The structure of definite automata

In [10], Perles et al. discussed the synthesis problem of definite automata. In this section we will also deal with this problem. Strongly definite automata are given as homomorphic images of shift registers, and a method to obtain all homomorphic images of a given automaton is well known [6]. Therefore it remains to determine the structure of definite automata which are not necessarily strongly connected. Let us first give

Definition 5.1. Let $A=(S, X)$ be a definite automaton. Then the subset $U=\{Sp^A | p\in X^{d(A)}\}$ of S is called the *core of* S.

Lemma 5.1. For all $x\in X$ we have $Ux^A \subseteq U$.

Proof. The lemma obviously holds for $d(A)=0$. Assume $d(A)\geqq 1$ and let $s\in U$. Then there exists a $p\in X^{d(A)}$ such that $s=Sp^A$. Let $p=x'p'$ with $x'\in X$ and $p'\in X^{d(A)-1}$. Then, for all $x\in X$, we have $sx^A=(Sp^A)x^A=(Sx'^A p'^A)x^A= =(Sx'^A)(p'x)^A=S(p'x)^A$, where $p'x\in X^{d(A)}$. Consequently, we have $sx^A\in U$.

Lemma 5.2. Let $C=(U, X)$, where $sx^C=sx^A$ for all $(s, x)\in U\times X$. Then C is a strongly definite automaton and $d(C)\leqq d(A)$.

Proof. Let $s\in U$. There exists a $p\in X^{d(A)}$ such that $s=Sp^A$. Therefore $s= =Sp^A=Up^A=Up^C$. This shows that C is a strongly connected automaton. Obviously, C is definite with $d(C)\leqq d(A)$.

Definition 5.2. $C=(U, X)$ is called the *core of* A. Moreover, $d(C)$ is the *radius of the core* and denoted by $r_c(A)$.

Definition 5.3. Let $A=(S, X)$ be a definite automaton and $C=(U, X)$ its core. Then $S-U$ is called the *shell of* S. Moreover, $\max\{|px| \,|\, s\in S-U, p\in X^*, x\in X, sp^A\in S-U$ and $s(px)^A\in U\}$ is called the *thickness of the shell* and denoted by $t_s(A)$.

The following result is obvious.

Proposition 5.1. $t_s(A)\leqq d(A)\leqq t_s(A)+r_c(A)$ and $r_c(A)\leqq d(A)$.

We characterize definite automata by means of $r_c(A)$ and $t_s(A)$.

Let $A=(S, X)$ be a definite automaton and $C=(U, X)$ its core. Let $T_0=U$ and $T_1=\{s\in S | sx^A\in T_0$ for all $x\in X\}$. We have:

Lemma 5.3. $T_0\subseteq T_1$ and if $S-U\neq\emptyset$ then $T_1-T_0\neq\emptyset$.

Proof. $T_0\subseteq T_1$ is obvious. Suppose that for all $s\in S-U$ we have $s\notin T_1$. Then for all $s\in S-U$ there exists some $x_1\in X$ such that $sx_1^A\notin T_0=U$. Since

$sx_1^A \in S-U$, by the same reason as above, there exists some $x_2 \in X$ such that $(sx_1^A)x_2^A \in T_0 = U$. By continuing this procedure, we have an infinite sequence $x_1, x_2, \ldots$ $\ldots, x_k, \ldots$ of elements of X such that $s(x_1x_2 \ldots x_k)^A \notin U$ for any positive integer k. This contradicts the definiteness of A. Hence $T_1 - T_0 \neq \emptyset$.

Now suppose that T_i is defined and $T_{i-1} \subseteq T_i$. Set $T_{i+1} = \{s \in S \mid sx^A \in T_i$ for all $x \in X\}$. Then, by the same way as in the proof of the above lemma, we obtain:

Lemma 5.4. $T_i \subseteq T_{i+1}$ and if $S - T_i \neq \emptyset$ then $T_{i+1} - T_i \neq \emptyset$.

It is obvious that there exists some positive integer i such that $T_i = T_{i+1}$ and $T_k = T_i$ for all $k \geqq i$. This means that in the case $S - U \neq \emptyset$ there exists a minimal positive integer n such that $T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_{n-1} \subset T_n = T_{n+1} = \ldots$ and $S = T_n$.

Definition 5.4. Let $A = (S, X)$ be a definite automaton and $\{T_i \mid 0 \leqq i \leqq n\}$ the set defined as above. Let $L_i = T_i - T_{i-1}$ for all i with $1 \leqq i \leqq n$. Then $\{L_i \mid 1 \leqq i \leqq n\}$ is called the *set of layers of the shell.*

Lemma 5.5. The number of layers coincides with $t_s(A)$.

Proof. Let $s \in S - U$. Then there exists some i with $1 \leqq i \leqq n$ such that $s \in L_i$. It is obvious that $sp^A \in U$ holds for all $p \in X^i$. This means that $t_s(A) \leqq n$. Now let $s \in L_n$. Then, by the definition of L_n, there exists some $x_n \in X$ such that $sx_n^A \in L_{n-1}$. By the same way as above, there exists some $x_{n-1} \in X$ such that $(sx_n^A)x_{n-1}^A \in L_{n-2}$. By the same procedure, we have a sequence $x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1$ of elements in X such that $s(x_nx_{n-1}x_{n-2} \ldots x_{k+1}x_k)^A \notin U$ for $2 \leqq k \leqq n$ and $s(x_n \ldots x_2x_1)^A \in U$. Consequently we have $t_s(A) \geqq n$. Thus $t_s(A) = n$.

Now we are ready to prove the following theorem.

Theorem 5.1. Let $A = (S, X)$ be a definite automaton with $(r_c(A), t_s(A)) = (r, t)$. Then S can be partitioned in $\{U(=L_0), L_i \mid 1 \leqq i \leqq t\}$ such that:

(i) $C = (U, X)$ is a strongly definite automaton with degree r, where $sx^C = sx^A$ for all $(s, x) \in U \times X$.
(ii) $sx^A \in U \cup L_1 \cup L_2 \cup \ldots \cup L_{i-1}$ for all $(s, x) \in L_i \times X$ with $1 \leqq i \leqq t$.
(iii) For all $s \in L_i$ with $1 \leqq i \leqq t$ there exists an $x_i \in X$ such that $sx_i^A \in L_{i-1}$.

Conversely, let $C = (U, X)$ be a strongly definite automaton with degree r and let $\{U(=L_0), L_i \mid 1 \leqq i \leqq t\}$ be a partition of a finite set S. Then each automaton $A = (S, X)$ whose state transition function satisfies the above conditions (i)—(iii) is a definite automaton with $(r_c(A), t_s(A)) = (r, t)$.

Proof. Let $C = (U, X)$ be the core of A and $\{L_i \mid 1 \leqq i \leqq t\}$ the set of layers fo the shell. The first part of the theorem is now obvious. The second part is obvious too.

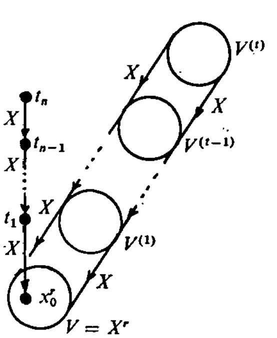
In Proposition 5.1 inequalities were given. We show that there is no relationship among $d(A)$, $r_c(A)$ and $t_s(A)$ beside these inequalities.

Proposition 5.2. Let d, r and t be nonnegative integers such that $t \leqq d \leqq t + r$ and $r \leqq d$. Then for all alphabets X with $|X| \geqq 2$ there exists a definite automaton $A = (S, X)$ such that $d = d(A)$, $r = r_c(A)$ and $t = t_s(A)$.

Proof. Let S be the disjoint union $S=V\cup V^{(1)}\cup\ldots\cup V^{(t)}\cup\{t_1, t_2, \ldots, t_n\}$. Here $V=X^r$, $V^{(i)}=\{v^{(i)}|\,v\in V\}$ are copies of V for $1\leqq i\leqq t$ and $t_1, \ldots, t_n$ are $n=d-r$ additional states. Choose $x_0\in X$ and define the state transition function of A as follows:

(i) For all $(v, x)\in V\times X=X^r\times X$ set $vx^A=vx^{A(r)}$.
(ii) For all $x\in X$ and $2\leqq i\leqq n$ set $t_i x^A=t_{i-1}$ and furthermore set $t_1x^A==x_0^r=x_0\ldots x_0\in X^r$.
(iii) For all $(v, x)\in V\times X$ and $2\leqq i\leqq t$ set $v^{(i)}x^A=(vx^{A(r)})^{(i-1)}$ and furthermore set $v^{(1)}x^A=vx^{A(r)}$.

This situation is depicted in Fig. 2. Obviously, A is a definite automaton. Let us first show $d=d(A)$. The case $d=0$ is trivial. Let $d\geqq 1$. If now $r=0$, then $V=\{e\}$, $n=t=d$ and $d=d(A)$. If now $t=0$, then $r=d$, $n=0$ and $d=d(A)$. Hence we can assume $t, d, r\geqq 1$. If now $n=0$, then $d=r$. Since $|X|\geqq 2$, we have $d=d(A)$. Let now $n\geqq 1$ and $p=p'x\in X^n$ with $p'\in X^{n-1}$, $x\in X$ and $x=x_0$. Then there exists a $p''\in X^{r-1}$ such that $(\{t_n\}\cup X^r)p^A\supseteq\{x_0^r\}\cup\{p''x\}$. It is easy to see that $|(\{x_0^r\}\cup\{p''x\})q^A|\neq 1$ for all $q\in X^{r-1}$. Consequently, $|S(pq)^A|>1$. This means that $d(A)>|pq|=n+r-1==d-1$. Now let $p\in X^d$. Then $p=p'p''$ with $p'\in X^n$ and $p''\in X^r$. From this $(\{t_1, t_2, \ldots, t_n\}\cup V)p^A\subseteq Vp''^A=p''$ follows. On the other hand, since $d\geqq t$, we have $V^{(i)}p^A=Vp^A=p''$ for all i with $1\leqq i\leqq t$. Therefore $Sp^A=p''$. This means that $d(A)\leqq d$. Hence $d(A)=d$. The core of A coincides with $A(r)$, hence $r_c(A)=r$, and since $n=d-r\leqq t$, we have $t_s(A)=t$.



Fig, 2

Acknowledgement. The authors would like to thank Dr. R. Parchmann for his valuable suggestion for Theorem 2.1. This work was done during the first author's stay at the Institut für Informatik of Hannover University by the DAAD grant. The first author would like to thank the German Academic Exchange Service for the opportunity to do this joint work with the second author.

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(Received Oct. 18, 1982)