The probabilistic behaviour of the *NFD* Bin Packing algorithm

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Introduction

In the classical one-dimensional bin-packing problem we are given a list $L=(a_1, a_2, \ldots, a_n)$ of numbers (items) in the interval (0,1], which must be packed into a minimum number of unit-capacity bins (i.e. bins that can contain items totalling at most 1). It is well known that this problem is NP-hard [4], and accordingly a number of approximation algorithms have been developed for its solution. Johnson et al. analysed the best-known heuristics from a worst-case point of view [6]. Their analysis of approximation rules concentrated on the derivation of worst-case bounds of the form

$$A(L) \leqq \alpha \cdot OPT(L)+\beta$$

where α and β are constant and $A(L)$ and $OPT(L)$ are the numbers of bins required to pack L by algorithm A and an optimization rule, respectively. The multiplicative constant is an asymptotic bound on $A(L)/OPT(L)$ and it is the main focus of the analysis. The least constant gives the tight bound of the algorithm.

Two of the best-known heuristics are First Fit (*FF*) and Next Fit (*NF*). In *FF*, we place a_1 in bin 1, and treat the remaining items in order, placing each in the first bin that still has enough room for it (if no opened bin has enough room, then we start a new bin). In *NF* we similarly place a_1 in bin 1, and if a_i will fit into the last-opened bin, then we put it in this bin; otherwise, we start a new bin (which will be the last-opened bin). *FFD* (First Fit Decreasing) and *NFD* (Next Fit Decreasing) differ from *FF* and *NF* only in that the list is initially sorted so that

$$a_1 \geqq a_2 \geqq \ldots \geqq a_n.$$

Johnson proved that the tight asymptotic bound for *FF* is 17/10, for *FFD* is 11/9, and for *NF* is 2. Baker [1] showed that the sum

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{b_i} = 1+\frac{1}{2}+\frac{1}{6}+\ldots \approx 1.691$$

where $b_1=1$ and $b_{i+1}=b_i(b_i+1)$ for $i\geqq 1$, is a tight asymptotic bound for *NFD*.

Numerous results have been achieved on a second line of research: analysis of the expected behaviour of a heuristic algorithm. In this approach, one assumes a density function for the items and establishes probabilistic properties of the heuristic, such as their expected performance. In the special case, lists consist of items independently and uniformly distributed in the interval (0, 1]. Frederickson [3] showed that the *FFD* rule is asymptotically optimal. Recently, Bentley [2] proved the unexpected result that *FF* is also asymptotically optimal. Hofri [5] and Ong [7] showed that for *NF*

$$\mathbf{E}(NF) = 1/6+2n/3$$

in this case, where $\mathbf{E}(NF)$ denotes the expected number of bins for the *NF* rule.

Results

Let we now assume that the items of $L=(a_1, a_2, \ldots, a_n)$ are independently and uniformly distributed in the interval (0, 1]. We then have

Lemma 1.

$$\overline{\lim_{n\to\infty}} \frac{\mathbf{E}(NFD)}{n/2} \leqq 2\left(\frac{\pi^2}{6}-1\right).$$

Proof. Let us sort the elements of L so that

$$a_{i_1} \geqq a_{i_2} \geqq \ldots \geqq a_{i_n}.$$

Let the number of elements in the interval

$$\left(\frac{1}{l+1}, \frac{1}{l}\right] \quad l = 1, 2, \ldots$$

be k_l.

Let us define the sliced NFD_r ($SNFD_r$) algorithm as follows: we pack the elements $> \frac{1}{r}$ in accordance with the *NFD* rule; we then complete the last-opened bin so that it will have at most $(r-1)$ elements; and finally, from the remaining items, we always pack r together. Clearly, for all L:

$$SNFD_r(L) \geqq NFD(L)$$

and

$$\lim_{r\to\infty} SNFD_r(L) = NFD(L).$$

Let $K_r = k_r+k_{r+1}+\ldots$.

Then, for the packing of L by $SNFD_r$, we need at most

$$k_1+\frac{k_2}{2}+\ldots+\frac{k_{r-1}}{r-1}+\frac{K_r}{r}+r$$

bins. Hence, for the expected value of bins with the $SNFD_r$ (for list $L=(a_1, a_2, \ldots, a_n)$):

$$\mathbf{E}(SNFD_r(L)) \leqq \sum_{k_1+k_2+\ldots+k_{r-1}+K_r=n} P(k_1, k_2, \ldots, k_{r-1}, K_r) \cdot \left(k_1+\frac{k_2}{2}+\ldots+\frac{k_{r-1}}{r-1}+\frac{K_r}{r}+r\right)$$

where

$$P(k_1, k_2, \ldots, k_{r-1}, K_r) = \frac{n!}{k_1!\,k_2!\ldots k_{r-1}!\,K_r!}\left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{6}\right)^{k_2}\cdots\left(\frac{1}{(r-1)r}\right)^{k_{r-1}}\left(\frac{1}{r}\right)^{K_r}.$$

Then

$$\mathbf{E}(SNFD_r(L)) \leqq \sum_{i=1}^{r-1}\frac{k_i}{i}\sum_{k_1+k_2+\ldots+k_{r-1}+K_r=n} P(k_1, k_2, \ldots, k_{r-1}, K_r)+\frac{K_r}{r}\sum_{k_1+k_2+\ldots+k_{r-1}+K_r=n} P(k_1, k_2, \ldots, k_{r-1}, K_r)+r =$$

$$= n\sum_{i=1}^{r-1}\frac{1}{i}\,\frac{1}{i(i+1)}\sum_{\substack{k_1+k_2+\ldots+k_{r-1}+K_r=n\\ k_i>0}}\frac{(n-1)!}{k_1!\,k_2!\ldots k_{i-1}!\,(k_i-1)!\,k_{i+1}!\ldots k_{r-1}!\,K_r!}\cdot \left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{6}\right)^{k_2}\cdots\left(\frac{1}{i(i+1)}\right)^{k_i-1}\cdots\left(\frac{1}{r}\right)^{K_r}+n\frac{1}{r}\frac{1}{r}\sum_{\substack{k_1+k_2+\ldots+k_{r-1}+K_r=n\\ K_r>0}}\frac{(n-1)!}{k_1!\ldots k_{r-1}!\,(K_r-1)!}\left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{6}\right)^{k_2}\cdots\left(\frac{1}{(r-1)r}\right)^{k_{r-1}}\left(\frac{1}{r}\right)^{K_r-1}+r=$$

$$= n\left(\sum_{i=1}^{r-1}\frac{1}{i^2(i+1)}+\frac{1}{r^2}\right)+r.$$

And hence for a fixed r

$$\overline{\lim_{n\to\infty}}\,\frac{\mathbf{E}(SNFD_r)}{n/2} \leqq 2\left(\frac{\pi^2}{6}-1\right).$$

Thus $\overline{\lim\limits_{n\to\infty}}\,\dfrac{\mathbf{E}(NFD)}{n/2} \leqq 2\left(\dfrac{\pi^2}{6}-1\right)$ which completes the proof of Lemma 1.

If we now fix r and pack the list $L=(a_1, a_2, \ldots, a_n)$ with the *NFD* rule so that we do not count the bins which contain elements of two different intervals of type $\left(\frac{1}{l+1}, \frac{1}{l}\right]$ or element $<\frac{1}{r}$, then

$$\mathbf{E}(NFD) \geqq \sum_{k_1+k_2+\ldots+k_{r-1}+K_r=n} P(k_1, k_2, \ldots, k_{r-1}, K_r) \cdot \left(k_1-1+\frac{k_2}{2}-1+\ldots+\frac{k_{r-1}}{r-1}-1\right).$$

From this, in a similar way as in Lemma 1, we get

$$\mathrm{E}(NFD) \geqq n\left(\sum_{i=1}^{r}\frac{1}{i^2}-1+\frac{1}{r}\right)-(r-1).$$

Thus, for a fixed r:

$$\varliminf_{n\to\infty}\frac{\mathrm{E}(NFD)}{\frac{n}{2}} \geqq 2\left(\sum_{i=1}^{r-1}\frac{1}{i^2}-1\right)+\frac{2}{r}.$$

The right side is a monotonously increasing function of r and

$$\lim_{r\to\infty}\left[2\left(\sum_{i=1}^{r-1}\frac{1}{i^2}-1\right)+\frac{2}{r}\right] = 2\left(\frac{\pi^2}{6}-1\right).$$

This leads to

Lemma 2.

$$\varliminf_{n\to\infty}\frac{\mathrm{E}(NFD)}{\frac{n}{2}} \geqq 2\left(\frac{\pi^2}{6}-1\right).$$

Then, from Lemma 1 and Lemma 2:

Theorem 1.

$$\lim_{n\to\infty}\frac{\mathrm{E}(NFD)}{\frac{n}{2}} = 2\left(\frac{\pi^2}{6}-1\right) \approx 1{,}29.$$

Let we now assume that the items of $L=(a_1, a_2, \ldots, a_n)$ are independently and uniformly distributed in the interval $(0, \alpha]$ $(0<\alpha\leqq 1)$, and let A be an integer such that

$$\frac{1}{A+1} < \alpha \leqq \frac{1}{A}.$$

In this case, $\mathrm{E}(OPT(L)) \geqq \frac{n\cdot\alpha}{2}$ and, in a totally similar way as for $\alpha=1$, we can get

Theorem 2.

$$\lim_{n\to\infty}\frac{\mathrm{E}(NFD)}{\frac{n\alpha}{2}} = \frac{2}{\alpha^2}\left(\sum_{i=A+1}^{\infty}\frac{1}{i^2}-\frac{1}{A}(1-\alpha)\right).$$

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References

[1] BAKER, B. S., COFFMAN, E. G., A tight asymptotic bound for Next-Fit-Decreasing Bin-Packing, Siam J. Alg. Disc. Meth. 2 (1981), 147—152.

[2] BENTLEY, J. L., JOHNSON, D. S., LEIGHTON, F. T., MCGEOCH, C. C., MCGEOCH, L. A., Some unexpected Expected Behaviour Results for Bin Packing Proc. ACM Symp. on Theory of Computing, 1984, 279—288.

[3] FREDERICKSON, G. N., Probabilistic Analysis for simple One and Two-dimensional Bin Packing Algorithms, Information Processing Letters, 11 (1980), 156—161.

[4] GAREY, M. R., JOHNSON, D. S., Computers and Intractibility: A Guide to the Theory of NP-Completeness, Freemann, San Francisco, CA, 1979.

[5] HOFRI, M., A Probabilistic Analysis of the Next-Fit Bin Packing Algoritm, J. of Algorithms, 5 (1984), 547—556.

[6] JOHNSON, D. S., DEMERS, A., ULLMANN, J. D., GAREY, M. R., GRAHAM, R. L., Worst -Case Performance Bounds for Simple One-Dimensional Bin Packing Algorithms, SIAM J. Comp. 3 (1974), 299—325.

[7] ONG, H. L., MAGAZINE, M. J., WEE, T. S., Probabilistic Analysis of Bin Packing Heuristics Operations Research, 32 (1984), 983—998.

(Received Apr. 10, 1985)