On compositions of root-to-frontier tree transformations

By S. VÁGVÖLGYI

0. Introduction

It is well known that the family of (nondeterministic) root-to-frontier tree transformations is not closed with respect to the composition, see [2]. In this paper we introduce the notion of $k$-synchronized root-to-frontier tree transducer. Transducers of this type are capable of inducing all the relations which are compositions of $k$ root-to-frontier tree transformations. Conversely, we shall show that any relation induced by a $k$-synchronized tree transducer is a composition of $k$ root-to-frontier tree transformations. We mention that similar results are obtained by M. Dauchet in his dissertation [1] using the theory of magmoids.

1. Preliminaires

In this chapter we shall review the basic notions and notations used in the paper and give a reformalized notation of root-to-frontier tree transducers.

Definition 1.1. An operator domain is a set $G$ together with a mapping $v: G \rightarrow \{0, 1, 2, \ldots\}$ that assigns to every $g \in G$ an arity, or rank, $v(g)$. For any $m \geq 0$,

$G^m = \{g \in G | v(g) = m\}$

is the set of $m$-ary operators.

From now on, by an operator domain we mean a finite one, that means $G$ is a finite set. The letters $F$ and $G$ always denote operator domains.

Definition 1.2. Let $Y$ be a set disjoint from the operator domain $G$. The set $T_G(Y)$ of $G$-trees over $Y$ is defined as follows:

1. $G^0 \cup Y \subseteq T_G(Y)$,
2. $g(p_1, \ldots, p_m) \in T_G(Y)$ whenever $m \geq 1$, $g \in G^m$ and $p_1, \ldots, p_m \in T_G(Y)$, and
3. every $G$-tree over $Y$ can be obtained by applying the rules (1) and (2) a finite number of times.
The set $T \subseteq T_G(Y)$ is called a G-forest over $Y$.

**Definition 1.3.** Let $p \in T_G(Y)$ be a G-tree over $Y$. The set sub $(p)$ of subtrees of $p$ is defined by the following rules:

1. sub $(p) = \{p\}$ if $p \in G^0 \cup Y$,
2. sub $(p) = \{p\} \cup \bigcup_{i=1}^{m} \text{sub}(p_i)$ if $p = g(p_1, \ldots, p_m)$ with $g \in G^m$ and $p_1, \ldots, p_m \in T_G(Y)$.

**Definition 1.4.** Let $p \in T_G(Y)$ be a tree. The root root $(p)$ and height $h(p)$ are defined as follows:

1. If $p \in G^0 \cup Y$, then root $(p) = p$, $h(p) = 0$.
2. If $p = g(p_1, \ldots, p_m) \ (m > 0)$, then root $(p) = g$ and $h(p) = \max\{h(p_i) | i = 1, \ldots, m\} + 1$.

**Definition 1.5.** Let $u \in N^*$ be a word over the set of natural numbers. The word $u$ induces a partial function $u : T_G(Y) \to T_G(Y)$ in the following way:

1. If $u = e$ then $u(p) = p$ for every $p \in T_G(Y)$, where $e$ denotes the empty word.
2. If $u = iv$, $i \in N$, $v \in N^*$ and $p \in T_G(Y)$, then $u(p) = \begin{cases} v(p_i) & \text{if } i = g(p_1, \ldots, p_m), g \in G^m, \ 1 \leq i \leq m \\ \text{else undefined} & \end{cases}$

The elements of $T_G(Y)$ may be visualized as tree like directed ordered labelled graphs. In this case every path from the root to a given node in the graph is determined by a word over $N$. For every word $u \in N^*$, if there exists a node $r$ such that $u$ is the path from the root of $p$ to $r$, then $u(p)$ denotes the subtree (subgraph) with root $r$.

**Definition 1.6.** Let $Y$ be a set disjoint from $G$. We may assume without loss of generality that $N^* \cap T_G(Y) = \emptyset$ and $G \cap N^* = \emptyset$ hold in the rest of the paper. The set $P_G(Y)$ of quasi G-trees over $Y$ is defined by the following rule:

$P_G(Y) = \{p \in T_G(Y \cup N^*) | \forall u \in N^* \text{ if } u(p) \in N^* \text{ then } u(p) = u\}$

**Definition 1.7.** The mapping $S : P_G(Y) \to 2^{N^*}$ assigns a subset $S(p)$ of $N^*$ to every quasi tree $p$ which is defined by

$S(p) = \{u(p) | u \in N^*\} \cap N^*$

It is clear that $S(p)$ is a finite set for every $p \in P_G(Y)$. The set $S(p)$ is also denoted by $S'_p$. Members of $S'_p$ are called arguments of $p$.

**Definition 1.8.** Let $Z$ be an arbitrary set and let $\varphi : S_p \to Z$ be a given function for a given quasi tree $p \in P_G(Y)$. Replacing every element $u$ of $S'_p$ by $\varphi(u)$ in the tree $p$ we obtain a G-tree over $Y \cup Z$, which is denoted by $\varphi[S_p, \varphi]$.

**Example.** Let $G = \{g_1, g_2\}$ be an operator domain with $v(g_1) = 1$, $v(g_2) = 2$ and let $Y = \{y_1, y_2, y_3\}$. The quasi tree $p = g_1(g_1(11), g_2(21), y_3)$ may be visualized by the graph on Fig. 1.
Let us define the mapping \( \varphi: \{11, 21\} \rightarrow \{y_2, y_3\} \) as follows:

\[
\varphi(11) = y_2, \quad \varphi(21) = y_3.
\]

The quasi tree \( p[S_p, \varphi] \) may be visualized by the graph on Fig. 2.

Binary relations \( T_r \subseteq T_F(X) \times T_G(Y) \) are called tree transformations. The composition \( \tau_1 \circ \tau_2 \) of the tree transformations \( \tau_1(\subseteq T_F(X) \times T_G(Y)) \) and \( \tau_2(\subseteq T_G(Y) \times T_H(Z)) \) is defined by

\[
\tau_1 \circ \tau_2 = \{(p, q) | (p, r) \in \tau_1, (r, q) \in \tau_2 \text{ for some } r\}.
\]

The composition \( \tau_1 \circ \tau_2 \circ ... \circ \tau_l \ | l \geq 3 \) of the tree transformations \( \tau_1, \tau_2, ..., \tau_l \) is defined by

\[
\tau_1 \circ \tau_2 \circ ... \circ \tau_l = (\tau_1 \circ ... \circ \tau_{l-1}) \circ \tau_l.
\]

**Definition 1.9.** A state set \( A \) is an operator domain consisting of unary operators only. If \( A \) is a state set and \( D \) is an arbitrary set then \( AD \) will denote the forest

\[
AD = \{a(d) | a \in A, d \in D\}.
\]

Moreover, if \( a \in A \) and \( d \in D \) then we generally write \( ad \) for \( a(d) \).

If \( A_1, ..., A_l \) are state sets \( (j \in \mathbb{N}) \) then \( A_1 \times ... \times A_l \) denotes the state set \( A_1 \times ... \times A_l \) which is the Cartesian product of the sets \( A_i \) \( (1 \leq i \leq l) \).

Elements of \( A_1 \times ... \times A_l \) are denoted by sequences \( a_1 ... a_l \), where \( a_i \in A_i \), \( i = 1, ..., j \).

For every non-negative integer \( l \), \( \{1, ..., l\} \) denotes the set \( \{l|1 \leq l \leq l\} \).

**Definition 1.10.** A root-to-frontier tree transducer (R-transducer) is a system \( \mathfrak{H} = (F, X, A, G, Y, A', \Sigma) \), where

1. \( F \) and \( G \) are operator domains.
2. \( A \) is an operator domain consisting of unary operators, the state set of \( \mathfrak{H} \).
   (It will be assumed that \( A \cap T_F(X) = \emptyset \) and that \( A \cap T_G(Y) = \emptyset \).
3. \( X \) and \( Y \) are finite sets.
4. \( A' \subseteq A \) is the set of initial states.
5. \( \Sigma \) is a finite set of productions (rewriting rules) of the following two types:
   (i) \( ax \rightarrow q(a \in A, x \in X, q \in T_G(Y)) \),
   (ii) \( af \rightarrow q[S_q, \varphi](q \in P_G(Y), f \in F^m, \varphi: S_q \rightarrow A \{1, ..., m\}) \).
\( \mathfrak{U} \) is said to be a deterministic \( R \)-transducer if \( A' \) is a singleton and there are no distinct productions in \( \Sigma \) with the same left-hand side.

**Definition 1.11.** Let \( \mathfrak{U} = (F, X, A, G, Y, A', \Sigma) \) be an \( R \)-transducer and let \( p_1, p_2 \in T_\mathfrak{U}(Y \cup N^* \cup AT_\mathfrak{U}(X \cup N^*)) \) be trees. We say that \( p_1 \) directly derives \( p_2 \) in \( \mathfrak{U} \), in symbols \( p_1 \Rightarrow p_2 \), if \( p_2 \) can be obtained from \( p_1 \) by

(i) replacing an occurrence of a subtree \( ax(\in AX) \) in \( p_1 \) by the right side \( q \) of a production \( ax \rightarrow q \) in \( \Sigma \), or by

(ii) replacing an occurrence of a subtree

\[ af(1, \ldots, m)(1, \ldots, m) \sigma] \in (f \in F^m, \sigma \in \{1, \ldots, m\}) \rightarrow T_\mathfrak{U}(X \cup N^*) \]

in \( p_1 \) by

\[ q[S_q, \beta] \]

where \( af \rightarrow q[S_q, \beta] \) is in \( \Sigma \) and \( \beta \) is a mapping \( \beta : S_q \rightarrow AT_\mathfrak{U}(X \cup N^*) \) such that for each \( s \in S_q \) if \( \phi(s) = ct(c \in A, t \in \{1, \ldots, m\}) \) then \( \beta(s) = cA(t) \).

Each application of steps (i) and (ii) is called a direct derivation in \( \mathfrak{U} \).

The reflexive-transitive closure of \( \Rightarrow \) is denoted by \( \Rightarrow^* \).

Using the notation \( \Rightarrow^* \) the transformation \( \tau_\mathfrak{U} \) induced by a root-to-frontier tree transducer \( \mathfrak{U} = (F, X, A, G, Y, A', \Sigma) \) is defined by:

\[ \tau_\mathfrak{U} = \{(p, q) \mid p \in T_\mathfrak{U}(X), q \in T_\mathfrak{U}(Y), ap \Rightarrow^* q \text{ for some } a \in A'\} \]

The range of a mapping \( \phi : A \rightarrow B \) is denoted by \( \text{rg}(\phi) \). Let \( U_0, U_1, \ldots, U_l \) be sets, and let \( V \) be a subset of the set \((U_0 \times U_1 \times \ldots \times U_l) \cup (U_0 \times U_1 \times \ldots \times U_{l-1}) \cup \ldots \cup (U_0 \times U_l) \cup U_0 \), where \( U_0 \times U_1 \times \ldots \times U_l \) the Cartesian product of the sets \( U_i \) \((0 \leq i \leq l) \). Then for an index \( j, (0 \leq j \leq l) \) \[ [V]_j \] denotes the set

\[ \{u_j \mid \exists (u_0, \ldots, u_j, \ldots, u_n) \in V, \ 0 \leq n \leq l, \ 0 \leq j \leq n\} \]

**Definition 1.12.** Let \( u \) be an element of \( N^* \). The mapping \( \omega_u : T_\mathfrak{U}(Y \cup N^*) \rightarrow T_\mathfrak{U}(Y \cup N^*) \) is defined as follows:

1. \( \omega_u(p) = p \) if \( p = y(\epsilon Y) \) or \( p = f(\epsilon G^0) \),
2. \( \omega_u(p) = up \) if \( p \in N^* \),
3. \( \omega_u(p) = f(\omega_u(p_1), \ldots, \omega_u(p_l)) \) if \( p = f(p_1, \ldots, p_l) \in G^l, l \equiv 1, p_i \in T_\mathfrak{U}(Y \cup N^*) \), \( i = 1, \ldots, l \).

2. Derivation sequences

In this chapter we shall deal with the description of derivations according to root-to-frontier tree transducers.

In the rest of the paper \( k \) denotes a natural number, not less than two, moreover let \( \mathfrak{U}_i = (G_{i-1}, Y_{i-1}, A_i, G_i, Y_i, A'_i, \Sigma_0) \) be \( R \)-transducers, \( 1 \equiv i \leq k \).

Now we give a procedure \( P \). The input of \( P \) is a derivation in the form

\[ a_j p_{j-1} \Rightarrow_\mathfrak{U} p_j (a_j \in A, p_{j-1} \in T_{G_{j-1}}(Y_{j-1}), p_j \in T_{G_j}(Y_j)) \]

for some \( j \in \{1, \ldots, k\} \) and a decomposition

\[ p_{j-1} = r_{j-1}[S_{r_{j-1}}, \phi_{j-1}] (r_{j-1} \in P_{G_{j-1}}(Y_{j-1}), \phi_{j-1} : S_{r_{j-1}} \rightarrow T_{G_{j-1}}(Y_{j-1})) \].
The procedure $P$ produces two derivations denoted by (2) and (3) which are defined by induction on the height of $r_{j-1}(\in T_{\alpha_{j-1}}(Y_{j-1} \cup N^*))$. The derivations (2) and (3) will have the following forms:

(2) $a_j r_{j-1}[S_{rj}, \varphi_{j-1}] \rightarrow a_j^* r_j[S_{rj}, \psi_j] \rightarrow a_j^* r_j[S_{rj}, \varphi_j] = p_j,$

for each $s_j \in S_{rj}, \psi_j(s_j) \rightarrow a_j^* \varphi_j(s_j)$ holds,

(3) $a_j r_{j-1} \rightarrow a_j^* r_j[S_{rj}, \bar{\psi}_j], (\bar{\psi}_j: S_{rj} \rightarrow A_j S_{rj-1}),$

and for each $s_j \in S_{rj}$ if $\bar{\psi}_j(s_j) = a_j s_j$ then $\psi_j(s_j) = a_j \varphi_{j-1}(s_j)$ holds.

Let $h(r_{j-1}) = 0$.

Case 1. $r_{j-1} = f, f \in G_{j-1}^0$. In this case $S_{r_{j-1}} = \emptyset, \varphi_{j-1} = \emptyset$ and $r_{j-1}[S_{rj-1}, \varphi_{j-1}] = f$. Thus $a_j^* f \rightarrow p_j \in S_{a_j}$, where $p_j \in T_{\alpha_j}(Y_j)$. Let $r_j = p_j$, thus $S_{rj} = \emptyset$. Let $\varphi_j = \emptyset, \psi_j = \emptyset, \bar{\psi}_j = \emptyset$. Thus the derivation (1) takes the following forms:

(2) $a_j r_{j-1}[S_{rj}, \varphi_{j-1}] \rightarrow a_j^* r_j[S_{rj}, \psi_j] \rightarrow a_j^* r_j[S_{rj}, \varphi_j] = p_j,$

(3) $a_j r_{j-1} \rightarrow a_j^* r_j[S_{rj}, \bar{\psi}_j].$

Case 2. $r_{j-1} = y(\in Y_{j-1})$. This case is the same as Case 1.

Case 3. $r_{j-1} = e(\in N^*)$. In this case $\varphi_{j-1}(e) = r_{j-1}[S_{rj-1}, \varphi_{j-1}]$. Let $r_j = e$, thus $S_{rj} = \{e\}$. Let the mappings $\psi_j: S_{rj} \rightarrow T_{\alpha_j}(Y_j)$ and $\varphi_j: S_{rj} \rightarrow T_{\alpha_j}(Y_j)$ be defined as follows:

$$\psi_j(e) = a_j r_{j-1}[S_{rj}, \varphi_{j-1}], \bar{\psi}_j(e) = a_j e, \psi_j(e) = p_j.$$ 

Thus $r_j[S_{rj}, \psi_j] = a_j r_{j-1}[S_{rj-1}, \varphi_{j-1}], r_j[S_{rj}, \bar{\psi}_j] = a_j e.$

Thus we have obtained the desired derivations (2) and (3), and $\bar{\psi}_j(e) = a_j e, \psi_j(e) = a_j \varphi_{j-1}(e)$, where $S_{rj} = \{e\}$. We have proved the basic step of the induction. Let

$$r_{j-1} = f(p_1, ..., p_l) = f(\omega_1(p_1), ..., \omega_l(p_l))$$

$$p_1, ..., p_l \in P_{\alpha_{j-1}}(Y_{j-1}), S_{rj-1} = 1 \cdot S_{p_1} \cup 2 \cdot S_{p_2} \cup ... \cup l \cdot S_{p_l},$$

where $i \cdot S_{p_i} = \{s|s \in S_{p_i}\}, i = 1, ..., l)$. 

$$r_{j-1}[S_{rj-1}, \varphi_{j-1}] = f(p_1[S_{p_1}, \mu_1], ..., p_l[S_{p_l}, \mu_l]),$$

where for each $i \in \{1, ..., l\}$, and $s \in S_{p_i}, \mu_i(s) = \varphi_{j-1}(is)$ holds. The production applied in the first step in derivation (1) must be of the form $a_j f \rightarrow q[S_q, e]$, where
\(q \in P_{G_j}(Y_j), \ f \in G_{j-1}\) for some \(j\), \(e : S_q \rightarrow A_j \{1, \ldots, l\}\). Consequently derivation (1) can be written in the following form:

\[a_j r_{j-1}(S_{j-1}, \varphi_{j-1}) \Rightarrow_{a_j} q[S_q, \varphi] \Rightarrow_{a_j} q[S_q, \tau] \]

\((\varphi : S_q \rightarrow A_j \text{rg } \varphi_{j-1})\), \(\tau : S_q \rightarrow T_{G_j}(Y_j)\),

where the mapping \(q\) satisfies the following formula: for every \(s \in S_q\) if \(e(s) = b_j t\) (1 \(\leq t \leq l\), \(b_j \in A_j\)) then \(q(s) = b_j p_t[S_{p_t}, \mu_t]\). This implies that \(q(s) = b_j p_t[S_{p_t}, \mu_t] \Rightarrow_{a_j} \Rightarrow_{a_j} \tau(s)\) holds. The desired derivations will take the following forms:

(2) \[a_j f(p_1[S_{p_1}, \mu_1], \ldots, p_t[S_{p_t}, \mu_t]) \Rightarrow_{a_j} q[S_q, \varphi] \Rightarrow_{a_j} q[S_q, \kappa] \Rightarrow_{a_j} q[S_q, \tau],\]

where \(\kappa : S_q \rightarrow T_{G_j}(Y_j \cup A_j T_{G_{j-1}}(Y_{j-1}))\), \(\tau : S_q \rightarrow T_{G_j}(Y_j)\).

(3) \[a_j f(\omega_1(p_1), \ldots, \omega_t(p_t)) \Rightarrow_{a_j} q[S_q, \varphi] \Rightarrow_{a_j} q[S_q, \bar{\kappa}],\]

where \(\bar{\kappa} : S_q \rightarrow A_j T_{G_{j-1}}(Y_{j-1} \cup N^+), \ \bar{\kappa} : S_q \rightarrow T_{G_j}(Y_j \cup A_j S_{q_{j-1}})\).

We shall define the mappings \(\kappa, \bar{\kappa}, \bar{\kappa}\). For each \(s \in S_q\) let us consider the derivation

(4) \(\varphi(s) = b_j p_t[S_{p_t}, \mu_t] \Rightarrow_{a_j} \tau(s)\),

where \(e(s) = b_j t \) holds.

Since \(h(p_j) < h(r_{j-1})\) we may apply the induction hypothesis to derivation (4) and decomposition \(p_t[S_{p_t}, \mu_t]\). The derivations (5) and (6) are obtained by applying procedure \(P\) to (4) and decomposition \(p_t[S_{p_t}, \mu_t]\).

(5) \(b_j p_t[S_{p_t}, \mu_t] \Rightarrow_{a_t} q[S_{q_x}, \eta_x] \Rightarrow_{a_t} q[S_{q_x}, \xi_x] = \tau(s)\),

where \(\eta : S_q \rightarrow A_j \text{rg } \mu_t, \ \epsilon : S_q \rightarrow T_{G_j}(Y_j))\), such that for every \(v \in S_{q_x}, \eta_x(v) \Rightarrow_{a_t} \xi_x(v)\) holds.

(6) \(b_j p_t \Rightarrow_{a_t} q[S_{q_x}, \eta_x](\eta_1 : S_{q_x} \rightarrow A_j S_{p_t})\),

and for every \(v \in S_{q_x}\) if \(\eta_x(v) = b_j z\) for some \(b_j \in A_j \) and \(z \in S_{p_t}\) then \(\eta_x(v) = b_j \mu_1(z)\).

In this case \(\kappa, \bar{\kappa}, \bar{\kappa}\) are defined by

\(\kappa(s) = q_x[S_{q_x}, \eta_x], \ \bar{\kappa}(s) = \omega_t(b_j p_t)\),

\(\bar{\kappa}(s) = \omega_t(q_x[S_{q_x}, \eta_x])\).

The derivation \(q(s) \Rightarrow_{a_t} \Rightarrow_{a_t} \tau(s)\) is the same as derivation (5).

The derivation \(\bar{\kappa}(s) \Rightarrow_{a_t} \bar{\kappa}(s)\) is obtained from derivation (6) by applying the mapping \(\omega_t\) to each step of derivation (6).
We give a procedure $S$. The input of $S$ is a derivation sequence $D = D_1, \ldots, D_k$ given in the following form:

- $D_1: a_1 p_0 \Rightarrow^*_{a_1} p_1, \ (p_0 \in T_{G_0}(Y_0), a_1 \in A_1, p_1 \in T_{G_1}(Y_1))$
- $D_2: a_2 p_1 \Rightarrow^*_{a_2} p_2, \ (a_2 \in A_2, p_2 \in T_{G_2}(Y_2))$
- $\ldots$
- $D_k: a_k p_{k-1} \Rightarrow^*_{a_k} p_k (a_k \in A_k, p_k \in T_{G_k}(Y_k))$

and a decomposition $p_0 = r_0[S_{r_0}, \phi_0]$. $S$ produces two derivation sequences denoted by $D^o = D^o_1, D^o_2, \ldots, D^o_k$ and $D^o = D^o_1, D^o_2, \ldots, D^o_k$. The derivation sequences $D^o$ and $\overline{D^o}$ will have the following forms:

- $D_1^o: a_1 r_0[S_{r_0}, \phi_0] \Rightarrow^*_{a_1} r_1[S_{r_1}, \psi_1] \Rightarrow^*_{a_1} r_1[S_{r_1}, \phi_1] = p_1 \ (r_1 \in P_{G_1}(Y_1), \psi_1: S_{r_1} \rightarrow A_1 \operatorname{rg} (\phi_0), \phi_1: S_{r_1} \rightarrow T_{G_1}(Y_1))$

and for each $s_1 \in S_{r_1}$ the derivation $\psi_1(s_1) \Rightarrow^*_a \phi_1(s_1)$ is valid,

- $D_2^o: a_2 r_1[S_{r_1}, \phi_1] \Rightarrow^*_{a_2} r_2[S_{r_2}, \psi_2] \Rightarrow^*_{a_2} r_2[S_{r_2}, \phi_2] = p_2 \ (r_2 \in P_{G_2}(Y_2), \psi_2: S_{r_2} \rightarrow A_2 \operatorname{rg} (\phi_1), \phi_2: S_{r_2} \rightarrow T_{G_2}(Y_2))$

and for each $s_2 \in S_{r_2}$ the derivation $\psi_2(s_2) \Rightarrow^*_a \phi_2(s_2)$ is valid,

- $\vdots$
- $D_k^o: a_k r_{k-1}[S_{r_{k-1}}, \phi_{k-1}] \Rightarrow^*_{a_k} r_k[S_{r_k}, \psi_k] \Rightarrow^*_{a_k} r_k[S_{r_k}, \phi_k] = p_k \ (r_k \in P_{G_k}(Y_k), \psi_k: S_{r_k} \rightarrow A_k \operatorname{rg} (\phi_{k-1}), \phi_k: S_{r_k} \rightarrow T_{G_k}(Y_k))$

and for each $s_k \in S_{r_k}$ the derivation $\psi_k(s_k) \Rightarrow^*_a \phi_k(s_k)$ is valid.

For every $j \in \{1, \ldots, k\}$ and $s_j \in S_{r_j}$ if $\overline{\psi}_j(s_j) = b_j s_{j-1}$ for some $b_j \in A_j$ and $s_{j-1} \in S_{r_{j-1}}$ then $\psi_j(s_j) = b_j \phi_{j-1}(s_{j-1})$. Applying the procedure $P$ to the derivation $D_1$ and the decomposition $p_0 = r_0[S_{r_0}, \phi_0]$ we obtain the derivations $D^o_1, \overline{D^o_1}$. Assume that the derivations $D^o_{j-1}, \overline{D^o_{j-1}}$ are constructed for an index $j (2 \leq j \leq k)$. Then the derivations $D^o_j, \overline{D^o_j}$ are obtained by applying the procedure $P$ to $D_j$ and decomposition $p_{j-1} = r_{j-1}[S_{r_{j-1}}, \phi_{j-1}], \overline{D^o_j}$, where the decomposition $r_{j-1}[S_{r_{j-1}}, \phi_{j-1}]$ of $p_{j-1}$ is given in the derivation $D^o_{j-1}$.

Let $\mathcal{M} = (G_{j-1}, Y_{j-1}, A_j, G_j, Y_j, A_j', s_{j-1})$ ($j = 1, \ldots, k$) be $R$-transducers. Let us denote the arity function of the operator domain $G_0$ by $v$. We fix these notations
for this chapter. Let \( D=D_1, \ldots, D_k \) be the following derivation sequence:

\[
D_1: a_1 p_0 \rightarrow^{*} a_1 p_1 (p_0 \in T_{G_0}(Y_0), p_1 \in T_{G_1}(Y_1), a_1 \in A_0),
\]
\[
D_2: a_2 p_1 \rightarrow^{*} a_2 p_2 (p_2 \in T_{G_2}(Y_2), a_2 \in A_2),
\]
\[
\vdots
\]
\[
D_k: a_k p_{k-1} \rightarrow^{*} a_k p_k (p_k \in T_{G_k}(Y_k), a_k \in A_k),
\]

moreover, we assume that \( p_0 = q_0[S_{q_0}, \gamma_0] \) holds for some \( q_0 \in P_{G_0}(Y_0), \gamma_0 : S_{q_0} \rightarrow T_{G_0}(Y_0) \).

Applying the procedure \( S \) to the derivation sequence \( D \) and decomposition \( p_0 = q_0[S_{q_0}, \varphi_0] \), we obtain derivation sequences \( D^q_0 \) and \( D^q_0 \).

\[
D^q_0: a_1 q_0 [S_{q_0}, \gamma_0] \rightarrow^{*} a_1 q_1 [S_{q_1}, \alpha_1] \rightarrow^{*} a_1 q_1 [S_{q_1}, \gamma_1] = p_1,
\]
\[
(q_1 \in P_{G_1}(Y_1), \alpha_1 : S_{q_1} \rightarrow A_1 \text{rg}(\gamma_0), \gamma_1 : S_{q_1} \rightarrow T_{G_1}(Y_1)),
\]
and for every \( s_1 \in S_{q_1}, \alpha_1(s_1) \rightarrow^{*} \gamma_1(s_1) \) holds.

\[
D^q_2: a_2 q_1 [S_{q_1}, \gamma_1] \rightarrow^{*} a_2 q_2 [S_{q_2}, \alpha_2] \rightarrow^{*} a_2 q_2 [S_{q_2}, \gamma_2] = p_2,
\]
\[
(q_2 \in P_{G_2}(Y_2), \alpha_2 : S_{q_2} \rightarrow A_2 \text{rg}(\gamma_1), \gamma_2 : S_{q_2} \rightarrow T_{G_2}(Y_2)),
\]
and for every \( s_2 \in S_{q_2}, \alpha_2(s_2) \rightarrow^{*} \gamma_2(s_2) \) holds.

\[
\vdots
\]
\[
D^q_k: a_k q_{k-1} [S_{q_{k-1}}, \gamma_{k-1}] \rightarrow^{*} a_k q_k [S_{q_k}, \alpha_k] \rightarrow^{*} a_k q_k [S_{q_k}, \gamma_k] = p_k,
\]
\[
(q_k \in P_{G_k}(Y_k), \alpha_k : S_{q_k} \rightarrow A_k \text{rg}(\gamma_{k-1}), \gamma_k : S_{q_k} \rightarrow T_{G_k}(Y_k)),
\]
and for every \( s_k \in S_{q_k}, \alpha_k(s_k) \rightarrow^{*} \gamma_k(s_k) \) holds.

\[
\vdots
\]
\[
D^q_j: a_j q_0 \rightarrow^{*} a_j q_1 [S_{q_1}, \bar{\alpha}_1], (\bar{\alpha}_1 : S_{q_1} \rightarrow A_1 S_{q_0}),
\]
\[
D^q_j: a_j q_1 \rightarrow^{*} a_j q_2 [S_{q_2}, \bar{\alpha}_2], (\bar{\alpha}_2 : S_{q_2} \rightarrow A_2 S_{q_1}),
\]
\[
\vdots
\]
\[
D^q_k: a_k q_{k-1} \rightarrow^{*} a_k q_k [S_{q_k}, \bar{\alpha}_k], (\bar{\alpha}_k : S_{q_k} \rightarrow A_k S_{q_{k-1}}),
\]

and for every \( j \in \{1, \ldots, k\} \) and \( s_j \in S_{q_j} \) if

\[
\bar{\alpha}_j(s_j) = b_j s_{j-1} \text{ for some } b_j \in A_j, s_{j-1} \in S_{q_{j-1}},
\]

then

\[
\alpha_j(s_j) = b_j \gamma_{j-1}(s_{j-1}).
\]

We shall define a set \( Z_{(D, \varphi_0)} \) and mappings

\[
\Omega_{(D, \varphi_0)}: Z_{(D, \varphi_0)} \rightarrow (A_1 \cup A_2 A_1 \cup \ldots \cup A_k \ldots A_1) \text{rg}(\gamma_0),
\]
\[
\theta_{(D, \varphi_0)}: S_{q_k} \rightarrow Z_{(D, \varphi_0)}
\]
and

\[
\psi_{(D, \varphi_0)}: S_{q_k} \rightarrow A_k \ldots A_1 T_{G_k}(Y_0)
\]
in the following way:

\[ Z(D,q_0) = \{(s_0, s_1, \ldots, s_j)| s_0 \in S_{q_0}, s_1 \in S_{q_1}, \ldots, s_j \in S_{q_j}, \]

\[ 1 \leq j \leq k \text{ and (} j = k \text{ or (}} j < k \text{ and there are no } s_{j+1} \in S_{q_{j+1}} \text{ and } b_{j+1} \in A_{j+1} \text{ such that } \tilde{a}_{j+1}(s_{j+1}) = b_{j+1}s_j) \]

\[ \text{and} \quad \tilde{a}_i(s_i) = b_is_{i-1} \text{ for } i = 1, \ldots, j. \]

For every \((s_0, s_1, \ldots, s_j) \in Z(D,q_0)\)

\[ \Omega(D,q_0)((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 \gamma_0(s_0) \]

iff

\[ \tilde{a}_i(s_i) = b_is_{i-1} \text{ for } i = 1, \ldots, j. \]

For every \(s_k \in S_{q_k}\), \(\theta(D,q_0)(s_k) = (s_0, s_1, \ldots, s_k)\) iff

\[ \tilde{a}_i(s_i) = b_is_{i-1}(b_i \in A_i) \text{ for } i = 1, \ldots, k. \]

For each \(s_k \in S_{q_k}\), \(\psi(D,q_0)(s_k) = b_k \ldots b_1 \gamma_0(s_0)\) iff

\[ \theta(D,q_0)(s_k) = (s_0, s_1, \ldots, s_k) \]

and

\[ \Omega(D,q_0)((s_0, s_1, \ldots, s_k)) = b_k \ldots b_1 \gamma_0(s_0). \]

One can see the equality \(\psi(D,q_0) = \theta(D,q_0) \circ \Omega(D,q_0)\) holds.

For the derivation sequence \(D\) and a decomposition

\[ p_0 = q_0[S_{q_0}, \gamma_0](q_0 \in P_{G_0}(Y_0), \gamma_0: S_{q_0} \rightarrow T_{G_0}(Y_0)) \]

we can determine the configuration

\[ K(D,q_0): (q_k[S_{q_k}, \psi(D,q_0)], \theta(D,q_0), Z(D,q_0), \Omega(D,q_0)). \]

For the sake of a unified formalism, in the sequel we use the following convention.

Let \(G\) be an operator domain with arity function \(\tilde{v}\), and let \(Y\) be a set disjoint with \(G\).

If \(u \in G^0 \cup Y\), then \(u(1, \ldots, \tilde{v}(u))\) means the \(G\)-tree \(u\) over \(Y\), moreover, \(u(1, \ldots, \tilde{v}(u))[[1, \ldots, \tilde{v}(u)]], \emptyset]\) means \(u\) for arbitrary \(\emptyset\).

We continue the analysis of derivation sequence \(D\). For each \(s_0 \in S_{q_0}\) the tree \(\gamma_0(s_0)\) can be written in the following form:

\[ \gamma_0(s_0) = u_0(1, \ldots, v(u_0))[[1, \ldots, v(u_0)], \emptyset], \]

where \(u_0 \in G_0 \cup Y_0\) and \(\emptyset: \{1, \ldots, v(u_0)\} \rightarrow T_{G_0}(Y_0)\). There are two cases.

1. Case \(Z(D,q_0) = \emptyset\). Take the quasi tree \(r_0 \in P_{G_0}(Y_0)\) defined by \(r_0 = q_0[S_{q_0}, \xi_0]\), where the mapping \(\xi_0: S_{q_0} \rightarrow T_{G_0}(Y_0 \cup N^*)\) is determined by the following formula:

for each \(s_0 \in S_{q_0}\)

\[ \xi_0(s_0) = \omega_{s_0}(u_0(1, \ldots, v(u_0))) \quad \text{if} \quad \gamma_0(s_0) = u_0(1, \ldots, v(u_0))[[1, \ldots, v(u_0)], \emptyset] \]

\[ (u_0 \in G_0 \cup Y_0, \emptyset: \{1, \ldots, v(u_0)\} \rightarrow T_{G_0}(Y_0)). \]

One can see \(K(D,q_0) = K(D,r_0)\) holds.

2. Case \(Z(D,q_0) \neq \emptyset\). Using these decompositions of the trees \(\gamma_0(s_0)\) we obtain the derivation sequence \(E = E_1, \ldots, E_k\) from \(D\). For every \(i \in \{1, \ldots, k\}\) the derivation \(E_i\)
is the same as $D_1$ disregarding the order of direct derivations in $D_1$. We shall introduce the derivation sequence $E = E_1, ..., E_k$ too.

$$E_1: a_1 q_0[S_{q_0}, y_0] \Rightarrow^*_u q_1[S_{q_1}, \alpha_1] \Rightarrow^*_u q_1[S_{q_1}, \beta_1] \Rightarrow^*_u$$

$$\Rightarrow^*_u q_1[S_{q_1}, \gamma_1] (q_1 \in P_{G_1}(Y_1), \alpha_1: S_{q_1} \rightarrow A_1 rg (y_0), \beta_1: S_{q_1} \rightarrow T_{G_1}(Y_1)) \Rightarrow^*_u q_1[S_{q_1}, \beta_1]$$

$$\Rightarrow^*_u q_1[S_{q_1}, \gamma_1] \Rightarrow^*_u q_1[S_{q_1}, \beta_1]$$

$$\Rightarrow^*_u q_1[S_{q_1}, \gamma_1] (q_1 \in P_{G_1}(Y_1), \alpha_1: S_{q_1} \rightarrow A_1 rg (y_0), \beta_1: S_{q_1} \rightarrow T_{G_1}(Y_1)) \Rightarrow^*_u q_1[S_{q_1}, \beta_1] \Rightarrow^*_u q_1[S_{q_1}, \beta_1]$$

$$(\xi_0: S_{q_0} \rightarrow T_{G_0}(Y_0 \cup N^*), \beta_1: S_{q_1} \rightarrow T_{G_1}(Y_1 \cup A_1 N^*))$$

$\xi_0$ is defined by the following formula: for each

$$s_0 \in S_{q_0} \text{ if } \gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

for some $u_0 \in G_0 \cup Y_0$ and mapping

$$\theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0) \text{ then } \xi_0(s_0) = \omega_0(u_0(1, ..., v(u_0))).$$

We shall define the mappings $\beta_1$ and $\beta_1$. For every $s_1 \in S_{q_1}$ let us consider the sub-derivation

1. $\alpha_1(s_1) \Rightarrow^*_u \gamma_1(s_1)$ of $D$.

Let us assume that $\alpha_1(s_1) = b_1 s_0$ and

$$\alpha_1(s_1) = b_1 \gamma_0(s_0) = b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

where

$$s_0 \in S_{q_0}, b_1 \in A_1, u_0 \in G_0 \cup Y_0, \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

Applying procedure $P$ to derivation (1) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

2. $b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$

Applying procedure $P$ to derivation (2) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

3. $b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$

Applying procedure $P$ to derivation (1) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

Applying procedure $P$ to derivation (2) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

Applying procedure $P$ to derivation (1) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

Applying procedure $P$ to derivation (2) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

Applying procedure $P$ to derivation (1) and decomposition

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

We shall define the mappings $\beta_1$ and $\beta_1$. For every $s_1 \in S_{q_1}$ let us consider the sub-derivation

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

We shall define the mappings $\beta_1$ and $\beta_1$. For every $s_1 \in S_{q_1}$ let us consider the sub-derivation

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$

We shall define the mappings $\beta_1$ and $\beta_1$. For every $s_1 \in S_{q_1}$ let us consider the sub-derivation

$$\gamma_0(s_0) = u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1),$$

we obtain derivations (2), (3).

$$b_1 u_0(1, ..., v(u_0)) \Rightarrow^*_u \gamma_1(s_1), \theta_0: \{1, ..., v(u_0)\} \rightarrow T_{G_0}(Y_0).$$
Let \( l \) be an index of the transducers in consideration such that \( 2 \leq l \leq k \). The derivations \( E_i \) and \( \bar{E}_i \) are the following:

\[
E_i: a_i q_{l-1} [S_{q_{l-1}}, \gamma_{l-1}] \Rightarrow^* q_i [S_{q_i}, \alpha_i] \Rightarrow^* q_i [S_{q_i}, \beta_i] \Rightarrow^* q_i [S_{q_i}, \gamma_i]
\]

\( (q_i \in P_{G_i}(Y_i), \alpha_i: S_{q_i} \rightarrow A_i, \gamma_i: S_{q_i} \rightarrow T_{G_i}(Y_i)) \),

\[
\beta_i: S_{q_i} \rightarrow T_{G_i}(Y_i) \cup A_i T_{G_{l-1}}(Y_{l-1}), \gamma_i: S_{q_i} \rightarrow T_{G_i}(Y_i).
\]

\[
\bar{E}_i: a_i q_{l-1} [S_{q_{l-1}}, \xi_{l-1}] \Rightarrow^* q_i [S_{q_i}, \bar{\beta}_i]
\]

\( (\xi_{l-1}: S_{q_{l-1}} \rightarrow T_{G_{l-1}}(Y_{l-1} \cup N^*), \bar{\beta}_i: S_{q_i} \rightarrow T_{G_i}(Y_i \cup A_i N^*)) \).

\( \xi_{l-1} \) is defined by the following formula: for every \( s_{l-1} \in S_{q_{l-1}} \) if \( \beta_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \delta_{l-1}] \) then \( \xi_{l-1}(s_{l-1}) = \omega_{q_{l-1}}(u_{l-1}) \). We shall define the mappings \( \beta_i \) and \( \bar{\beta}_i \). For every \( s_i \in S_{q_i} \) let us consider the subderivation

(1) \( \alpha_i(s_i) \Rightarrow^* \gamma_i(s_i) \) of \( D_i \).

Let us assume that

\[
\bar{\alpha}_i(s_i) = b_i s_{l-1} \quad \text{and} \quad \alpha_i(s_i) = b_i \gamma_{l-1}(s_{l-1}) = b_i u_{l-1}[S_{u_{l-1}}, \delta_{l-1}],
\]

where

\( s_{l-1} \in S_{q_{l-1}}, b_i \in A_i, u_{l-1} \in P_{G_{l-1}}(Y_{l-1}) \),

and the decomposition \( \gamma_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \delta_{l-1}] \) of \( \gamma_{l-1}(s_{l-1}) \) is the same as in \( E_{l-1} \). Applying the procedure \( P \) to derivation (1) and decomposition \( \gamma_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \delta_{l-1}] \) we obtain derivations (2), (3).

(2) \( b_i u_{l-1}[S_{u_{l-1}}, \delta_{l-1}] \Rightarrow^* u_i [S_{u_i}, \delta_i] \Rightarrow^* u_i [S_{u_i}, \gamma_i] = \gamma_i(s_i) \),

where

\( u_i \in P_{G_i}(Y_i), \delta_i: S_{u_i} \rightarrow A_i T_{G_{l-1}}(Y_{l-1}), \gamma_i: S_{u_i} \rightarrow T_{G_i}(Y_i) \),

and for every \( v_i \in S_{u_i} \) the derivation \( \delta_i(v_i) \Rightarrow^* \gamma_i(v_i) \) is valid.

(3) \( b_i u_{l-1} \Rightarrow^* u_i [S_{u_i}, \bar{\delta}_i] \), where \( \bar{\delta}_i: S_{u_i} \rightarrow A_i S_{u_{l-1}} \)

and for each \( v_i \in S_{u_i} \) if

\( \bar{\delta}_i(v_i) = c_i t_{l-1}(c_i \in A_i, t_{l-1} \in S_{u_{l-1}}) \)

then

\( \delta_i(v_i) = c_i \delta_{l-1}(t_{l-1}). \)

In this case \( \beta_i \) and \( \bar{\beta}_i \) are defined by

\( \beta_i(s_i) = u_i [S_{u_i}, \delta_i], \bar{\beta}_i(s_i) = \omega_{q_{l-1}}(u_i [S_{u_i}, \delta_i]). \)
Take the quasi-tree $r_0 \in P_{G_0}(Y_0)$ defined by $r_0 = q_0[S_{q_0}, \xi_0]$, where the mapping $\xi_0 \colon S_{q_0} \rightarrow T_{G_0}(Y_0 \cup N^*)$ as in $E_1$. Let $\lambda_0 \colon S_{r_0} \rightarrow T_{G_0}(Y_0)$ be the mapping such that 
\[
\lambda_0(s_0) = \emptyset(i) \quad \text{if} \quad \gamma_0(s_0) = u_0(1, \ldots, v(u_0)) \{ \} \{ v(0) \}, \emptyset_0, \]
where 
\[
s_0 \in S_{q_0}, \ u_0 \in \mathbb{G}_0 \cup Y_0, \ \emptyset_0 : \{ 1, \ldots, v(u_0) \} \rightarrow T_{G_0}(Y_0), \ \emptyset \in \{ 1, \ldots, v(u_0) \}.
\]
For these $r_0$ and $\lambda_0$ we have that $q_0[S_{q_0}, \gamma_0] = r_0[S_{r_0}, \lambda_0]$ holds. We take the quasi tree $r_1 \in P_{G_1}(Y_1)$ which is defined by $r_1 = q_1[S_{q_1}, \xi_1]$, $\xi_1 \colon S_{q_1} \rightarrow T_{G_1}(Y_1 \cup N^*)$, for every $s_1 \in S_{q_1}$, $\xi_1(s_1) = \omega_1(u_1)$ if $\beta_1(s_1) = u_1[S_{u_1}, \delta_1]$. It can be seen that 
\[
S_{r_1} = \{ s_1 t_1 | s_1 \in S_{q_1}, \xi_1(s_1) = \omega_1(u_1), t_1 \in S_{u_1} \}
\]
holds. Let us define the mappings $\eta_1 \colon S_{r_1} \rightarrow A_1 T_{G_0}(Y_0), \ \bar{\eta}_1 \colon S_{r_1} \rightarrow A_1 S_{r_0}$ and $\lambda_1 \colon S_{r_1} \rightarrow T_{G_1}(Y_1)$ as follows: for each $s_i \in S_{q_1}$, let us consider its unique decomposition $s_i = s_i t_i$, where $s_i \in S_{q_1}$, $s_i s_i = b_i s_i$ for some $b_i \in A_1$ and 
\[
s_0 \in S_{q_0}, \ \xi_1(s_i) = \omega_1(u_1), t_i \in S_{u_1}, \ \beta_1(s_i) = u_1[S_{u_1}, \delta_1].
\]
\[
\bar{\beta}_1(s_i) = \omega_0(u_1[S_{u_1}, \delta_1]), \ \gamma(s_i) = u_1[S_{u_1}, \delta_1]
\]
and $\eta_1$, $\beta_1$, $\bar{\beta}_1$, $\gamma_1$, $\delta_1$, $\bar{\delta}_1$, $\emptyset_1$ as in $E_1$, $\bar{E}_1$.
Let $\eta_1(s_1 t_1) = \delta_1(t_1), \ \bar{\eta}_1(s_1 t_1) = \omega_0(\delta_1(t_1)), \ \lambda_1(s_1 t_1) = \emptyset_1(t_1)$. The derivation $\delta_1(t_1) = \gamma_1(\delta_1(t_1))$ holds, which implies that the derivation $\eta_1(s_1 t_1) = \delta_1(t_1)$ is valid. Thus we obtain the derivations $E'_1$ and $\bar{E}'_1$ from $E_1$ and $\bar{E}_1$, respectively.
\[
E'_1 : a_1 r_0[S_{r_0}, \lambda_0] = \bar{\eta}_1 r_1[S_{r_1}, \eta_1] = \bar{\delta}_1 r_1[S_{r_1}, \lambda_1],
\]
and for each $v_1 \in S_{r_1}$ if $\bar{\eta}_1(v_1) = c_1 v_0$ for some $c_1 \in A_2$, $v_0 \in S_{r_0}$, then $\eta_1(v_1) = c_1 \lambda_0(v_0)$. For each $\bar{E}'_1 \in \bar{E}_1$, $l \in k$ we take the quasi trees $r_1 \in P_{G_1}(Y_1)$ which is defined by 
\[
\bar{r}_1 = q_1[S_{q_1}, \xi_1], \ \bar{\xi}_1 : S_{q_1} \rightarrow T_{G_1}(Y_1 \cup N^*)
\]
for every 
\[
s_i \in S_{q_1}, \ \bar{\xi}_1(s_i) = \omega_1(u_i) \quad \text{if} \quad \bar{\beta}_1(s_i) = u_1[S_{u_i}, \delta_i].
\]
It can be seen that 
\[
S_{r_1} = \{ s_1 t_1 | \bar{\xi}_1(s_i) = \omega_1(u_i), t_i \in S_{u_i} \}
\]
holds. Let us define the mappings $\eta_1 \colon S_{r_1} \rightarrow A_1 T_{G_{l-1}}(Y_{l-1}), \ \bar{\eta}_1 \colon S_{r_1} \rightarrow A_1 S_{r_{l-1}}$ and $\lambda_1 \colon S_{r_1} \rightarrow T_{G_{l}}(Y_{l})$ as follows: for each $s_i \in S_{r_1}$, let us consider its unique decomposition 
\[
(s_i, \beta_1, \bar{\beta}_1, \gamma_1, \delta_i, \bar{\delta}_i, \emptyset_1) \quad \text{as in} \quad E_1, \bar{E}_1 \quad \text{and} \quad \bar{\xi}_1(s_i) = b_i s_{i-1} \quad \text{for some} \quad b_i \in A_1 \quad \text{and} \quad s_{i-1} \in S_{q_{i-1}}.
\]
In this case $\eta_1$ and $\lambda_1$ are defined by $\eta_1(s_{i-1}) = \delta_1(t), \ \bar{\eta}_1(s_{i-1}) = \omega_1(t), \ \lambda_1(s_{i-1}) = \emptyset_1(t)$. The derivation $\delta_1(t) = \gamma_1(\delta_1(t))$ holds, which implies that the derivation $\eta_1(s_{i-1}) = \delta_1(t)$ is valid. Thus we obtain the derivations.
$E'$ and $E'_i$ from $E_i$ and $E_i$, respectively.

$$E'_i: a_t r_{i-1}[S_{r_{i-1}}, \lambda_{i-1}] \Rightarrow s_t r_{i}[S_{r_{i}}, \eta_{i}],$$

and for each $v_i \in S_{r_{i}}$ if $\eta_i(v_i) = c_t v_{i-1}$ for some $c_t \in A_t$, $v_{i-1} \in S_{r_{i-1}}$, then $\eta_i(s_t) = c_t \lambda_{i-1}(v_{i-1})$.

For the sequence of root-to-frontier tree transducers $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ we shall define the sets $\Sigma(l)$ and $V_f$, $(0 \leq l \leq k)$ in the following way:

$$\Sigma(0) = \{u_0 | u_0 \in G_0 \cup Y_0\},$$

$$V_0 = \Sigma(0);$$

$$\Sigma(1) = \{(b_1, u_0, u_1[S_{u_1}, \varphi_1], q_1, W_1, \tau_1) | b_1 u_0 \rightarrow u_1[S_{u_1}, \varphi_1] \in \Sigma_1, \text{ and the second component of } \sigma_1 \text{ is } u_0\}.$$  

Let $j$ be an index such that $2 \leq j \leq k$, and assume that for each $i$ $(1 \leq i < j)$ the sets $\Sigma(i)$ and $V_i$ are defined, and that for each $\sigma_i = (b_i \ldots b_1, u_0, u_1[S_{u_1}, \varphi_1], q_i, W_i, \tau_i) \in \Sigma(i)$ holds. We shall define $\Sigma(j)$ and $V_j$ as follows:

$$\Sigma(j) = \{(b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], q_j, W_j, \tau_j) | (b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], e_{j-1}, W_{j-1}, \tau_{j-1}) \in \Sigma(j-1),$$

$$b_j u_{j-1} \rightarrow s_j u_j[S_{u_j}, \varepsilon_j] \text{ holds, where } u_j \in P_{\varepsilon_j}(Y_j),$$

$$\varepsilon_j: S_{u_j} \rightarrow A_j S_{u_{j-1}},$$

$$\varphi_j: S_{u_j} \rightarrow A_j A_{j-1} \ldots A_1 \{1, \ldots, v(u_0)\};$$

$$\varphi_j(t_j) = c_j c_{j-1} \ldots c_1 t_0 \text{ if } \varepsilon_j(t_j) = c_j t_{j-1} \text{ and}$$

$$\varphi_{j-1}(t_{j-1}) = c_{j-1} \ldots c_1 t_0,$$

$$W_j = \{(t_0, \ldots, t_{j-1}, t_j) | e_j(t_j) = c_j t_{j-1}, c_j \in A_j, \quad \varphi_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1})\} \cup \{(t_0, \ldots, t_{j-1}) \in W_{j-1} | \text{ there are no } t_j \text{ in } S_{u_j} \text{ and } c_j \in A_j \text{ such that} \quad e_j(t_j) = c_j t_{j-1}\} \cup \{(t_0, \ldots, t_l) \in W_{j-1} | 1 \leq l \leq j-2\},$$

$$e_j: S_{u_j} \rightarrow W_j; \quad e_j(t_j) = (t_0, \ldots, t_{j-1}, t_j) \text{ if}$$

$$\varepsilon_j(t_j) = c_j t_{j-1} \text{ and } \varphi_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1}).$$
We say that the element \((b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varepsilon_j, W_j, \tau_j)\) of \(\Sigma(j)\) is generated by the derivation \(b_j u_{j-1} \Rightarrow^{*} u_j[S_{u_j}, \varepsilon_j]\) and element \((b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varepsilon_{j-1}, W_{j-1}, \tau_{j-1})\) of \(\Sigma(j-1)\).

It can be seen that for each element \((b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varepsilon_j, W_j, \tau_j)\) of \(\Sigma(j)\), \(\varphi_j = \varphi_j \circ \tau_j\) hold.

We define mappings \(\kappa_i\colon [Z(p, q_0)] \rightarrow \Sigma(i)\) for \(0 \leq i \leq k\). Let \(s_0 \in [Z(p, q_0)]_0\), which means \(s_0 \in S_{q_0}\). \(\kappa_0(s_0)\) is defined by

\[
\kappa_0(s_0) = \text{root}(\gamma_0(s_0)) = u_0 \quad \text{if}
\]

\[
\gamma_0(s_0) = u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0), \delta_0\}(\delta_0, \{1, \ldots, v(u_0)\} \rightarrow T_{G_0}(Y_0)).
\]

Let \(s_1 \in [Z(p, q_0)]_1\), that is, \(s_1 \in S_{q_1}\). Let us consider the decomposition \(\kappa_1(s_1) = b_{1} u_{0}(1, \ldots, v(u_0))[\{1, \ldots, v(u_0), \delta_0\}(\delta_0, \{1, \ldots, v(u_0)\} \rightarrow T_{G_0}(Y_0))\).

Applying the procedure \(P\) to the subderivation \((1)\) \(\kappa_1(s_1) \Rightarrow^{*} \kappa_2(s_2)\) of \(D_1\) and decomposition \(u_0(1, \ldots, v(u_0))\), \(\delta_0\), \(v_0\), we obtain derivations (2) and (3).

(2) \(b_1 u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0), \delta_0\} \Rightarrow_{\beta_1} u_1[S_{u_1}, \delta_1] \Rightarrow^{*} \)

\[
=_{\beta_1} u_1[S_{u_1}, \delta_1] = \gamma_1(s_1), \quad \text{where} \quad u_1 \in P_{G_1}(Y_1),
\]

\(\delta_1 \colon S_{u_1} \rightarrow A_1 T_{G_0}(Y_0), \quad \delta_1 \colon S_{u_1} \rightarrow T_{G_1}(Y_1),\)

and for each \(v_1 \in S_{u_1}, \delta_2(v_1) \Rightarrow_{\beta_1} \delta_2(v_1)\) holds.

(3) \(b_1 u_0 \Rightarrow_{\beta_1} u_1[S_{u_1}, \delta_1], \delta_2(v_1) = \delta_1(v_1) = \delta_2(v_1) = \delta_0(v_0), \delta_0(v_0)\), then \(\delta_2(v_1) = \delta_0(v_0)\).

\(\beta_1(s_1) = u_1[S_{u_1}, \delta_1], \quad \beta_1(s_1) = \omega_{q_0}(u_1[S_{u_1}, \delta_1])\)

for \(\beta_1, \beta_1\) given in derivations \(E_1, E_1\).

Let \(\kappa_i(s_i)\) be the element of \(\Sigma(1)\) generated by the production \(b_1 u_0 \Rightarrow u_1[S_{u_1}, \delta_1]\).

Assume that \(\kappa_i\) is defined for every \(0 \leq i \leq j-1\). Then the mapping \(\kappa_j(2 \leq j \leq k)\) is defined in the following way: for each \(s_j \in [Z(p, q_0)] = S_{q_0}\), \(\beta_j(s_j) = b_j s_{j-1}\) for some \(b_j \in A_j\) and \(s_{j-1} \in [Z(p, q_0)]_{j-1}\). Thus \(\kappa_j(s_j) = b_j \gamma_{j-1}(s_{j-1})\). \(\kappa_j(s_{j-1})\) has the form \((b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varepsilon_{j-1}, W_{j-1}, \tau_{j-1})\) \(\in \Sigma(m)(J-1)\). Let us consider the decomposition \(\gamma_{j-1}(s_{j-1}) = u_{j-1}[S_{u_{j-1}}, \delta_{j-1}]\) of \(\gamma_{j-1}(s_{j-1})\) which is the same as in \(E_{j-1}\).
Applying the procedure \( P \) to the subderivation

(1) \( \gamma_j(s_j) \Rightarrow \gamma_j(s_j) \) of \( D_j \) and decomposition \( u_{j-1}[S_{u_{j-1}}, \delta_{j-1}] \) we obtain derivations (2) and (3).

(2) \( b_j u_{j-1}[S_{u_{j-1}}, \delta_{j-1}] \Rightarrow \gamma_j(s_j) \) \( u_j[S_{u_j}, \delta_j] \Rightarrow \gamma_j(s_j) \)\( u_j[S_{u_j}, \delta_j] \Rightarrow \gamma_j(s_j) \),

where

\[
\begin{align*}
u_j & \in \mathcal{P}_j(Y_j), \\
\delta_j : S_{u_j} & \rightarrow A_j T_{G_{j-1}}(Y_{j-1}), \\
\delta_j : S_{u_j} & \rightarrow T_{G_j}(Y_j),
\end{align*}
\]

and for every \( \nu_j \in S_{u_j} \) the derivation \( \delta_j(\nu_j) \Rightarrow \gamma_j(\nu_j) \) is valid.

(3) \( b_j u_{j-1}[S_{u_{j-1}}, \delta_{j-1}] \Rightarrow \gamma_j(s_j) \) \( u_j[S_{u_j}, \delta_j] \Rightarrow \gamma_j(s_j) \)\( u_j[S_{u_j}, \delta_j] \Rightarrow \gamma_j(s_j) \),

where \( \delta_j : S_{u_j} \rightarrow A_j S_{u_{j-1}} \) and for each \( \nu_j \in S_{u_j} \) if \( \delta_j(\nu_j) = c_j t_{j-1} \) (\( c_j \in A_j, t_{j-1} \in S_{u_{j-1}} \)) then \( \delta_j(\nu_j) = c_j \delta_{j-1}(t_{j-1}) \).

Let \( \gamma_j(s_j) \) be the element of \( \Sigma(j) \) generated by derivation (3) and \( \gamma_{j-1}(s_{j-1}) \) (\( \in \Sigma(j-1) \)).

We associate the configuration

\[
K(D, r_0) = (r_k[S_{r_k}, \psi(D, r_0)], \Theta(D, r_0), Z(D, r_0), \Omega(D, r_0))
\]

with the derivation sequence \( D \) and decomposition \( r_0 = r_0[S_{r_0}, \lambda_0] \).

Using the derivation sequences \( E_i \) and \( 
\]

(1) \( r_s = q_k[S_{q_k}, \xi_k] \), which was established in \( E_k \), moreover we know that for each \( s_k \in S_{q_k}, \xi_k(s_k) = \omega_{s_k}(u_k) \), where

\[
\begin{align*}
\xi_k(s_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k), \\
\omega_{s_k}(u_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k).
\end{align*}
\]

(2) \( Z(D, r_0) = \{(s_0, 0, 1, \ldots, s_j, t_j) | (s_0, s_1, \ldots, s_l) \in Z(D, r_0) \}
\]

for some \( l \) (\( 1 \leq j \leq l \leq k \)) and \( \xi_i(s_i) = (b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], q_i, W_i, t_i) \) and \( (t_0, t_1, \ldots, t_j) \in W_j \).

(3) For every \( s_k \in S_{r_k} \) let us consider its unique decomposition \( s_k = s_k t_k \), where \( s_k \in S_{q_k}, \xi_k(s_k) \) has the form

\[
\begin{align*}
\xi_k(s_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k), \\
\omega_{s_k}(u_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k).
\end{align*}
\]

If \( \Theta(D, r_0)(s_k) = (s_0, s_1, \ldots, s_k) \) and \( \Theta(D, r_0)(t_k) = (t_0, t_1, \ldots, t_k) \) then

\[
\Theta(D, r_0)(s_k t_k) = (s_0, s_1, t_1, \ldots, s_k t_k).
\]

(4) Let \( s_k \in S_{r_k} \) be arbitrary, and consider its unique decomposition \( s_k = s_k t_k \), where \( s_k \in S_{q_k}, \xi_k(s_k) \) has the form

\[
\begin{align*}
\xi_k(s_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k), \\
\omega_{s_k}(u_k) & = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, t_k).
\end{align*}
\]

and \( t_k \in S_{u_k} \). Then if \( \varphi_k(t_k) = c_k \ldots c_1 t_0 \) and

\[
\psi(D, r_0)(s_k) = u_0(1, \ldots, v(u_0))[1, \ldots, v(u_0), \varphi_0]
\]

\( u_0 \in G_0 \cup Y_0, \varphi_0 : \{1, \ldots, v(u_0) \} \rightarrow T_{G_0}(Y_0) \), then

\[
\psi(D, r_0)(s_k t_k) = c_k \ldots c_1 \varphi_0(t_0).
\]
(5) From the definition of $Z_{(D, r_0)}$ it follows that for every $(s_0, s_1, ..., s_j) \in Z_{(D, r_0)}$ there is a vector $(s_0, s_1, ..., s_j) \in Z_{(D, r_0)}$ for some $l(\geq j)$ such that

$$\lambda_l(s_l) = (b_1 ... b_1, u_0, u_l[S_{u_l}, \varphi_l], q_l, W_l, \tau_l),$$

and

$$\tilde{s}_0 = s_0 t_0, \; \tilde{s}_1 = s_1 t_1, ..., \tilde{s}_j = s_j t_j$$

hold for some $(t_0, t_1, ..., t_j) \in W_l$.

If $\tau_l((t_0, t_1, ..., t_j)) = c_j ... c_1 t_0$ and

$$\Omega_{(D, r_0)}((s_0, s_1, ..., s_l)) = b_1 ... b_1 u_0(1, ..., v(u_0))\{1, ..., v(u_0)\}, \varrho_0$$

for some $u_0 \in G_0 \cup Y_0$ and $\varrho_0 : \{1, ..., v(u_0)\} \rightarrow T_{\varrho_0}(Y_0)$, then

$$\Omega_{(D, r_0)}((\tilde{s}_0, \tilde{s}_1, ..., \tilde{s}_j)) = c_j ... c_1 \varrho_0(t_0).$$

3. $k$-synchronized $R$-transducers

In this chapter we shall introduce the notion of a $k$-synchronized $R$-transducer and prove that the relations induced by this type of transducers are exactly those relations which can be obtained by compositions of $k$ relations induced by root-to-frontier tree transducers.

**Definition 3.1.** A $k$-synchronized $R$-transducer is a system

$$\mathcal{B} = (G_0, G_1, ..., G_k, Y_0, Y_1, ..., Y_k, A_1, ..., A_k, A_1', ..., A_k', \Sigma_{\mathcal{B}}, V),$$

where

1. $k \geq 2$,
2. $G_0, G_1, ..., G_k$ are operator domains,
3. $A_1, ..., A_k$ are state sets, for $i = 1, ..., k$,
   $$A_i \cap T_{\varrho_0}(Y_0) = \emptyset, \quad \text{and} \quad A_k \cap T_{\varrho_k}(Y_k) = \emptyset.$$
4. $A_1' \subseteq A_1, ..., A_k' \subseteq A_k$ are the sets of initial states,
5. $\Sigma_{\mathcal{B}}$ is a finite set of productions, which is a disjoint union

$$\Sigma_{\mathcal{B}} = \Sigma_{\mathcal{B}}(0) \cup \Sigma_{\mathcal{B}}(1) \cup \cdots \cup \Sigma_{\mathcal{B}}(k),$$

where $V = V_0 \cup V_1 \cup \cdots \cup V_k$, where $V_0 = \Sigma_{\mathcal{B}}(0)$, and for $i = 1, ..., k$,

$$V_i \subseteq \Sigma_{\mathcal{B}}(0) \times \Sigma_{\mathcal{B}}(1) \times \cdots \times \Sigma_{\mathcal{B}}(i)$$

and $|V_j| = \Sigma_{\mathcal{B}}(j)$.

$\Sigma_{\mathcal{B}}(0) = \{u_0 | u_0 \in G_0 \cup Y_0\}$ and the members $\sigma_j$ of the production sets $\Sigma_{\mathcal{B}}(j)$ ($j \geq 1$) have the form:

$$\sigma_j = (b_j ... b_1, u_0, u_j[S_{u_j}, \varphi_j], q_j, W_j, \tau_j), \quad \text{where}$$

$$b_i \in A_i \quad \text{for} \quad i = 1, ..., j, \quad u_0 \in G_0 \cup Y_0,$$

$$u_j \in P_{\varphi_j}(Y_j), \varphi_j : S_{u_j} \rightarrow A_j ... A_1 \{1, ..., v(u_0)\}.$$
$W_j$ is a finite subset of $(N^*)^2 \cup \ldots \cup (N^*)^{j+1}$, where $(N^*)^1 = N^*$ and for each $l \geq 1$, $(N^*)^{l+1} = (N^*)^l \times N^*$.

$q_j$: $S_{u_j} \rightarrow W_j$, $t_j$: $W_j \rightarrow (A_j \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1)[W_j]_0$,

and the following requirements are satisfied:

a) $j=1$

i) $W_1 = \{ (t_0, t_1) | t_1 \in S_{u_1}, \varphi_1(t_1) = c_1 t_0 \text{ for some } c_1 \in A_1 \}$,

ii) for every $t_1 \in S_{u_1}$ if $\varphi_1(t_1) = c_1 t_0$ then $\varphi_1(t_1) = (t_0, t_1)$,

iii) $\varphi_1 = q_1 \circ t_1$,

iv) $V_1 = \{ (u_0, \sigma_1) | u_0 \in G_0 \cup Y_0 \}$, and the second component of $\sigma_1(\xi S_0(1))$ is $u_0$.

b) $j>1$ if $(u_0, \sigma_1, \ldots, \sigma_{j-1}, \sigma_j) \in V_j$ and $\sigma_{j-1}$ has the form $(b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \sigma_{j-1}], q_{j-1}, W_{j-1}, t_{j-1})$ then $(u_0, \sigma_1, \ldots, \sigma_{j-1}) \in V_{j-1}$ and there is a mapping $e_j$: $S_{u_j} \rightarrow A_j[W_{j-1}]_{j-1}$ such that i)—iv) hold:

i) $W_j = \{ (t_0, \ldots, t_{j-1}, t_j) | e_j(t_j) = c_j t_{j-1}, c_j \in A_j, t_j \in S_{u_j}, t_{j-1} \in S_{u_{j-1}},$ 

\[ q_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1}, t_j) \} \cup $\cup \{ (t_0, \ldots, t_l) \in W_{j-1} | 1 \leq l \leq j-2 \} \cup$

\[ \cup \{ (t_0, \ldots, t_{j-1}) \in W_{j-1} \} \text{ there are no } t_j \in S_{u_j} \text{ and } c_j \in A_j$

\[ \text{such that } e_j(t_j) = c_j t_{j-1} \}.$

ii) For

$\tau_j, \tau_j|_{W_j \cap W_{j-1}} = \tau_{j-1}|_{W_j \cap W_{j-1}}$ and

if $(t_0, \ldots, t_{j-1}, t_j) \in W_j$, $e_j(t_j) = c_j t_{j-1}$ \((c_j \in A_j, t_{j-1} \in [W_{j-1}]_{j-1})\) and

\[ \tau_{j-1}(t_0, \ldots, t_{j-1}) = c_{j-1} \ldots c_1 t_0 \text{ then } \tau_j((t_0, \ldots, t_{j-1}, t_j)) = c_j \ldots c_1 t_0 \}

iii) For each $t_j \in S_{u_j}$ if

\[ e_j(t_j) = c_j t_{j-1}(c_j \in A_j, t_{j-1} \in [W_{j-1}]_{j-1}) \text{ and }$ 

\[ q_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1}) \text{ then } \varphi_j(t_j) = (t_0, \ldots, t_{j-1}, t_j) \}

iv) $\varphi_j = q_j \circ t_j$.

(One can see that for each $t_j \in S_{u_j}$, $e_j(t_j) = c_j t_{j-1}$ \((c_j \in A_j, t_{j-1} \in S_{u_{j-1}})\) iff

\[ \varphi_j(t_j) = (t_0, \ldots, t_{j-1}, t_j) \text{ and } \tau_j((t_0, \ldots, t_{j-1}, t_j)) = c_j \ldots c_1 t_0 \}

In the rest of the paper we shall denote the arity function of $G_0$ by $v$.

**Definition 3.2.** Let $B$ be a $k$-synchronized R-transducer as in Definition 3.1. A configuration of $B$ is a system \((q[S_q], \psi, \Theta, Z, \Omega)\), where $q \in P_{G_k}(Y_k)$, $\psi$: $S_q \rightarrow A_k \ldots A_1 T_{G_0}(Y_0)$, $\Theta$: $S_q \rightarrow Z$; for each $s_k \in S_q$, $\Theta(s_k) = (s_0, \ldots, s_{k-1}, s_k)$ for some $s_0, \ldots, s_{k-1} \in N^*$.

$Z$ is a finite subset of $(N^*)^2 \cup \ldots \cup (N^*)^k \cup (N^*)^{k+1}$ such that the following two conditions hold:

i) for $j=0, \ldots, k$ and arbitrary $s_j, \bar{s}_j \in [Z]_j$ if $s_j = \bar{s}_j \bar{s}_j$, then $s_j = \bar{s}_j$ and $\bar{s}_j = e$,
ii) for each \( s_k \in S_q \), \( \Theta(s_k) \) is the only element of \( Z \) which has the form \( (s_0, ..., s_{k-1}, s_k) \) for some \( s_0, ..., s_{k-1} \in N^* \).

\( \Omega : A_k A_{k-1} \cdots A_1 \cup A_{k-1} \cdots A_1 \cup \cdots \cup A_1 \rightarrow T_{G_0}(Y_0) \) is a mapping such that \( \psi = \Theta \circ \Omega \) holds, that is, the diagram in Figure 3 is commutative.

\[ S_q \xrightarrow{\Psi} (A_k A_{k-1} \cdots A_1 \cup A_{k-1} \cdots A_1 \cup \cdots \cup A_1)_T \]

\( \circ \)

\[ Z \]

**Figure 3**

A configuration \( (q[S_q, \psi], \Theta, Z, \Omega) \) is said to be a starting configuration, if \( q \) is the quasi tree \( e \in N^* \) (empty word) and \( \psi(e) \in A_k \cdots A_1 \cup T_{G_0}(Y_0) \), moreover \( Z = \{ (e, ..., e) \} \).

**Definition 3.3.** Let \( K_1 = (q[q[S_q, \psi^1], \Theta^1, Z^1, \Omega^1]) \) and \( K_2 = (q^2[q[S_q^2, \psi^2], \Theta^2, Z^2, \Omega^2]) \) be configurations of a \( k \)-synchronized \( R \)-transducer \( \mathcal{B} = (G_0, G_1, ..., G_k, Y_0, Y_1, ..., Y_k, A_1, ..., A_k, A'_1, ..., A'_k, \Sigma_0, V) \). It is said that there is a transition from \( K_1 \) to \( K_2 \) in \( \mathcal{B} \) which is denoted by \( K_1 \Rightarrow_\mathcal{B} K_2 \) if there are mappings \( \chi_j : [Z^1]^j \rightarrow \Sigma_0(j) \) for \( j = 0, 1, ..., k \) such that the following requirements hold:

1. For each \( (s_0, s_1, ..., s_j) \in Z^1 \) \( (1 \leq j \leq k) \) if
   \[
   \Omega^1((s_0, s_1, ..., s_j)) = b_j \cdots b_1 u_0(1, ..., v(u_0)) \{ 1, ..., v(u_0) \} \}
   \forall 0 \] for some \( u_0 \in G_0 \cup Y_0, b_j \cdots b_1 \in A_j \cdots A_1 \) and \( \partial_0 : \{ 1, ..., v(u_0) \} \rightarrow T_{G_0}(Y_0) \) then
   \[
   \chi_0(s_0) = u_0, \chi_i(s_i) = (b_i \cdots b_1, u_0, u_i[S_u, \varphi], \partial_i, W_i, \tau_i)
   \]
   for some \( u_i, \varphi_i, W_i \) and \( \tau_i \) \( (i = 1, 2, ..., j) \), and \( (\chi_0(s_0), \chi_1(s_1), ..., \chi_j(s_j)) \in V_j \).

2. \( q^2 = q^1[q[S_q^1, \xi]] \) for the mapping \( \xi : S_q^1 \rightarrow T_{G_0}(Y_k \cup N^*) \) which is defined by the following formula: for every \( s_k \in S_q^1 \), \( \xi(s_k) = \omega_{\xi}(u_k) \) if \( \chi_k(s_k) = (b_k \cdots b_1, u_0, u_k[S_u, \varphi_k], \partial_k, W_k, \tau_k) \).

3. \( Z^2 = \{(s_0 t_0, s_1 t_1, ..., s_j t_j) | (s_0, s_1, ..., s_j) \in Z^1 \} \) for some \( l, (1 \leq j \leq l \leq k) \) and \( \xi_i(s_i) = (b_i \cdots b_1, u_0, u_i[S_u, \varphi_i], \partial_i, W_i, \tau_i) \) and \( (t_0, t_1, ..., t_j) \in W_j \).

4. For each \( s_k \in S_q^2 \) consider its unique decomposition \( \delta_k = s_k t_k \), where \( s_k \in S_q^2, \chi_k(s_k) \) has the form \( \chi_k(s_k) = (b_k \cdots b_1, u_0, u_k[S_u, \varphi_k], \partial_k, W_k, \tau_k) \), \( \xi_k(s_k) = \omega_{\xi}(u_k) \), and \( t_k \in S_{\omega}. \) If \( \Theta^1(s_k) = (s_0, s_1, ..., s_k) \) and \( \Theta^2(s_k) = (t_0, t_1, ..., t_k) \) then \( \Theta^2(s_k) = (s_0 t_0, s_1 t_1, ..., s_k t_k) \).

5. Let \( s_k \in S_q^2 \) be arbitrary and consider its unique decomposition \( s_k = s_k t_k \), where \( s_k \in S_q^2, \chi_k(s_k) \) has the form \( \chi_k(s_k) = (b_k \cdots b_1, u_0, u_k[S_u, \varphi_k], \partial_k, W_k, \tau_k) \).
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\[ \tau_k, \xi_k(s_k) = \omega_k(u_k) \text{ and } t_k \in S_{u_k}. \]

If \( \varphi_k(t_k) = c_k \ldots c_1 t_0 \) and

\[ \psi^1(s_k) = u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0)\}, \vartheta_0], \]

\[ (u_0 \in G_0 \cup Y_0, \vartheta_0: \{1, \ldots, v(u_0)\} \to T_{G_0}(Y_0)), \]

then \( \psi^2(s_k) = c_k \ldots c_1 \vartheta_0(t_0). \)

(6) For every \( (\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_j) \in Z^2 \) there is a vector \( (s_0, s_1, \ldots, s_l) \in Z^1 \) for some \( l \) \((1 \leq j \leq l \leq k)\) such that \( \xi_i(s_i) = (b_1 \ldots b_l, u_0, u_1[S_{u_1}, \varphi_1], q_l, W_l, \tau_i) \), and \( \tilde{s}_0 = s_0 t_0, \tilde{s}_1 = s_1 t_1, \ldots, \tilde{s}_j = s_j t_j \) hold for some \( (t_0, t_1, \ldots, t_j) \in W_j. \)

If \( \tau_i((t_0, t_1, \ldots, t_j)) = c_j \ldots c_1 t_0 \) and

\[ \Omega^1((s_0, s_1, \ldots, s_l)) = b_1 \ldots b_l u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0)\}, \vartheta_0] \]

for some \( u_0 \in G_0 \cup Y_0 \) and \( \vartheta_0: \{1, \ldots, v(u_0)\} \to T_{G_0}(Y_0) \) then

\[ \Omega^2((\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_j)) = c_j \ldots c_1 \vartheta_0(t_0). \]

Notice, that given configuration \( K_i \) and mappings \( \xi_i \), for \( i = 0, \ldots, k \) satisfying condition (1), uniquely determine configuration \( K_0. \)

The reflexive and transitive closure of relation \( \Rightarrow_\beta \) between configurations is denoted by \( \Rightarrow^*_\beta. \)

Definition 3.4. Take a \( k \)-synchronized \( R \)-transducer

\[ \mathfrak{B} = (G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A'_1, \ldots, A'_k, \Sigma_{\mathfrak{B}}, V). \]

Then the relation

\[ \tau_\mathfrak{B} = \{(q, q) \mid q \in T_{G_0}(Y_0), q \in T_{G_k}(Y_k), \]

\[ K_0 = (e, [\varphi_0: e \mapsto b_p], \Theta^0, Z^0, \Omega^0) \Rightarrow_\beta (q, \emptyset, \emptyset, \emptyset) \]

for some starting configuration \( K_0. \}

is called the transformation induced by \( \mathfrak{B}. \)

Configurations of the form \( (q, \emptyset, \emptyset, \emptyset) \), where \( q \in T_{G_k}(Y_k) \), are said to be final.

Theorem 3.5. Let \( r_i = (G_{i-1}, Y_{i-1}, A_i, G_i, Y_i, A'_i, \Sigma_{\mathfrak{B}}) \) \((i = 1, \ldots, k, k \geq 2)\) be \( R \)-transducers. Then there is a \( k \)-synchronized \( R \)-transducer \( \mathfrak{B} \) such that \( \tau_\mathfrak{B} = \tau_{r_1} \circ \ldots \circ \tau_{r_k}. \)

Proof: We construct a \( k \)-synchronized \( R \)-transducer \( \mathfrak{B} \) as follows:

\[ \mathfrak{B} = (G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A'_1, \ldots, A'_k, \Sigma_{\mathfrak{B}}, V), \]

where \( \Sigma_{\mathfrak{B}}(0) = \Sigma(0), \ldots, \Sigma_{\mathfrak{B}}(k) = \Sigma(k) \) for the sets \( \Sigma(i) \), which are defined in the previous chapter. \( V = V_0 \cup V_1 \cup \ldots \cup V_k \), where the sets \( V_0, V_1, \ldots, V_k \) are defined in the previous chapter. We may assume without loss of generality that \( A_k \ldots A_1 \cap T_{G_0}(Y_0) = \emptyset \) and that, for \( i = 1, \ldots, k, A_i \ldots A_1 \cap T_{G_0}(Y_0) = \emptyset. \) Thus \( \mathfrak{B} \) satisfies requirement (3) of Definition 3.1.

First we shall prove the inclusion

\[ \tau_{r_1} \circ \ldots \circ \tau_{r_k} \subseteq \tau_\mathfrak{B}. \]
Assume that \((p_0, p_k) \in \tau_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \cdots \circ \tau_{\mathcal{B}}\). Then there are initial states \(a_1 \in A'_1, \ldots, a_k \in A'_k\) and there is a derivation sequence \(D: a_1 p_0 \Rightarrow^* a_1 p_1, a_2 p_1 \Rightarrow^* a_2 p_2, \ldots, a_k p_{k-1} \Rightarrow^* a_k p_k\), where \(p_i \in T_{\mathcal{G}_k}(Y_i)\) for \(i = 0, \ldots, k\).

Take an arbitrary decomposition \(p_0 = q_0[S_{q_0}, \gamma_0]\) of the tree \(p_0\), where \(q_0 \in \mathcal{P}_{\mathcal{G}_0}(Y_0)\) and \(\gamma_0: S_{q_0} \rightarrow T_{\mathcal{G}_0}(Y_0)\). We have constructed a configuration
\[
K(D, q_0) = (q_0[S_{q_0}, \gamma_0], \Theta_{D, q_0}, Z_{D, q_0}, \Omega_{D, q_0})
\]
for \(D\) and \(q_0\) in Chapter 2.

One can see that \(K(D, q_0)\) is a configuration of the \(k\)-synchronized \(R\)-transducer \(\mathcal{B}\).

Let \(r_0 = q_0[S_{r_0}, \xi_0]\) for the mapping \(\xi_0: S_{q_0} \rightarrow T_{\mathcal{G}_0}(Y_0 \cup N^*)\) which is defined by \(\xi_0(s_0) = \omega_{q_0}(u_0(1, \ldots, v(u_0)))\) for each \(s_0 \in S_{q_0}\), where \(y_0(s_0) = u_0(1, \ldots, v(u_0))[1, \ldots, v(u_0)], \gamma_0\),
\[
(u_0 \in \mathcal{G}_0 \cup Y_0, \gamma_0: \{1, \ldots, v(u_0)\} \rightarrow T_{\mathcal{G}_0}(Y_0)).
\]

\(K(D, r_0)\) is again a configuration of \(\mathcal{B}\).

It follows from the definition of the relation \(\Rightarrow_{\mathcal{B}}\) that
\[
K(D, q_0) = K(D, r_0) \quad \text{or} \quad K(D, q_0) \Rightarrow_{\mathcal{B}} K(D, r_0)
\]
holds.

Let \(p^0, p^1, \ldots, p^l \in \mathcal{P}_{\mathcal{G}_0}(Y_0)\) be quasi trees for \(l = h(p) + 1\) such that for every \(i\)
\[
(0 \equiv i \equiv l), \quad p_i = p^i[S_{p^i}, \gamma^i] \quad (\gamma^i: S_{p^i} \rightarrow T_{\mathcal{G}_0}(Y_0)),
\]
where
\[
i) p^0 = e, \quad \gamma^0(e) = p_0, \quad \text{and}
\]
\[
ii) p^{i+1} = p^i[S_{p^i}, \xi^{i+1}] \quad \text{for the mapping} \quad \xi^{i+1}: S_{p^i} \rightarrow T_{\mathcal{G}_0}(Y_0 \cup N^*)\quad \text{such that}
\]
\[
\xi^{i+1}(s^i) = \omega_{s^i}(u_0(1, \ldots, v(u_0))),
\]
where
\[
\gamma^i(s^i) = u_0(1, \ldots, v(u_0))[1, \ldots, v(u_0)], \gamma_0\]
for some \(u_0 \in \mathcal{G}_0 \cup Y_0\) and \(\gamma_0: \{1, \ldots, v(u_0)\} \rightarrow T_{\mathcal{G}_0}(Y_0)\). In this case every \(s^{i+1} \in S_{p^{i+1}}\) has a unique decomposition \(s^{i+1} = s^i t^i\),
\[
s^i \in S_{p^i}, \gamma^i(s^i) = u_0(1, \ldots, v(u_0))[1, \ldots, v(u_0)], \gamma_0\]
for some
\[
u_0 \in \mathcal{G}_0 \cup Y_0, \gamma_0: \{1, \ldots, v(u_0)\} \rightarrow T_{\mathcal{G}_0}(Y_0)\quad \text{and} \quad t^i \in \{1, \ldots, v(u_0)\}.
\]
Then \(\gamma^{i+1}(s^{i+1}) = \gamma_0(t^i)\).

We know that \(K(D, p^i) = K(D, p^{i+1})\) or \(K(D, p^i) \Rightarrow_{\mathcal{B}} K(D, p^{i+1})\) holds for \(i = 0, \ldots, l-1\).

It has remained to prove that \(K(D, p^i)\) is a starting configuration and \(K(D, p^l)\) is a final configuration of \(\mathcal{B}\). The first part of the statement trivially holds. Since \(p^l = p_0\) and \(p_0 \in \mathcal{P}_{\mathcal{G}_0}(Y_0)\), \(S_{p^0}\) must be the empty set, thus \(K(D, p^l) = (p_k, \emptyset, \emptyset, \emptyset)\). We have proved that \((p_0, p_k) \in \tau_{\mathcal{B}}\).

We shall prove the reverse inclusion:
\[
\tau_{\mathcal{B}} \subseteq \tau_{\mathcal{B}_1} \circ \tau_{\mathcal{B}_2} \circ \cdots \circ \tau_{\mathcal{B}_k}.
\]
Let $K_0 \Rightarrow K_1 \Rightarrow \ldots \Rightarrow K_n$ be a sequence of transitions in $\mathcal{B}$, where $n \geq 1$, $K_0 = (e, \psi^0, \Theta^0, \mathcal{Z}^0, \mathcal{O}^0)$ is a starting configuration and $K_i = (q_i[S_{q_i}, \psi_i], \Theta_i, \mathcal{Z}_i, \mathcal{O}_i)$ for $i = 1, \ldots, n$. Assume that $\psi^0(e) = a_k \ldots a_1 p$. Let $p^1, p^2, \ldots, p^l$ be the sequence of quasi trees constructed in the first part of the proof, where $p^l = p$. It can be seen that $n \geq l$. Then there is a derivation sequence $D = D_1, \ldots, D_k$ such that the following equalities hold:

i) $p_k = q^n$,

ii) $\psi(D, p^n) = \psi^n$,

iii) $Z^n = Z(D, p^n)$,

iv) $\Theta^n = \Theta(D, p^n)$,

v) $Q^n = Q(D, p^n)$,

where the sets $Z(D, p^n)$, $\Theta(D, p^n)$ and the mappings

$\psi(D, p^n): S_{p^n} \to (A_k \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1) S_{p^n}$,

$\Theta(D, p^n): Z(D, p^n) \to (A_k \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1) \text{rg}(g^n)$

are defined as follows:

1) $Z(D, p^n) = \{(s_0, s_1, \ldots, s_j) | s_0 \in S_{p^n}, s_1 \in S_{p_1}, \ldots, s_j \in S_{p_j}, 1 \leq j \leq k$,

and $(j=k$ or $(j<k$ and there are no $s_{j+1} \in S_{q_{j+1}}$ and $b_{j+1} \in A_{j+1}$ such that $\eta_{j+1}(s_{j+1}) = b_{j+1} s_j))$ and $\eta_i(s_i) = b_i s_{i-1} (b_i \in A_i)$ for $i = 1, \ldots, j}.$

2) For every $(s_0, s_1, \ldots, s_j) \in Z(D, p^n)$ $(1 \leq i \leq k)$, $\Omega(D, p^n)((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 g^n(s_0)$ iff $\eta_i(s_i) = b_i s_{i-1} (b_i \in A_i)$ for $i = 1, \ldots, j$.

3) For every $s_k \in S_{p^n}$, $\Theta(D, p^n)(s_k) = (s_0, s_1, \ldots, s_k)$ iff

$\eta_i(s_i) = b_i s_{i-1} (b_i \in A_i)$ for $i = 1, \ldots, k$.

4) $\psi(D, p^n) = \Theta(D, p^n) \circ \Omega(D, p^n)$.

We proceed by induction on $n$. Let $n = 1$. In this case $p^1 = u_0(1, \ldots, v(u_0))$, $u_0 = \text{root}(p)$. From the definition of the transition in $\mathcal{B}$ it follows that there are mappings $\chi_i: \{e\} \to \Sigma_B(i)$ $(i = 0, 1, \ldots, k)$ such that $\chi_0(e) = u_0$,

$\chi_1(e) = (a_1, u_0, u_1[S_{u_1}, \varphi_1], q_1, W_1, \tau_1)$,

and so on,

$\chi_k(e) = (a_k \ldots a_1, u_0, u_k[S_{u_k}, \varphi_k], q_k, W_k, \tau_k)$,
and \((\kappa_0(e), \kappa_1(e), \ldots, \kappa_k(e)) \in V_k\), and configuration \(K_0\) and mappings \(\kappa_i\) \((i=0, \ldots, k)\) determine the configuration \(K_1\).

According to the construction of the transducer \(B\) and the definition of the transition in \(B\), there is a derivation sequence \(D=D_1, \ldots, D_k\),

\[
D_1: \ a_1 u_0(1, \ldots, v(u_0)) = \kappa_{\eta_1} u_1[S_{u_1}, \varphi_1],
\]

for some \(\eta_i: S_{u_i} \rightarrow A_i S_{u_{i-1}}, \ (i = 2, \ldots, k)\)

such that the following equalities hold:

i) \(q^1 = u_k\),

ii) \(\psi_{(D, p^1)} = \psi^1\),

iii) \(Z^1 = W_k = Z_{(D, p^1)}\),

iv) \(\Theta^1 = \varphi_k = \Theta_{(D, p^1)}\),

v) \(\Omega^1 = \tau_k = \Omega_{(D, p^1)}\).

The proof of the basic step is complete.

Assume that the statement is true for \(n-1\). It means that there is a derivation sequence

\[
D_1: \ a_1 p^{n-1} = \kappa_{\eta_1} p_1[S_{p_1}, \eta_1]\ (p_1 \in P_{G_1}(Y_1), \eta_1: S_{p_1} \rightarrow A_1 S_{p^{n-1}}),
\]

\[
D_2: \ a_2 p_1 = \kappa_{\eta_2} p_2[S_{p_2}, \eta_2]\ (p_2 \in P_{G_2}(Y_2), \eta_2: S_{p_2} \rightarrow A_2 S_{p_1}),
\]

\[
\vdots
\]

\[
D_k: \ a_k p_{k-1} = \kappa_{\eta_k} p_k[S_{p_k}, \eta_k]\ (p_k \in P_{G_k}(Y_k), \eta_k: S_{p_k} \rightarrow A_k S_{p_{k-1}})
\]

such that the following equalities hold:

i) \(p_k = q^{n-1}\),

ii) \(\psi_{(D, p^{n-1})} = \psi^{n-1}\),

iii) \(Z^{n-1} = Z_{(D, p^{n-1})}\),

iv) \(\Theta^{n-1} = \Theta_{(D, p^{n-1})}\),

v) \(\Omega^{n-1} = \Omega_{(D, p^{n-1})}\).

Because of the transition \(K_{n-1} = \kappa_{\eta} K_n\) there are mappings \(\kappa_i: [Z^{n-1}]_i \rightarrow \Sigma_B(i)\) \((i=0, \ldots, k)\) which satisfy condition (1) in Definition 3.3.

Take the sequence \(r_0, \ldots, r_k\) of quasi trees given as follows:

\[
r_0 = p^n, \quad \text{for} \ i = 2, \ldots, k \quad \text{let} \ r_i = p_i[S_{p_i}, \varepsilon_i],
\]

where \(\varepsilon_i: S_{p_i} \rightarrow T_{G_i}(Y_i \cup N^*)\) such that for every \(s \in S_{p_i}, \varepsilon_i(s) = \omega_i(u_i)\) holds, where \(u_i[S_{u_i}, \varphi_i]\) is the third component of

\[
\kappa_i(s) = (b_i, \ldots, b_1, u_0, u_i[S_{u_i}, \varphi_i], q_i, W_i, \tau_i).
\]
Let $\xi_i: S_{r_i} \rightarrow A_i S_{r_i-1}, \ (i \in \{1, ..., k\})$ be the mapping satisfying the following requirements: we know that for each $s_i \in S_{r_i}$ there is a unique decomposition $s_i=s_i t_i$ of $s_i$, where $s_i \in S_{p_i}$, $\pi_i(s_i) = (b_i \ldots b_1, u_0, u_i [S_{u_i}, \phi_i], q_i, W_i, \tau_i)$ and $t_i \in S_{u_i}$. If $\gamma(t_i) = (t_0, \ldots, t_{i-1}, t_i) \in W_i$ and $\tau_i(t_0, \ldots, t_{i-1}, t_i) = c_1 \ldots c_{i-1} t_0$ for some $c_1 \in A_1, \ldots, c_i \in A_i$, and $\eta_i(s_i) = b_i s_{i-1}$ for some $s_{i-1} \in S_{p_i-1}$, then $\xi_i(s_i t_i) = c_1 c_2 \ldots t_1 t_i$. We obtain that for $i = 1, \ldots, k$, $E_i: a_i r_{i-1} = \xi_i^* r_i [S_{r_i}, \xi_i]$ holds.

From the definition of the transition in $\mathcal{B}$ and from the definitions of $r_k$, $Z_{(E, p^* n)}$, $\Theta_{(E, p^* n)}$, $\Omega_{(E, p^* n)}$, $\psi_{(E, p^*)}$ it follows that

i) $r_k = q^n$,

ii) $\psi_{(E, p^* n)} = \psi^n$,

iii) $Z^n = Z_{(E, p^* n)}$,

iv) $\Theta^n = \Theta_{(E, p^* n)}$,

v) $\Omega^n = \Omega_{(E, p^* n)}$.

Assume that $(p, q) \in \tau_{\mathcal{B}}$. Then there are configurations $K_0, \ldots, K_n \ (n \geq 1)$ such that $K_0$ is a starting configuration, $K_0 = (e \in \{e\}, \psi^0, \Theta^0, Z^0, \Omega^0)$ where $\psi^0(e) = a_k \ldots a_1 p$ for some $a_k \in A_k', \ldots, a_1 \in A_1'$, $K_n$ is a final configuration, $K_n = (q, \emptyset, \emptyset, \emptyset)$, moreover, $K_{i-1} = \pi_i K_i$ holds for $i = 1, \ldots, n$.

According to the above proposition there is a derivation sequence $D = D_1, \ldots, D_k$,

\begin{align*}
D_1: & \quad a_1 p^i \Rightarrow^*_{\mathcal{B}_1} p_1 [S_{p_1}, \eta_1], (p_1 \in P_{G_1} (Y_1), \eta_1: S_{p_1} \rightarrow A_1 S_{p_1}), \\
D_2: & \quad a_2 p_1 \Rightarrow^*_{\mathcal{B}_2} p_2 [S_{p_2}, \eta_2], (p_2 \in P_{G_2} (Y_2), \eta_2: S_{p_2} \rightarrow A_2 S_{p_2}), \\
& \vdots \\
D_k: & \quad a_k p_{k-1} \Rightarrow^*_{\mathcal{B}_k} p_k [S_{p_k}, \eta_k], (p_k \in P_{G_k} (Y_k), \eta_k: S_{p_k} \rightarrow A_k S_{p_k-1})
\end{align*}

such that the following equalities hold:

i) $p_k = q$,

ii) $\psi_{(D, p^* n)} = \psi^n = \emptyset$,

iii) $Z^n = Z_{(D, p^* n)}$,

iv) $\Theta^n = \Theta_{(D, p^* n)}$,

v) $\Omega^n = \Omega_{(D, p^* n)}$.

Thus $Z_{(D, p^* n)} = 0, \Theta_{(D, p^* n)} = 0, \Omega_{(D, p^* n)} = 0$. According to the definition of $Z_{(D, p^* n)}$, $S_{p_i} = [Z_{(D, p^* n)}]_i, \ (i = 1, \ldots, k)$. Thus $P_i \in T_{G_i} (Y_i)$ for $i = 1, \ldots, k$. One can see that $a_i p^i [S_{p_i}, \gamma] = \#_{\mathcal{B}_i} P_i$ holds. Thus $(p^i [S_{p_i}, \gamma], q) \in \tau_{\mathcal{B}_1} \circ \tau_{\mathcal{B}_2} \circ \cdots \circ \tau_{\mathcal{B}_k}$. The proof of the theorem is complete.

**Theorem 3.6.** Let $\mathcal{B} = (G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A_1', \ldots, A_k', \Sigma_{\mathcal{B}}, \mathcal{V})$ be a $k$-synchronized $R$-transducer. Then there are $R$-transducers $\mathcal{A}_1, \ldots, \mathcal{A}_k$ such that $\tau_{\mathcal{B}} = \tau_{\mathcal{A}_1} \circ \cdots \circ \tau_{\mathcal{A}_k}$.
Proof. The production sets \( \Sigma_0(i) \) for \( i = 1, \ldots, k-1 \) are considered to be operator domains with arity function \( v^i: \Sigma_0(i) \rightarrow \{0, 1, 2, \ldots\} \) as follows: for \( 1 \leq i \leq k-1 \),

\[
\sigma = (b_1 \ldots b_1, u_0, u_i | S_{u_i} | \varphi_i, \varphi_i, W_i, \tau_i) \in \Sigma_0(i)
\]

let \( v^i(\sigma) = |S_{u_i}| \), where \( |S_{u_i}| \) denotes the cardinality of the set \( S_{u_i} \).

Remember, the arity function of the operator domain \( G_0 \) is denoted by \( v \).

Convention: Let \( S \subseteq N^* \). If \( S \neq \emptyset \), then \( \xi_0, \xi_1: S \rightarrow S \) denote the identity function. If \( S = \emptyset \), then \( \xi_0, \xi_1: S \rightarrow S \) denote the empty function.

For every \( j \) \( (1 \leq j \leq k) \) if \( S \neq \emptyset \) then \( \xi_j: S \rightarrow \{1, \ldots, |S|\} \) denotes the function whose value on \( s \in S \) is the ordinal number of \( s \) in \( S \) with respect to the lexicographic ordering. If \( S = \emptyset \) then \( \xi_j: S \rightarrow S \) denotes the empty function. Thus \( \xi_j \) always denotes a bijective function which is determined by its domain.

Take the \( R \)-transducer

\[
\Omega_1 = (G_0, Y_0, A_1, \Sigma_0(1), \emptyset, \Sigma_{a_1}, A_1),
\]

where

\[
\Sigma_{a_1} = \{ b_1u_0 \rightarrow \sigma_1(1, \ldots, v^i(\sigma_1)) \} (1, \ldots, v^i(\sigma_1)), \beta_1] \]

\( \sigma_1 \) has the form \((b_1, u_0, u_i | S_{u_i} | \varphi_i, \varphi_i, W_i, \tau_i) \in \Sigma_0(1) \),

\( (\sigma_0, (b_1, u_0, u_i | S_{u_i} | \varphi_i, \varphi_i, W_i, \tau_i)) \in V_1 \) and the mapping \( \beta_1: \{1, \ldots, v^i(\sigma_1)\} \rightarrow A_1 \{1, \ldots, v(\sigma_0)\} \) is defined as follows:

Let \( \xi_0: \{1, \ldots, v(\sigma_0)\} \rightarrow \{1, \ldots, v(\sigma_0)\} \), \( \xi_1: S_{u_i} \rightarrow \{1, \ldots, |S_{u_i}|\} \).

For each \( t_i \in S_{u_i} \), \( \beta_1(\xi_1(t_i)) = c_1 \xi_0(t_i) \) iff \( \varphi_i(t_i) = (t_0, t_i) \) and \( \tau_i((t_0, t_i)) = c_1 t_0 \).

Thus for each \( t_i \in S_{u_i} \), \( \beta_1(\xi_1(t_i)) = c_1 \xi_0(t_i) \) iff \( \varphi_i(t_i) = c_1 t_0 \).

For \( j = 2, \ldots, k-1 \) consider the \( R \)-transducer \( \Omega_j = (\Sigma_0(j-1), \emptyset, A_j, \Sigma_{a_j}, A_j^j) \), where the production set \( \Sigma_{a_j} \) is defined as follows:

\[
\Sigma_{a_j} = \{ b_j \sigma_{j-1} \rightarrow \sigma_j(1, \ldots, v^j(\sigma_j)) \} (1, \ldots, v^j(\sigma_j)), \beta_j] \]

There is an element \((\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in V \) such that \( \sigma_{j-1} \) has the form \((b_{j-1} \ldots b_1, u_0, u_{j-1} | S_{u_{j-1}} | \varphi_{j-1}, \varphi_{j-1}, W_{j-1}, \tau_{j-1}) \).

\( \sigma_j \) has the form \((b_j \ldots b_1, u_0, u_j | S_{u_j} | \varphi_j, \varphi_j, W_j, \tau_j) \).

There is a mapping \( e_j: S_{u_j} \rightarrow A_j \{W_{j-1}\} \) such that conditions i)—iv) in part (5.b) of Definition 3.1 hold.

The mapping \( \beta_j: \{1, \ldots, v^j(\sigma_j)\} \rightarrow A_j \{1, \ldots, v^{j-1}(\sigma_{j-1})\} \) is defined as follows:

Let \( \xi_{j-1}: S_{u_{j-1}} \rightarrow \{1, \ldots, |S_{u_{j-1}}|\} \), \( \xi_j: S_{u_j} \rightarrow \{1, \ldots, |S_{u_j}|\} \). For every \( t_j \in S_{u_j} \), \( \beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})(c_j \in A_j, t_{j-1} \in S_{u_{j-1}}) \) iff \( \varphi_j(t_j) = (t_0, \ldots, t_{j-1}, t_j) \) and \( \tau_j((t_0, \ldots, t_{j-1}, t_j)) = c_j t_0 \) (Thus for every \( t_j \in S_{u_j} \), \( \beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})(c_j \in A_j, t_{j-1} \in S_{u_{j-1}}) \) iff \( \varphi_j(t_j) = c_j t_0 \)).

Take the \( R \)-transducer \( \Omega_k = (\Sigma_0(k-1), \emptyset, A_k, G_k, Y_k, \Sigma_{a_k}, A_k^k) \) where the production set \( \Sigma_{a_k} \) is defined as follows:

\[
\Sigma_{a_k} = \{ b_k \sigma_{k-1} \rightarrow u_k | S_{u_k} | \beta_k \}
\]

There is an element \((\sigma_0, \ldots, \sigma_{k-1}, \sigma_k) \in V \) such that \( \sigma_{k-1} \) has the form \((b_{k-1} \ldots b_1, u_0, u_{k-1} | S_{u_{k-1}} | \varphi_{k-1}, \varphi_{k-1}, W_{k-1}, \tau_{k-1}) \).
σ_k has the form \((b_k ... b_1, u_0, u_i[S_u], \varphi), \varepsilon_k, W_k, \tau_k\). There is a mapping \(\varepsilon_k: S_{u_k} \rightarrow A_k[\sigma_k]_{-1}k-1\) such that conditions i)—iv) in part (5) of Definition 3.3 hold. The mapping \(\beta_k: S_{u_k} \rightarrow A_k(1, ... , v^k-(\sigma_k-i))\) is defined as follows: Let \(\xi_{k-1}: S_{u_{k-1}} \rightarrow \{1, ... , |S_{u_{k-1}}|\} \), \(\xi_k: S_{u_k} \rightarrow S_{u_k}\). For every \(t_k \in S_{u_k}\), \(\beta_k(t_k) = c_k \xi_{k-1}(t_{k-1})\) iff \(\varphi_k(t_k) = (t_0, ... , t_{k-1}, t_k)\) and \(\tau_k((t_0, ... , t_{k-1}, t_k)) = (c_k, ... , t_0).\) Thus for every \(t_k \in S_{u_k}\), \(\beta_k(t_k) = c_k \xi_{k-1}(t_{k-1})(c_k \in A_k, t_{k-1} \in S_{u_{k-1}})\)

\(\varepsilon_k(t_k) = c_k(t_k).\)

We may assume without loss of generality that for

\(i = 2, ... , k-1, A_i \cap T_{\Xi} (1)(\emptyset) = \emptyset, A_i \cap T_{\Xi} (0)(\emptyset) = \emptyset, A_k \cap T_{\Xi} (1)(\emptyset) = \emptyset.\)

Thus \(\mathfrak{U}_1, ... , \mathfrak{U}_k\) satisfy requirement (2) of Definition 1.10.

We shall prove that \(\tau_{\Xi} = \tau_{\Xi} \circ \tau_{\Xi} \circ ... \circ \tau_{\Xi}\). Let \(\mathcal{C}\) be the \(k\)-synchronized \(R\)-transducer that can be obtained from \(\mathfrak{U}_1, ... , \mathfrak{U}_k\) by the construction of Theorem 3.5.

In this case

\[\mathcal{C} = (G_0, \Sigma_\Xi(1), ... , \Sigma_\Xi(k-1), G_k, Y_0, \emptyset, ... , \emptyset, Y_k, A_1, ... , A_k, A'_1, ... , A'_k, \Sigma_\varepsilon, \overline{V}).\]

We may assume without loss of generality that for \(i = 1, ... , k, A_i \cap T_{\Xi}(Y_i) = \emptyset\)

and \(A_i \cap T_{\Xi}(\emptyset) = \emptyset, A_i \cap T_{\Xi}(\emptyset) = \emptyset, A \cap T_{\Xi}(\emptyset) = \emptyset.\)

Thus \(\mathfrak{U}_1, ... , \mathfrak{U}_k\) satisfy the requirements of Definition 3.1.

By Theorem 3.5, \(\tau_{\Xi} = \tau_{\Xi} \circ \tau_{\Xi} \circ ... \circ \tau_{\Xi}\), so it is sufficient to prove that \(\tau_{\Xi} = \tau_{\Xi}\).

In order to prove this equality we shall introduce bijective mappings \(\gamma_j: \Sigma_\Xi(j) \rightarrow \Sigma_\Xi(j)\) for \(j = 0, ... , k-1\) and a surjective mapping \(\gamma_k: \Sigma_\Xi(k) \rightarrow \Sigma_\Xi(k)\), and we shall show that for \(j = 0, ... , k\) the mappings \(\gamma_0, ... , \gamma_j\) satisfy assumption (1) and that for \(j = 0, ... , k\) the mapping \(\gamma_j\) satisfies assumption (2).

1. There are two cases.

Case 1. \(0 \leq j \leq k-1\). In this case if \((\sigma_0, ... , \sigma_j) \in V_j\) then \((\gamma_0(\sigma_0), ... , \gamma_j(\sigma_j)) \in \overline{V}_j\), and if \((\overline{\sigma}_0, ... , \overline{\sigma}_j) \in \overline{V}_j\) then \((\gamma_0^j(\overline{\sigma}_0), ... , \gamma_j^j(\overline{\sigma}_j)) \in \overline{V}_j\).

Case 2. \(j = k\). In this case if \((\sigma_0, ... , \sigma_k) \in V_k\) then \((\gamma_0(\sigma_0), ... , \gamma_k(\sigma_k)) \in \overline{V}_k\), and if \((\overline{\sigma}_0, ... , \overline{\sigma}_{k-1}, \overline{\sigma}_k) \in \overline{V}_k\) then there is a unique \(\sigma_k \in \Sigma_\Xi(k)\) such that \(\gamma_k(\sigma_k) = \overline{\sigma}_k\) and \((\gamma_0^j(\overline{\sigma}_0), ... , \gamma_k^{k-1}(\overline{\sigma}_{k-1}), \sigma_k) \in V_k\).

2. There are three cases.

Case 1. \(j = 0\). In this case \(\Sigma_\Xi(0) = \Sigma_\Xi(0)\) and \(\gamma_0\) is the identity function.

Case 2. \(1 \leq j \leq k\). Let \((\sigma_0, ... , \sigma_j) \in V_j\) and \(\sigma_0 = u_0\). Assume that \(\sigma_i (1 \leq i \leq j)\) has the form \((b_i ... b_1, u_0, u_i[S_u], \varphi), \varepsilon_i, W_i, \tau_i)\). Let \(\xi_i: \{1, ... , v(u_0)\} = [W]_0 \rightarrow \{1, ... , v(u_0)\}, \xi_i: S_u = [W]_i \rightarrow \{1, ... , |S_u|\} \) for \(i = 1, ... , j\). Then \(\gamma_j(\sigma_j)\) has the form \(\gamma_j(\sigma_j) = (b_j ... b_1, u_0, \sigma_j(1, ... , v(\sigma_j)), \{1, ... , v(\sigma_j), \varphi), \varepsilon_j, W_j, \tau_j)\) and the following hold:

i) \([W]_0 = [W]_0\) and for \(i = 1, ... , j-1, [W]_i = \{1, ... , |W]_i|\} = \{1, ... , |S_u|\} = \overline{W} \varepsilon_i(\xi_i)\).

ii) \((t_0, t_1, ... , t_i) \in W_j\) iff \((\xi_0(t_0), \xi_1(t_1), ... , \xi_i(t_i)) \in \overline{W}_j (1 \leq i \leq j)\)

\((t_0 \in N^*, t_1 \in N^*, ... , t_i \in N^*)\).
iii) For every
\[ t_j \in S_{u_j}, \varphi_j(t_j) = c_j \ldots c_1 t_0 \text{ iff } \bar{\varphi}_j(\xi_j(t_j)) = c_j \ldots c_1 \xi_0(t_0), \]
\((t_0 \in S_{u_0}, c_1 \in A_1, \ldots, c_j \in A_j).\)

iv) For each
\[ t_j \in S_{u_j}, q_j(t_j) = (t_0, t_1, \ldots, t_j) \text{ iff } \bar{q}_j(\xi_j(t_j)) = (\xi_0(t_0), \xi_1(t_1), \ldots, \xi_j(t_j)). \]

v) For every
\[ (t_0, t_1, \ldots, t_l) \in W_j \text{ (} 1 \leq l \leq j), \tau_j((t_0, t_1, \ldots, t_l)) = c_l \ldots c_1 t_0 \text{ iff } \]
\[ \tau_j((\xi_0(t_0), \xi_1(t_1), \ldots, \xi_l(t_l))) = c_l \ldots c_1 \xi_0(t_0), \]
\((t_0 \in \{1, \ldots, v(u_0)\}, c_1 \in A_1, \ldots, c_j \in A_j).\)

**Case 3.** \(j = k.\) Let \((\sigma_0, \sigma_1, \ldots, \sigma_k) \in V_k\) and \(\sigma_0 = u_0.\)

Assume that \(\sigma_l \ (1 \leq l \leq k)\) has the form
\[ (b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], q_l, W_l, \tau_l). \]

Let
\[ \xi_0: \{1, \ldots, v(u_0)\} = [W_k]_0 \rightarrow \{1, \ldots, v(u_0)\}, \]
\[ \xi_l: S_{u_l} = [W_k]_l \rightarrow \{1, \ldots, |S_{u_l}| \} \text{ for } l = 1, \ldots, k-1, \xi_k: S_{u_k} = [W_k]_k = S_{u_k}. \]

Then \(\gamma_k(\sigma_k)\) has the form \(\gamma_k(\sigma_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar{\varphi}_k, \bar{W}_k, \bar{\tau}_k)\) and the following hold:

i) \([W_k]_0 = [W_k]_0, \text{ for } i = 1, \ldots, k-1, [W_k]_i = \{1, \ldots, [W_k]_i\} = \{1, \ldots, |S_{u_i}| \} = \text{rg} (\xi_i) \]
and \([W_k]_k = [W_k]_k = S_{u_k} = \text{rg} (\xi_k).\)

ii) \((t_0, t_1, \ldots, t_l) \in W_k \text{ iff } (\xi_0(t_0), \xi_1(t_1), \ldots, \xi_l(t_l)) \in W_k \]
\((1 \leq l \leq k, t_0, t_1, \ldots, t_l \in N^*).\)

iii) For every
\[ t_k \in S_{u_k}, \varphi_k(t_k) = c_k \ldots c_1 t_0 \text{ iff } \bar{\varphi}_k(\xi_k(t_k)) = c_k \ldots c_1 \xi_0(t_0), \]
\((t_0 \in S_{u_0}, c_1 \in A_1, \ldots, c_k \in A_k).\)

iv) For each
\[ t_k \in S_{u_k}, q_k(t_k) = (t_0, t_1, \ldots, t_k) \text{ iff } \bar{q}_k(\xi_k(t_k)) = (\xi_0(t_0), \xi_1(t_1), \ldots, \xi_k(t_k)). \]

v) For every
\[ (t_0, t_1, \ldots, t_l) \in W_k \text{ (} 1 \leq l \leq k), \tau_k((t_0, t_1, \ldots, t_l)) = c_l \ldots c_1 t_0 \text{ iff } \]
\[ \tau_k((\xi_0(t_0), \xi_1(t_1), \ldots, \xi_l(t_l))) = c_l \ldots c_1 t_0, \]
\((t_0 \in \{1, \ldots, v(u_0)\}, c_1 \in A_1, \ldots, c_k \in A_k).\)

We shall define mappings \(\gamma_j: \Sigma_{\pi}(j) \rightarrow \Sigma_{\sigma}(j)\) according to the construction of \(\Sigma_{\pi}(j)\) and \(\Sigma_{\sigma}(j).\)
Let $j = 0$. Since $\Sigma_0(0) = \Sigma_0(0)$, let $\gamma_0$ be the identity mapping.

Let $j = 1$. In this case

$\Sigma_0(1) = \{ (b_1, u_0, \sigma_1(1, \ldots, v^1(\sigma_1)), \{1, \ldots, v^1(\sigma_1)\}, \bar{\sigma_1}, W_1, \bar{\epsilon}_1) \}$

i) $(u_0, \sigma_1) \in V$ such that $\sigma_1$ has the form $(b_1, u_0, u_1[S_{u1}, \varphi_1], \bar{\sigma_1}, W_1, \bar{\epsilon}_1)$.

ii) $\gamma_0(u_0) = u_0$, the production

$$b_1 u_0 \rightarrow \sigma_1(1, \ldots, v^1(\sigma_1))\{1, \ldots, v^1(\sigma_1)\}, \beta_1 \in \Sigma_{u1},$$

where the mapping $\beta_1: \{1, \ldots, v^1(\sigma_1)\} \rightarrow A_1 \{1, \ldots, v(u_0)\}$ is defined as follows:

Let

$$\xi_0: \{1, \ldots, v(u_0)\} \rightarrow \{1, \ldots, v(u_0)\}, \xi_1: S_{u1} \rightarrow \{1, \ldots, |S_{u1}|\}.$$

For each

$$t_1 \in S_{u1}, \beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0) \text{ iff } \varphi_1(t_1) = (t_0, t_1) \text{ and } \tau_1((t_0, t_1)) = c_1 t_0.$$

(Thus for each $t_1 \in S_{u1}$, $\beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0)$ iff $\varphi_1(t_1) = c_1 t_0$.)

iii) $\bar{\sigma_1} = \beta_1$.

iv) $\bar{\epsilon}_1: \{1, \ldots, v^1(\sigma_1)\} \rightarrow W_1$;

for every

$$\xi_1(t_1) \in \{1, \ldots, v^1(\sigma_1)\}, \text{ if } \beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0) \text{ (}c_1 \in A_1, t_0 \in \{1, \ldots, v(u_0)\}\text{) then}

$$\bar{\sigma_1}(\xi_1(t_1)) = (\xi_0(t_0), \xi_1(t_1)).$$

v) $\bar{\epsilon}_1: W_1 \rightarrow A_1 \{1, \ldots, v(u_0)\}$;

for every

$$((\xi_0(t_0), \xi_1(t_1)) \in W_1, \text{ if } \beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0) \text{ (}c_1 \in A_1, t_0 \in \{1, \ldots, v(u_0)\}\text{) then}

$$\bar{\epsilon}_1(\xi_0(t_0), \xi_1(t_1)) = c_1 \xi_0(t_0).$$

It can be seen that

$$\bar{\nu}_1 = \{(y_0(\sigma_0), \bar{\sigma_1})|\sigma_0 = u_0 \in \Sigma_0(0), \bar{\sigma_1} \in \Sigma_0(1)\}$$

has the form

$$(b_1, u_0, \sigma_1(1, \ldots, v^1(\sigma_1)), \{1, \ldots, v^1(\sigma_1)\}, \bar{\sigma_1}, W_1, \bar{\epsilon}_1)$$

and $\bar{\sigma_1}$ is generated by the production

$$b_1 u_0 \rightarrow \sigma_1(1, \ldots, v^1(\sigma_1))\{1, \ldots, v^1(\sigma_1)\}, \beta_1 \in \Sigma_{u1}.$$

We define $\gamma_1: \bar{\Sigma}_0(1) \rightarrow \Sigma_1(1)$ as follows:

Let $\sigma_1 = (b_1, u_0, u_1[S_{u1}, \varphi_1], \bar{\sigma_1}, W_1, \bar{\epsilon}_1) \in \Sigma_1(1)$, then by the construction of $\mathfrak{A}_1$ and $C$ there is a unique production $b_1 u_0 \rightarrow \sigma_1(1, \ldots, v^1(\sigma_1))\{1, \ldots, v^1(\sigma_1)\}, \beta_1 \in \Sigma_{u1}$ which generates a unique $\bar{\sigma_1} \in \Sigma_1(1)$. We define $\gamma_1(\sigma_1)$ to be $\bar{\sigma_1}$. One can see by the definition of $\Sigma_1(1)$ that $\gamma_1$ is onto, hence $\gamma_1$ is bijective.

It is routine work to check according to the construction of $\mathfrak{A}_1$ and $\Sigma_1(1)$ that $\gamma_0, \gamma_1$ satisfy condition (1) and that $\gamma_1$ satisfies condition (2).

Let $j$ be an index between 2 and $k - 1$. We can assume that $\Sigma_0(0), \Sigma_1(1), \ldots, \Sigma_j(j - 1)$ and $\gamma_0, \ldots, \gamma_{j - 1}$ are defined such that $\gamma_0, \ldots, \gamma_{j - 1}$ satisfy condition (1) and that $\gamma_{j - 1}$ satisfies condition (2).
We know that $E_{a}(J)$ is the set $E_{a}(J) = \{(b_{j-1} \ldots b_{1}, u_{0}, \sigma_{j}(1, \ldots, \nu^{j}(\sigma_{j}))[\{1, \ldots, \nu^{j}(\sigma_{j})\}, \varphi_{j}], \bar{q}_{j}, W_{j}, \tau_{j}\}$

i) There is an element $(\sigma_{0}, \ldots, \sigma_{j-1}, \sigma_{j}) \in V$ such that $\sigma_{0}=u_{0}, \sigma_{j-1}$ has the form $(b_{j-1} \ldots b_{1}, u_{0}, u_{j-1}[S_{uj-1}, \varphi_{j-1}], \varphi_{j-1}, W_{j-1}, \tau_{j-1}),$ $\sigma_{j}$ has the form $(b_{j-1} \ldots b_{1}, u_{0}, u_{j}[S_{uj}, \varphi_{j}], \varphi_{j}, W_{j}, \tau_{j}).$ There is a mapping $e_{j}: S_{uj} \to A_{j}[W_{j-1}]_{j-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold.

ii) $\gamma_{j}(\sigma_{j-1}) = (b_{j-1} \ldots b_{1}, u_{0}, \sigma_{j-1}(1, \ldots, \nu^{j-1}(\sigma_{j-1}))[\{1, \ldots, \nu^{j-1}(\sigma_{j-1})\}, \varphi_{j-1}],$ $\bar{q}_{j-1}, W_{j-1}, \tau_{j-1}) \in E_{a}(j-1)$

and the production $b_{j} \sigma_{j-1} \to \sigma_{j}(1, \ldots, \nu^{j}(\sigma_{j}))[\{1, \ldots, \nu^{j}(\sigma_{j})\}, \beta_{j}]$ is in $\Sigma_{ui}$, where the mapping $\beta_{j}: \{1, \ldots, \nu^{j}(\sigma_{j})\} \to A_{j}[\{1, \ldots, \nu^{j}(\sigma_{j})\}]$ is defined as follows:

Let $\xi_{j-1}: S_{uj-1} \to \{1, \ldots, |S_{uj-1}|\}$, $\xi_{j}: S_{uj} \to \{1, \ldots, |S_{uj}|\}$. For every $t_{j} \in S_{uj}$, $\beta_{j}(t_{j})=c_{j}\xi_{j-1}(t_{j-1})$ iff $\xi_{j}(t_{j})=(t_{0}, \ldots, t_{j-1}, t_{j})$ and $\tau_{j}(t_{0}, \ldots, t_{j-1}, t_{j})=c_{j}c_{t_{j}0}$. (Thus for each $t_{j} \in S_{uj}$, $\beta_{j}(\xi_{j}(t_{j}))=c_{j}\xi_{j-1}(t_{j-1})$ iff $\xi_{j}(t_{j})=c_{j}t_{j-1}$.)

iii) $\bar{q}_{j} = \{(t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}), \xi_{j}(t_{j}))|\beta_{j}(\xi_{j}(t_{j})) = c_{j}\xi_{j-1}(t_{j-1}),$ $\bar{q}_{j-1}(\xi_{j-1}(t_{j-1})) = (t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}))\} \cup$ $\cup \{(t_{0}, \ldots, t_{j-1}) \in W_{j-1} | \text{there are no } t_{j} \text{ in } \{1, \ldots, \nu^{j}(\sigma_{j})\} \text{ and } c_{j} \in A_{j} \text{ such that } \beta_{j}(t_{j}) = c_{j}t_{j-1}\} \cup$ $\cup \{(t_{0}, \ldots, t_{j}) \in W_{j-1} | 1 \equiv l \equiv j-2\}.$

iv) $\bar{q}_{j}: \{1, \ldots, \nu^{j}(\sigma_{j})\} \to W_{j}$ satisfies the following requirement: for every $t_{j} \in S_{uj}$ if $\beta_{j}(\xi_{j}(t_{j})) = c_{j}\xi_{j-1}(t_{j-1})$ and $\bar{q}_{j-1}(\xi_{j-1}(t_{j-1})) = (\xi_{0}(t_{0}), \ldots, \xi_{j-1}(t_{j-1}))$ then $(\xi_{0}(t_{0}), \ldots, \xi_{j-1}(t_{j-1}), \xi_{j}(t_{j})) \in W_{j}$ and $\bar{q}_{j}(\xi_{j}(t_{j})) = (\xi_{0}(t_{0}), \ldots, \xi_{j-1}(t_{j-1}), \xi_{j}(t_{j})).$

v) For $\bar{t}_{j}: W_{j} \to A_{j} \ldots A_{1}\{1, \ldots, v(u_{0})\}$,

$\bar{t}_{j}|W_{j} \cap W_{j-1} = \bar{t}_{j}|W_{j} \cap W_{j-1}$ and if $$(t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}), \xi_{j}(t_{j})) \in W_{j},$$

$\beta_{j}(\xi_{j}(t_{j})) = c_{j}\xi_{j-1}(t_{j-1})$ and $\bar{q}_{j-1}(\xi_{j-1}(t_{j-1})) = (t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}))$ then $\bar{t}_{j}(t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}), \xi_{j}(t_{j})) = c_{j}\bar{t}_{j-1}(t_{0}, \ldots, t_{j-2}, \xi_{j-1}(t_{j-1}))$. 
vi) \( \overline{\phi}_j = \overline{\phi}_j \circ \overline{\tau}_j. \)

It can be seen that

\[ \mathcal{V}_j = \{(\gamma_0(\sigma_0), \ldots, \gamma_{j-1}(\sigma_{j-1}), \overline{\phi}_j) \} \quad \text{for} \quad i = 0, \ldots, j-1 \]

\( \sigma_i \in \Sigma_B(i), (\gamma_0(\sigma_0), \ldots, \gamma_{j-1}(\sigma_{j-1})) \in \mathcal{V}_{j-1}, \)

\( \sigma_0 \) has the form \( u_0, \) and \( \sigma_{j-1} \) has the form \( (b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varphi_{j-1}, W_{j-1}, \tau_{j-1}) \). There is an element

\[ \sigma_j = (b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varphi_j, W_j, \tau_j) \in \Sigma_B(j) \]

such that \( (\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in \mathcal{V}_j \), and there is a mapping \( \varepsilon_j: S_{u_j} \rightarrow A_j[W_{j-1}]_{j-1} \) such that conditions i)—iv) in part (5).b of Definition 3.1. hold.

\[ \overline{\sigma}_j = (b_j, \ldots, b_1, u_0, \sigma_j(1, \ldots, v^j(\sigma_j))[\{1, \ldots, v^j(\sigma_j)\}, \overline{\phi}_j, \overline{\tau}_j] \]

satisfies the requirements ii)—vi) of \( \Sigma_B(j). \}

We define \( \gamma_j: \Sigma_B(j) \rightarrow \Sigma_B(j) \) as follows: Let us consider the set

\[ \Gamma_j = \{\gamma: \Sigma_B(j) \rightarrow \Sigma_B(j) \mid \text{for each} \quad \sigma_j \in \Sigma_B(j), \gamma(\sigma_j) = \overline{\sigma}_j \}

has the form

\[ (b_j \ldots b_1, u_0, \sigma_j(1, \ldots, v^j(\sigma_j))[\{1, \ldots, v^j(\sigma_j)\}, \overline{\phi}_j, \overline{\tau}_j] \]

and there is a vector \( (\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in \mathcal{V}_j \) such that \( \sigma_0 \) has the form \( u_0, \sigma_{j-1} \) has the form \( (b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varphi_{j-1}, W_{j-1}, \tau_{j-1}) \), \( \sigma_j \) has the form \( (b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varphi_j, W_j, \tau_j) \), and there is a mapping \( \varepsilon_j: S_{u_j} \rightarrow A_j[W_{j-1}]_{j-1} \) such that conditions i)—iv) in part (5).b of Definition 3.1 hold, and \( \overline{\sigma}_j \) satisfies the requirements ii)—vi) of \( \Sigma_B(j). \}

One can see that if \( \overline{\gamma}_j \in \Gamma_j \) then \( \overline{\gamma}_j \) is injective and \( \overline{\gamma}_j \) satisfies condition (2) because of the construction of \( \mathcal{U}_j \) and \( \Sigma_B(j) \). Using this fact one can see that \( |\Gamma_j| = 1. \)

Let \( \overline{\gamma}_j \) be the only element of \( \Gamma_j \). One can see that \( \overline{\gamma}_j \) is bijective, and \( \gamma_0, \ldots, \gamma_{j-1} \) satisfy condition (1). Let \( j = k \). We can assume that \( \Sigma_B(0), \Sigma_B(1), \ldots, \Sigma_B(k-1) \) and \( \gamma_0, \ldots, \gamma_{k-1} \) are defined such that \( \gamma_0, \ldots, \gamma_{k-1} \) satisfy condition (1) and that \( \gamma_{k-1} \) satisfies condition (2).

We know that \( \Sigma_B(k) \) is the set

\[ \Sigma_B(k) = \{(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \overline{\phi}_k, \overline{\tau}_k)\} \]

i) there is an element \( (\sigma_0, \ldots, \sigma_{k-1}, \sigma_k) \in \mathcal{V}_k \) such that \( \sigma_0 = u_0, \sigma_{k-1} \) has the form \( (b_{k-1} \ldots b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \overline{\phi}_{k-1}, \overline{\tau}_{k-1}) \), \( \sigma_k \) has the form \( (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \overline{\phi}_k, \overline{\tau}_k) \). There is a mapping \( \varepsilon_k: S_{u_k} \rightarrow A_k[W_{k-1}]_{k-1} \) such that conditions i)—iv) in part (5).b of Definition 3.1 hold.

\[ \gamma_{k-1}(\sigma_{k-1}) = (b_{k-1} \ldots b_1, u_0, \sigma_{k-1}(1, \ldots, v^{k-1}(\sigma_{k-1}))[\{1, \ldots, v^{k-1}(\sigma_{k-1})\}, \overline{\phi}_{k-1}], \overline{\phi}_{k-1}, \overline{\tau}_{k-1}) \in \Sigma_B(k-1) \]
and the production
\[ b_k \sigma_{k-1} \rightarrow u_k [S_{u_k}, \beta_k] \] is in \( S_{u_k} \),
where the mapping
\[ \beta_k: S_{u_k} \rightarrow A_k \{1, \ldots, v^{k-1}(\sigma_{k-1})\} \]
is defined as follows: Let
\[ \xi_{k-1}: S_{u_{k-1}} \rightarrow \{1, \ldots, |S_{u_{k-1}}|\}, \xi_k: S_{u_k} \rightarrow S_{u_k} \]
For every
\[ t_k \in S_{u_k}, \beta_k(t_k) = c_k \xi_{k-1}(t_{k-1}) \text{ iff } \xi_k(t_k) = (t_0, \ldots, t_{k-1}, t_k) \text{ and } \]
\[ t_k((t_0, \ldots, t_{k-1}, t_k)) = c_k \ldots c_1 t_0. \]
(Thus for each \( t_k \in S_{u_k}, \beta_k(\xi_{k-1}(t_{k-1})) = c_k \xi_{k-1}(t_{k-1}) \text{ iff } \xi_k(t_k) = c_k t_{k-1}. \)

iii) \( W_k = \{ (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) | \beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1}), \xi_k(\xi_k(t_k)) = (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1})) \} \}
\[ \cup \{(t_0, \ldots, t_{i-2}, t_{i-1}) \in W_{k-1} | \text{ there are no } t_k \in S_{u_k} \text{ and } c_k \in A_k \text{ such that } \beta_k(t_i) = c_k t_{i-1} \} \cup \]
\[ \{(t_0, \ldots, \tilde{t}_{i-1}, t_i) \in W_{k-1} | 1 \leq i \leq k-2 \}. \]

iv) \( \tilde{\beta}_k: S_{u_k} \rightarrow W_k \) satisfies the following requirement: for every
\[ t_k \in S_{u_k}, \text{ if } \beta_k(\xi_{k-1}(t_{k-1})) = c_k \xi_{k-1}(t_{k-1}) \text{ and } \]
\[ \tilde{\beta}_{k-1}(\xi_{k-1}(t_{k-1})) = (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1})) \text{ then } \]
\[ (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) \in W_k \text{ and } \]
\[ \tilde{\beta}_k(\xi_k(t_k)) = (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)). \]

v) For \( v_k: W_k \rightarrow (A_k \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1) \{1, \ldots, v(u_0)\}, \quad v_k|W_k \cap W_{k-1} = \tau_{k-1}|W_k \cap W_{k-1} \)
and if
\[ (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) \in W_k, \beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1}), \text{ and } \]
\[ \tilde{\beta}_{k-1}(\xi_{k-1}(t_{k-1})) = (t_0, \ldots, t_{k-2}, \xi_{k-1}(t_{k-1})) \text{ then } \]
\[ v_k = \tilde{\beta}_k \circ \xi_k \cup \tau_{k-1} \]

It can be seen that:
\[ V_{k-1} = \{ (y_0(\sigma_0), \ldots, y_{k-1}(\sigma_{k-1})) | \xi_k(\sigma_k) \} \text{ for } i = 0, \ldots, k-1, \sigma_i \in \Sigma_{u_i}(i), \]
\[ (y_0(\sigma_0), \ldots, y_{k-1}(\sigma_{k-1})) \in V_{k-1}, \sigma_0 \text{ has the form } u_0(i) \ldots (1, \ldots, 1, \ldots, 1). \]
and \( \sigma_{k-1} \) has the form \((b_{k-1}, \ldots, b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1})\).

There is an element \( \sigma_k = (b_{k-1}, \ldots, b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k) \in \Sigma_\vartheta(k) \) such that \((\sigma_0, \ldots, \sigma_k) \in T_\vartheta \), and there is a mapping \( \varepsilon_k : S_{u_k} \rightarrow A_k[W_{k-1}]_{k-1} \) such that conditions i)—iv) in part (5),b of Definition 3.1 hold, and \( \sigma_k = (b_{k-1}, \ldots, b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k) \) satisfies the requirements ii)—vi) of \( \Sigma_\vartheta(k) \).

We define \( \gamma_k : \Sigma_\vartheta(k) \rightarrow \Sigma_\vartheta(k) \) as follows: Let us consider the set

\[ \Gamma_k = \{ \gamma : \Sigma_\vartheta(k) \rightarrow \Sigma_\vartheta(k) \mid \text{for each } \sigma_k \in \Sigma_\vartheta(k), \gamma(\sigma_k) = \overline{\sigma_k} \}
\]

has the form \((b_{k-1}, \ldots, b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1})\), \( \sigma_k \) has the form \((b_{k-1}, \ldots, b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k) \) and there is a mapping \( \varepsilon_k : S_{u_k} \rightarrow A_k[W_k] \) such that conditions i)—iv) in part (5),b of Definition 3.1 hold, and \( \overline{\sigma_k} \) satisfies the requirements ii)—vi) of \( \Sigma_\vartheta(k) \).

One can see that if \( \gamma_k \in \Gamma_k \), then \( \gamma_k \) satisfies condition (2) because of the constructions of \( \Psi_k \) and \( \Sigma_\vartheta(k) \). Using this fact we can see that \(|\Gamma_k| = 1 \). Let \( \gamma_k \) be the only element of \( \Gamma_k \). One can see that \( \gamma_k \) is surjective. Using the fact that \( \gamma_k \) satisfies condition (2), one can easily prove that the mappings \( \gamma_0, \ldots, \gamma_k \) satisfy condition (1).

Finally we shall prove, using the fact that for \( j = 0, \ldots, k \) the mappings \( \gamma_0, \ldots, \gamma_j \) satisfy condition (1) and for \( j = 0, \ldots, k \) the mapping \( \gamma_j \) satisfies condition (2), that \( \tau_\vartheta = \tau_\vartheta \).

Assume that \( K_0 = (e[\{e\}, \psi_0 : e \rightarrow b_1], \Theta_0, Z^0, \Omega^0) \) is a starting configuration of \( \mathcal{B} \) and that for a configuration \( K_1 = (q_1[S_{q_1}, \psi^1], \Theta^1, Z^1, \Omega^1) \), \( K_0 \rightarrow^*_\vartheta K_1 \) holds. Then \( K_0 \) is a starting configuration of \( \mathcal{C} \) as well. We shall show that there is a configuration

\[ K_1 = (q_1[S_{q_1}, \psi^1], \Theta^1, Z^1, \Omega^1) \]

of \( \mathcal{C} \) with bijective correspondences

\[ \alpha_0 : [Z^1]_0 \rightarrow [Z^1]_0 \]

(\*)

\[ \vdots \]

\[ \alpha_k : [Z^1]_k \rightarrow [Z^1]_k \]

such that \( \alpha_0 \) and \( \alpha_k \) is the identity function and \( K_0 \rightarrow^*_\vartheta K_1 \) holds, moreover

i) for every \( s_k \in S_{q_1}, \Theta^1(s_k) = (s_0, s_1, \ldots, s_k) \) iff \( \Theta^1(\alpha_k(s_k)) = (\alpha_0(s_0), \alpha_1(s_1), \ldots, \alpha_k(s_k)) \) and

\[ (1 \leq j \leq k, \ (s_0, s_1, \ldots, s_j) \in Z^1 \ ] \text{ iff } (\alpha_0(s_0), \alpha_1(s_1), \ldots, \alpha_j(s_j)) \in (N^*)^j \]

and

ii) for every

\[ (s_0, s_1, \ldots, s_j) \in Z^1 \ (1 \leq j \leq k), \ (s_0, s_1, \ldots, s_j) = \Omega^1(\alpha_0(s_0), \alpha_1(s_1), \ldots, \alpha_j(s_j)) \].

Conversely, if \( K_0 \rightarrow^*_\vartheta K_1 \) holds then there is a configuration \( K_1 \) of \( \mathcal{B} \) and there are bijective functions (\*) such that \( \alpha_0 \) and \( \alpha_k \) are identity functions and i), ii), iii) hold. Hence if \( K_1 \) is final then \( \overline{K_1} \) is final and vice versa, thus \( \tau_\vartheta = \tau_\vartheta \) follows.
First we shall prove the first part of the statement, the second part can be proved similarly. We prove by induction on the length of the transition $K_0 \Rightarrow^*_a K_1$.

a) The length of $K_0 \Rightarrow^*_a K_1$ is zero, $(K_1 = K_0)$. Trivial.

b) Assume that the statement is true for $K_0 \Rightarrow^*_a K_2$, $K_0 \Rightarrow^*_a K_1$ and for the functions $(\ast)$ and that $K_0 \Rightarrow^* K_2 = (q^2[S_{q^2}, \varphi^2], \Theta^2, Z^2, \Omega^2)$ holds.

By the definition of the relation $\Rightarrow^*_{q^2}$, there are mappings $\kappa_i : [Z^1] \rightarrow \Sigma_{q^2}(i)$ for $i = 0, 1, \ldots, k$ such that for every $(s_0, s_1, \ldots, s_j) \in Z^1$ (1 \(j\) \(k\)) if

$$
\Omega^1((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 u_0(1, \ldots, v(u_0)) \{(1, \ldots, v(u_0)), \vartheta_0\}
$$

then $\kappa_i(s_0) = u_i$, $\kappa_i(s_1)$ has the form $(b_i \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_l, \tau_l)$ for $i = 1, \ldots, j$, moreover $(\kappa_i(s_0), \kappa_i(s_1), \ldots, \kappa_i(s_j)) \in V_j$. Take the mappings $\kappa_i([Z^1] \rightarrow \Sigma_{q^2}(i))$ for $i = 0, 1, \ldots, k$ defined by $\kappa_i(\iota(s_i)) = \gamma_i(\kappa_i(s_i))$ for each $s_i \in [Z^1]$. Notice that $\kappa_i$ is well defined, because $\kappa_i$ and $\gamma_i$ are bijective. By the induction hypothesis for each $(s_0, s_1, \ldots, s_j) \in Z^1$ (1 \(j\) \(k\)), $\Omega^1((s_0, s_1, \ldots, s_j)) = \Omega_i((\kappa_0(s_0), \kappa_1(s_1), \ldots, \kappa_j(s_j)))$. Since $\kappa_0 = \vartheta_0$ and for each $s_i \in Z^1$ the first two components of $\kappa_i(s_i)$ are equal to the first two components of $\kappa_i(\iota(s_i))$ for $i = 1, \ldots, k$, moreover for every $(s_0, s_1, \ldots, s_j) \in V_j$ (1 \(j\) \(k\)) it follows that the mappings $\kappa_i (i = 0, 1, \ldots, k)$ satisfy condition (1) in Definition 3.3. The mappings $\kappa_i (i = 0, 1, \ldots, k)$ uniquely determine a configuration $K_0 = (q^2[S_{q^2}, \varphi^2], \Theta^2, Z^2, \Omega^2)$ of C such that $K_0 \Rightarrow^* K_2$ holds. First we show that $q^2[S_{q^2}, \varphi^2] = q^2[S_{q^2}, \varphi^2]$. By the transition $K_0 \Rightarrow^* K_2$ we know that $q^2 = q^2[S_{q^2}, \varphi^2]$, where $\delta : S_{q^1} \rightarrow \mathcal{T}_{G_k}(Y_k \cup N^*)$ satisfies the following formula: for every $s_i \in S_{q_1}$ if $\kappa_i(s_i) = (b_k \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_k, \tau_k)$ then $\delta(s_i) = \omega_{s_i}(u_0)$. By the induction hypothesis and the transition $K_0 \Rightarrow^* K_2$ we obtain that $q^2 = q^2[S_{q^2}, \varphi^2]$, where $\delta : S_{q^1} \rightarrow \mathcal{T}_{G_k}(Y_k \cup N^*)$ satisfies the following formula: for every $s_i \in S_{q_1}$ if $\kappa_i(s_i) = (b_k \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_k, \tau_k)$ and $\delta(s_i) = \omega_{s_i}(u_0)$ then $\kappa_i(s_i) = \gamma_i(\kappa_i(s_i)) = \kappa_i(s_i) = (b_k \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_k, \tau_k)$ for some $\vartheta_0, W_k$ and $\tau_k$, moreover $\delta(s_i) = \omega_{s_i}(u_0) = \delta(s_i)$, thus $q^2 = q^2$.

Again by the transition $K_0 \Rightarrow^* K_2$ and $K_0 \Rightarrow^* K_2$ we have that $\psi^2$ and $\psi^2 : S_{q^2} \rightarrow A_1 \ldots A_j G_k(Y_0)$ satisfy the following conditions:

Let $\tilde{s}_i \in S^2$ be arbitrary and consider its unique decomposition $\tilde{s}_i = s_k t_k$, where $s_k \in S_{q_1}$, $\delta(s_k) = \omega_{s_k}(u_k)$ for some $u_k \in P_{G_k}(Y_k)$, $t_k \in S_{q_0}$ and $\kappa_k(s_k)$ has the form $\kappa_k(s_k) = (b_k \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_k, \tau_k)$. Then if $\varphi_k(t_k) = c_k \ldots c_1 t_0$, $c_0 \ldots c_1 \in A_k \ldots A_1$, $t_0 \in \{1, \ldots, v(u_0)\}$ and $\psi^1(s_k) = u_0(1, \ldots, v(u_0)) \{(1, \ldots, v(u_0)), \vartheta_0\}$, $(u_0 \in G_0 \cup Y_0, \vartheta_0) = (1, \ldots, v(u_0)) \rightarrow \mathcal{T}_{G_k}(Y_0)$ then $\psi^2(s_k t_k) = c_k \ldots c_1 \vartheta_0(t_0)$. We have obtained that $\psi^2 = \tilde{\psi}^2$.

$Z^2 = \{(s_0 t_0, \ldots, s_j t_j)(s_0, \ldots, s_j) \in Z^1, \ j \leq l \leq k\}$

$$
\kappa_0(s_0) = (b_i \ldots b_1, u_0, u_0[S_{u_0}, \varphi_0], \vartheta_0, W_l, \tau_l) \text{ and } (t_0, t_1, \ldots, t_j) \in W_l)\}
$$

$Z^2 = \{((\alpha_0(s_0) t_0, \ldots, \alpha_j(s_j) t_j) (\alpha_0(s_0), \ldots, \alpha_j(s_j)) \in Z^1, \ j \leq l \leq k, \}

and $(t_0, \ldots, t_j)$ is a member of the fifth component of $\kappa_i(\iota(s_i))\}$
Now we define the mappings $\alpha^j : [Z]^j \to [Z]^j$, \((i = 0, \ldots, k)\) as follows: Let $\alpha^0$ and $\alpha^k$ be identity mappings, and for $i = 1, \ldots, k - 1$ take an arbitrary element $s_t, t_i \in [Z]^j$, where $x_i(s_t) = (b_i \ldots b_t, u_0, u_t[S_{ut}, \phi_t], \xi_t, W_t, \tau_t)$, then we define $\alpha^j(s_t, t_i)$ to be $\alpha^j(s_t) \xi_i(t_i)$, where $\xi_i : S_{ut} \to \{1, \ldots, |S_{ut}|\}$.

We have to show that for $i = 1, \ldots, k - 1$ $\alpha^j$ is a bijective function. Let $s_t, t_i = \bar{s}_t, \bar{t}_i \in [Z]^j$ and assume that $s_t \neq \bar{s}_t$. Then one of $s_t$ or $\bar{s}_t$ is a proper initial segment of the other one, which contradicts the definition of the configuration, thus $a^2$ is a well defined function.

Assume that $\alpha^2(s_t, t_i) = \alpha^2(\bar{s}_t, \bar{t}_i)$ such that $s_t \neq \bar{s}_t$ or $t_i \neq \bar{t}_i$. If $s_t \neq \bar{s}_t$ then $\alpha^2(s_t, t_i) = \alpha^2(\bar{s}_t, \bar{t}_i)$ and $\xi_i$ is a function whose range is $N$ thus $\alpha^2(s_t) \xi_i(t_i) \neq \alpha^2(\bar{s}_t) \xi_i(t_i)$. If $s_t = \bar{s}_t$ and $t_i \neq \bar{t}_i$ then $\alpha^2(s_t, t_i) = \alpha^2(s_t, \bar{t}_i) \neq \alpha^2(\bar{s}_t, \bar{t}_i) = \alpha^2(\bar{s}_t, \bar{t}_i)$ since $\xi_i(t_i) \neq \xi_i(\bar{t}_i)$ thus we obtained that $\alpha^2$ is injective.

Let $s_t, \bar{s}_t \in [Z]^j$, then there is an element $(s_0 t_0, \ldots, s_i t_i, \ldots, s_j t_j) \in Z^2$, where $j \geq i$. By the construction of $Z^2$, $(s_0, \ldots, s_i, \ldots, s_j, \ldots, \bar{s}_j) \in Z^1$ for some $\bar{s}_j + 1, \ldots, \bar{s}_j \in N^*$, $1 \equiv j \equiv l \equiv k$, and

$$z_l(s_l) = \{(b_1 \ldots b_l, s_0, \sigma_l(1, \ldots, \nu_l(\sigma_l), \bar{q}_l, \bar{W}_l, \bar{\tau}_l) \text{ if } l \leq k, \}
\{(b_k \ldots b_l, s_0, s_l[S_{ul}, \phi_l], \bar{q}_l, \bar{W}_l, \bar{\tau}_l) \text{ if } l = k, \}
$$
and $(i_0, \ldots, i_l, \ldots, i_j) \in W_l$. By the induction hypothesis there is an element $(s_0, i_0, \ldots, s_l, i_l, \ldots, s_j, i_j)$ of $Z^1$ such that

$$(a_0(s_0), \ldots, a_i(s_i), \ldots, a_j(s_j), \ldots, a_k(s_k)) = (\bar{s}_0, \ldots, \bar{s}_i, \ldots, \bar{s}_j, \ldots, \bar{s}_l).$$

Since $z_l(s_l) = \gamma_l(a_l(s_l))$ by definition, we can apply condition (2) (ii) stated for $\gamma_l$, which tells us that $(i_0, \ldots, i_l, \ldots, i_j) \in W_l$ iff there is a $(\xi_l^{-1}(i_0), \ldots, \xi_l^{-1}(i_l), \ldots, \xi_l^{-1}(i_j)) \in W_l$ for $\xi_0, \ldots, \xi_j$ defined in the condition. Thus

$$(s_0 \xi_l^{-1}(i_0), \ldots, s_l \xi_l^{-1}(i_l), \ldots, s_j \xi_j^{-1}(i_j)) \in Z^2$$

moreover $\alpha^2(s_l, \xi_l^{-1}(i_l)) = a_l(s_l) \xi_l(\xi_l^{-1}(i_l)) = a_l(s_l) t_l = s_t t_i.$

hence $\alpha^2$ is surjective ($i = 1, \ldots, k - 1$). Thus we have proved that $\alpha^2$ is bijective ($i = 0, \ldots, k$).

Let $\bar{s}_k \in S^k$ be arbitrary and consider its unique decomposition $\bar{s}_k = s_k t_k$ where $s_k \in S^q$, $x_k(s_k)$ has the form $(b_k \ldots b_t, u_0, u_t[S_{ut}, \phi_t], \xi_k, W_k, \tau_k)$, $\delta(s_k) = \omega_k(u_0), t_k \in S^q$. In this case $z_k(s_k) = (\gamma_k(x_k(s_k)))$ has the form $(b_k \ldots b_t, u_0, u_t[S_{ut}, \phi_t], \bar{q}_k, \bar{W}_k, \bar{\tau}_k)$. Using condition (2) (iv) stated for $\gamma_k$, $\gamma_k(t_k) = (t_0, t_1, \ldots, t_k) \iff \bar{q}_k(t_k) = (\xi_k(t_0), \xi_k(t_1), \ldots, \xi_k(t_k))$ for $\xi_0, \xi_1, \ldots, \xi_k$ defined in the condition.

Using the induction hypothesis $\Theta^1(s_k) = (s_0, s_1, \ldots, s_k)$ iff $\Theta^1(s_k) = (a_0(s_0), \ldots, a_k(s_k))$. By the definition of $\Theta^2$ and $\Theta^2$ we obtain that

$$\Theta^2(s_k) = (s_0 t_0, s_1 t_1, \ldots, s_k t_k) \iff \Theta^2(s_k) = (a_0(s_0) \xi_0(t_0), a_1(s_1) \xi_1(t_1), \ldots, a_k(s_k) \xi_k(t_k)) =
(a_0(s_0) t_0, a_1^2(s_1 t_1), \ldots, a_k^2(s_k t_k)).$$

Thus we have proved that condition i) holds for the mappings $\alpha^0, \ldots, \alpha^k$.

Let $(s_0, s_1, \ldots, s_j, t_j) \in Z^2$ be arbitrary, where $1 \equiv j \equiv k$ and $(s_0, s_1, \ldots, s_l) \in Z^1$ for some $s_j + 1, \ldots, s_l \in (N^*)$, $(j \equiv l \equiv k)$, moreover $\alpha_l(s_l) = (b_1 \ldots b_1, u_0, u_l[S_{ul}, \phi_l],$
\( q_i, W_i, \tau_i \) and \( (t_0, \ldots, t_j) \in W_i \). By the induction hypothesis, \((x_0(s_0), \ldots, x_i(s_i)) \in Z^1\).

By the definition of \( \tilde{z}_i(x_i(s_i)) = \gamma_i(x_i(s_i)) \), i.e.,

\[
\tilde{z}_i(x_i(s_i)) = \begin{cases} 
(b_1 \ldots b_l, u_0, \sigma_1(1, \ldots, \nu'(\sigma_1))[1, \ldots, \nu'(\sigma_1)], \tilde{\varphi}_1, \tilde{q}_i, W_i, \tilde{\tau}_i) & \text{if } l < k, \\
(b_k \ldots b_1, u_0, u_k[S_{u_k} \varphi_k], \tilde{q}_k, W_k, \tilde{\tau}_k) & \text{if } l = k.
\end{cases}
\]

We can apply condition (2)(ii) stated for \( y \), which tells us that \((\sigma_0, \ldots, \sigma_j, \nu'(\sigma_1)), \tilde{\varphi}_1, W_i, \tilde{\tau}_i) \in Z^2\).

Conversely, let \((\sigma_0, \ldots, \nu'_{s_2})(s_j(t_j)) \in Z^2\) be arbitrary. By the construction of the set \( Z^2 \) there are two vectors \((s_0, \ldots, s_j, \tilde{s}_j) \in Z^1 \) and \((t_0, \ldots, t_j) \in ((N^*)^j) \) such that \( s_i = s_i(t_i) \) for \( i = 0, \ldots, j \), and \((t_0, \ldots, t_j) \in \tilde{\nu}(s_j) \) is in the fifth component of \( \tilde{z}_s(x_i(s_i)) \).

By the induction hypothesis, \((\sigma_0, \ldots, \sigma_j, \nu'(\sigma_1)), \tilde{\varphi}_1, W_i, \tilde{\tau}_i) \in Z^2\).

We know that \( \tilde{z}_i(x_i(s_i)) = \gamma_i(x_i(s_i)) \). According to condition (2)(ii) stated for \( y \), we obtain that \((\sigma_0, \ldots, \sigma_j, \nu'(\sigma_1)), \tilde{\varphi}_1, W_i, \tilde{\tau}_i) \in Z^2\).

We have proved that condition ii) holds for the mappings \( \sigma_0, \ldots, \sigma_j \).

It has remained to prove that condition iii) holds for \( \sigma_0, \ldots, \sigma_j \). Let \((s_0, s_0, \ldots, s_{j+1}, s_j, \tilde{s}_j, \ldots, \tilde{s}_j) \in Z^2 \) be arbitrary, where \( 1 \leq j \leq k \), \((s_0, \ldots, s_j, \tilde{s}_j) \in Z^1 \) for some \( s_{j+1}, \ldots, s_{j+1}, \ldots, s_j \in \tilde{\nu}(s_j) \), and \( x_i(s_i) = (b_1 \ldots b_l, u_0, u_l[S_{u_l} \varphi_l], \tilde{q}_l, W_l, \tilde{\tau}_l) \) and \((t_0, \ldots, t_j) \in W_i \). For some \( \nu_0 : (1, \ldots, t_0) \to (V_0, \nu(t_0), 1) \), for some \( \nu_0 : (1, \ldots, t_0) \to (V_0, \nu(t_0), 1) \), for some \( c_j \in A_j, \ldots, c_j \in A_j, \) and \((t_0, \ldots, t_j) = (c_j \ldots c_j, \nu(t_0)) \), \((\nu_0, \ldots, \nu(t_0), \ldots, \nu(t_0)) \), \((\nu_0, \ldots, \nu(t_0), \ldots, \nu(t_0)) \) holds. By the induction hypothesis, \((\alpha_0(s_0), \ldots, \alpha_j(s_j) \in Z^2\) and

\[
\tilde{z}_s(x_i(s_i)) = \gamma_i(x_i(s_i)) = \begin{cases} 
(b_1 \ldots b_l, u_0, u_l[S_{u_l} \varphi_l], \tilde{q}_l, W_l, \tilde{\tau}_l) & \text{if } l < k, \\
(b_l \ldots b_1, u_0, \nu'(\sigma_1), \nu'(\sigma_1)) \tilde{\varphi}_1, W_l, \tilde{\tau}_l) & \text{if } l = k.
\end{cases}
\]

We can apply condition 2(v) stated for \( y \), which tells us that \((\tau_i((t_0, \ldots, t_j)) = c_j \ldots c_j, \nu(t_0)) \) for the mappings \( \tau_0, \ldots, \tau_j \) defined in the condition.

Thus, \((\tau_i((t_0, \ldots, t_j)) = c_j \ldots c_j, \nu(t_0)) \) holds. By the induction hypothesis, \( \Omega(\sigma_0(s_0), \ldots, \sigma_j(s_j), \ldots, \alpha_j(s_j)) = b_1 \ldots b_l, u_0, u_l[S_{u_l} \varphi_l], \tilde{q}_l, W_l, \tilde{\tau}_l) \). By the definition of \( \Omega^2 \) and \( \alpha_j(t_i) = (0, \ldots, 0, k) \),

\[
\Omega^2((\sigma_0(s_0), \ldots, \alpha_j(s_j) \nu(t_0)) = \begin{cases} 
\Omega^2((\sigma_0(s_0), \ldots, \alpha_j(s_j) \nu(t_0)) & c_j \ldots c_j, \nu(t_0) \end{cases}
\]

Thus \( \Omega^2((\sigma_0(s_0), \ldots, \alpha_j(s_j) \nu(t_0)) \) holds. The proof of the first part of the statement is complete. The second part of the statement can be proved by induction on the length of the transition \( K_0 \Rightarrow K_1 \).

a) The length of \( K_0 \Rightarrow \) is zero, \( (K_1 \Rightarrow K_0) \) Trivial.

b) Assume that the statement is true for \( K_0 \Rightarrow K_1 \) and for the functions \( (\ast) \) that \( K_1 \Rightarrow K_2 \) holds. By the definition of the relation \( \Rightarrow \) there are mappings
On the compositions of root-to-frontier tree transformations

\[ \tilde{x}_i: [Z^1] \rightarrow \Sigma_{\Theta}(i) \] for \( i = 0, 1, \ldots, k \) such that for every \((s_0, s_1, \ldots, s_j) \in Z^1 (1 \leq j \leq k)\) if 
\[ \Omega^1((s_0, s_1, \ldots, s_j)) = b_j b_1 u_0(1, \ldots, v(u_0)) \{1, \ldots, v(u_0)\}, \delta_0 \] \( (b_j \in A_j, \ldots, b_1 \in A_1, u_0 \in G_0 \cup Y_0, y_0: \{1, \ldots, v(u_0)\} \rightarrow T_{y_0}(Y_0) \) then \( \tilde{x}_0(s_0) = u_0 \); \( i = 1, \ldots, j \), \( x_i(s_i) \) has the form

\[ \{(b_1, \ldots, b_1, u_0, \sigma_i(1, \ldots, v(\sigma_i))), \delta_i, \bar{\delta}_i, \bar{W}_i, \tilde{\tau}_i\} \text{ if } 1 \leq i \leq k - 1, \]

\[ \{(b_k, \ldots, b_1, u_k, u_k[S_{uk}, \varphi_k, q_k, W_k, \tau_k]\} \text{ if } i = k, \]

moreover \((\tilde{x}_0(s_0), \ldots, \tilde{x}_j(s_j)) \in \bar{V}_j\).

Take the mappings \( x_i: [Z^1] \rightarrow \Sigma_{\Theta}(i) \) for \( i = 0, \ldots, k - 1 \) defined by \( x_i(s_i) = v_i^{-1}((x_i(s_i)), \gamma_i) \) for each \( s_i \in [Z^1] \). Notice that \( x_i \) is well defined, because \( \alpha_i \) and \( \gamma_i \) are bijective. According to Definition 3.2 for each \( s_k \in [Z^1]_k \), \( \Theta^1(s_k) \) is the only element of \( Z^1 \) which has the form \((s_0, \ldots, s_{k-1}, \tilde{s}_k)\) for some \( s_0, \ldots, \tilde{s}_k \in N^* \). We know that \((\tilde{x}_0(s_0), \ldots, \tilde{x}_{k-1}(s_{k-1}), \tilde{x}_k(s_k)) \in \bar{V}_k\). We can apply condition (1) stated for \( \gamma_0, \ldots, \gamma_{k-1}, \gamma_k \), which tells us that there is a unique \( \sigma_k \in \Sigma_{\Theta}(k) \) such that \( \gamma_k(\sigma_k) = \tilde{x}_k(s_k) \) and \( (\gamma_i^{-1}(\alpha_i(\sigma_i)), \ldots, \gamma_k^{-1}(\alpha_k(\sigma_k)), \sigma_k) \in V_k \). Let \( x_k(s_k) = \sigma_k \). By the induction hypothesis for each \((s_0, s_1, \ldots, s_j) \in Z^1 (1 \leq j \leq k)\), \( \Omega^1((s_0, s_1, \ldots, s_j)) = \Omega^1((\alpha_0(x_0(s_0)), \alpha_1(s_1), \ldots, \alpha_j(s_j))) \). Since \( x_0 = \tilde{x}_0 \) and for each \( s_i \in [Z^1]_i \) the first two components of \( x_i(s_i) \) are equal to the first two components of \( \tilde{x}_i(x_i(s_i)) \) for \( i = 1, \ldots, k \); moreover for every \((s_0, s_1, \ldots, s_j) \in Z^1 (1 \leq j \leq k)\), \( (x_0(s_0), x_1(s_1), \ldots, x_j(s_j)) \in \bar{V}_j \) it follows that the mappings \( x_i (i = 0, 1, \ldots, k) \) satisfy condition (1) in the Definition 3.3.

The mappings \( x_i (i = 0, 1, \ldots, k) \) uniquely determine a configuration \( K_2 = \{q \rightarrow \Sigma_{\Theta}(q'), \Theta^2, Z^2, \Omega^2\} \) such that \( K_1 \Rightarrow K_2 \) holds.

From now on the proof of the second part of the statement is similar to the proof of the first part.

The proof of the theorem is complete.

### 4. Example

Let us consider the following two \( R \)-transducers:

\[ \mathcal{U}_1 = (G_0, Y_0, A_1, G_1, Y_1, A'_1, \Sigma_{\Theta_1}), \]

where

\[ G_0 = G_0' = \{g_0\}, Y_0 = \{x_0\}, \]

\[ G_1 = G_1' = \{g_1\}, Y_1 = \{x_1, y_1\}, \]

\[ A_1 = \{a_1, b_1, c_1\}, A'_1 = \{a_1\}, \]

\[ \Sigma_{\Theta_1} = \{b_1 x_0 \rightarrow y_1, b_1 x_0 \rightarrow x_1, \}
\]

\[ a_1 g_0 \rightarrow g_1(1, 2)\{\{1, 2\}, \varphi_{11}: 1 \rightarrow b_1 1; \varphi_{12}: 2 \rightarrow b_1 2\}, \]

\[ a_1 g_0 \rightarrow g_1(1, 2)\{\{1, 2\}, \varphi_{12}: 1 \rightarrow b_1 1; \varphi_{11}: 2 \rightarrow c_1 2\}. \]

\[ \tau_{\Theta_1} = \{(g_0(x_0, x_0), g_1(y_1, y_1)), (g_0(x_0, x_0), g_1(x_1, x_1)), \}
\]

\[ (g_0(x_0, x_0), g_1(x_1, y_1)), (g_0(x_0, x_0), g_1(y_1, x_1)). \]

\[ \mathcal{U}_2 = (G_1, Y_1, A_2, G_2, Y_2, A'_2, \Sigma_{\Theta_2}), \]

where
\( G_2 = G_2^2 = \{g_2\}, Y_2 = \{x_2, y_2, z_2\}\)

\( A_2 = \{a_2, b_2\}, A'_2 = \{a_2\}\)

\( \Sigma_{a_2} = \{a_2 g_1 \to g_2(1, 2)[\{1, 2\}, \varphi_{21}: 1 \to b_2 \; 1; \; \varphi_{22}: 2 \to b_1 \; 1]\),

\( b_2 x_1 \to y_2, b_2 x_1 \to z_2, b_2 y_1 \to x_2\).

One can see that

\[
\tau_{a_2} \circ \tau_{a_2} = \left\{ (g_0(x_0, x_0), g_2(x_2, x_2)), (g_0(x_0, x_0), g_2(y_2, y_2)),

(g_0(x_0, x_0), g_2(y_2, z_2)), (g_0(x_0, x_0), g_2(z_2, y_2)),

(g_0(x_0, x_0), g_2(z_2, z_2)) \right\}.
\]

We construct the 2-synchronized \( R \)-transducer \( \mathcal{B} \) according to the Theorem 3.5:

\( \mathcal{B} = (G_0, G_1, Y_0, Y_1, A_1, A_2, A'_1, A'_2, \Sigma_a, V) \),

where

\( \Sigma_{a_0}(0) = V_0 = G_0 \cup Y_0 \),

\( \Sigma_{a_0}(1) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \),

where \( \sigma_1 = (b_1, x_0, y_1, \emptyset, \emptyset, \emptyset) \),

\( \sigma_2 = (b_1, x_0, x_1, \emptyset, \emptyset, \emptyset) \),

\( \sigma_3 = (a_1, g_0, g_1(1, 2)[\{1, 2\}, \varphi_3: 1 \to b_1 \; 1; \; \varphi_3: 2 \to b_1 \; 2],

\quad \varphi_3: 1 \to (1, 1); \; \varphi_3: 2 \to (2, 2), \{1, 1\}, \{2, 2\} ),

\quad \tau_3: (1, 1) \to b_1 \; 1; \; \tau_3: (2, 2) \to b_1 \; 2),

\( \sigma_4 = (a_1, g_0, g_1(1, 2)[\{1, 2\}, \varphi_4: 1 \to b_1 \; 1; \; \varphi_4: 2 \to c_1 \; 2],

\quad \varphi_4: 1 \to (1, 1); \; \varphi_4: 2 \to (2, 2), \{1, 1\}, \{2, 2\} ),

\quad \tau_4: (1, 1) \to b_1 \; 1; \; \tau_4: (2, 2) \to c_1 \; 2). \)

\( V_1 = \{(x_0, \sigma_1), (x_0, \sigma_2), (g_0, \sigma_3), (g_0, \sigma_4)\} \).

\( \Sigma_{a_0}(2) = \{\sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9\} \),

where \( \sigma_5 = (b_2b_1, x_0, x_2, \emptyset, \emptyset, \emptyset) \),

\( \sigma_6 = (b_2b_1, x_0, y_2, \emptyset, \emptyset, \emptyset) \),

\( \sigma_7 = (b_2b_1, x_0, z_2, \emptyset, \emptyset, \emptyset) \),

\( \sigma_8 = (a_2a_1, g_0, g_2(1, 2)[\{1, 2\}, \varphi_8: 1 \to b_2b_1 \; 1; \; \varphi_8: 2 \to b_2b_1 \; 1],

\quad \varphi_8: 1 \to (1, 1, 1); \; \varphi_8: 2 \to (1, 1, 2), \{1, 1\}, \{1, 2\}, \{2, 2\} ),

\quad \tau_8: (1, 1, 1) \to b_2b_1 \; 1; \; \tau_8: (1, 1, 2) \to b_2b_1 \; 1; \; \tau_8: (2, 2) \to b_2b_1 \; 1),

\( \sigma_9 = (a_2a_1, g_0, g_2(1, 2)[\{1, 2\}, \varphi_9: 1 \to b_2b_1 \; 1; \; \varphi_9: 2 \to b_2b_1 \; 1],

\quad \varphi_9: 1 \to (1, 1, 1); \; \varphi_9: 2 \to (1, 1, 2), \{1, 1\}, \{1, 2\}, \{2, 2\} ),

\quad \tau_9: (1, 1, 1) \to b_2b_1 \; 1; \; \tau_9: (1, 1, 2) \to b_2b_1 \; 1; \; \tau_9: (2, 2) \to c_1 \; 1). \)

\( V_2 = \{(x_0, \sigma_1, \sigma_3), (x_0, \sigma_2, \sigma_4), (x_0, \sigma_2, \sigma_7), (g_0, \sigma_3, \sigma_5), (g_0, \sigma_4, \sigma_9)\} \).
Let us consider configurations $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ of $\mathcal{B}$, where $K_0$ is a starting configuration, $K_2, K_3, K_4, K_5, K_6$ are final configurations.

$K_0 = (a_2 a_1 g_0(x_0, x_0), \Theta_0: e \mapsto (e, e, e), \{(e, e, e) \mapsto a_2 a_1 g_0(x_0, x_0)),$

$K_1 = (g_2(b_2 b_1, x_0), \Theta_1: 1 \mapsto (1, 1, 1); \Theta_1: 2 \mapsto (1, 1, 2),$

$\{(1, 1, 1), (1, 1, 2), (2, 2)}, \Omega_1: (1, 1, 1) \mapsto b_2 b_1 x_0;$

$\Omega_1: (1, 1, 2) \mapsto b_2 b_1 x_0; \Omega_1: (2, 2) \mapsto b_1 x_0),$

$K_2 = (g_2(x_2, x_2), \Theta, \Theta, \Theta),$

$K_3 = (g_2(y_2, y_2), \Theta, \Theta, \Theta),$

$K_4 = (g_2(z_2, y_2), \Theta, \Theta, \Theta),$

$K_5 = (g_2(y_2, x_2), \Theta, \Theta, \Theta),$  

$K_6 = (g_2(z_2, x_2), \Theta, \Theta, \Theta).$

All the transitions from configuration $K_0$ in $\mathcal{B}$ which are ended by final configuration are the following:

$K_0 \Rightarrow_\mathcal{B} K_1 \Rightarrow_\mathcal{B} K_2,$

$K_0 \Rightarrow_\mathcal{B} K_1 \Rightarrow_\mathcal{B} K_3,$

$K_0 \Rightarrow_\mathcal{B} K_1 \Rightarrow_\mathcal{B} K_4,$

$K_0 \Rightarrow_\mathcal{B} K_1 \Rightarrow_\mathcal{B} K_5,$

$K_0 \Rightarrow_\mathcal{B} K_1 \Rightarrow_\mathcal{B} K_6.$

The transition $K_0 \Rightarrow_\mathcal{B} K_1$ is determined by the mappings:

$x_0: \{e\} \rightarrow \Sigma_0(0); x_0(e) = g_0,$

$x_1: \{e\} \rightarrow \Sigma_1(1); x_1(e) = \sigma_3,$

$x_2: \{e\} \rightarrow \Sigma_2(2); x_2(e) = \sigma_8.$

The transition $K_1 \Rightarrow_\mathcal{B} K_2$ is determined by the mappings:

$x_0: \{1, 2\} \rightarrow \Sigma_0(0); x_0(1) = x_0; x_0(2) = x_0,$

$x_1: \{1, 2\} \rightarrow \Sigma_1(1); x_1(1) = \sigma_1; x_1(2) = \sigma_2,$

$x_2: \{1, 2\} \rightarrow \Sigma_2(2); x_2(1) = \sigma_5; x_2(2) = \sigma_5.$

The transition $K_1 \Rightarrow_\mathcal{B} K_3$ is determined by the mappings:

$x_0: \{1, 2\} \rightarrow \Sigma_0(0); x_0(1) = x_0; x_0(2) = x_0,$

$x_1: \{1, 2\} \rightarrow \Sigma_1(1); x_1(1) = \sigma_2; x_1(2) = \sigma_1,$

$x_2: \{1, 2\} \rightarrow \Sigma_2(2); x_2(1) = \sigma_6; x_2(2) = \sigma_6.$
The transition $K_1 \Rightarrow K_4$ is determined by the mappings:

$x_0: \{1, 2\} \rightarrow \Sigma_B(0); x_0(1) = x_0; x_0(2) = x_0$

$x_1: \{1, 2\} \rightarrow \Sigma_B(1); x_1(1) = \sigma_2; x_1(2) = \sigma_2$

$x_2: \{1, 2\} \rightarrow \Sigma_B(2); x_2(1) = \sigma_6; x_2(2) = \sigma_7$

The transition $K_1 \Rightarrow K_5$ is determined by the mappings:

$x_0: \{1, 2\} \rightarrow \Sigma_B(0); x_0(1) = x_0; x_0(2) = x_0$

$x_1: \{1, 2\} \rightarrow \Sigma_B(1); x_1(1) = \sigma_2; x_1(2) = \sigma_2$

$x_2: \{1, 2\} \rightarrow \Sigma_B(2); x_2(1) = \sigma_7; x_2(2) = \sigma_7$

The transition $K_1 \Rightarrow K_6$ is determined by the mappings:

$x_0: \{1, 2\} \rightarrow \Sigma_B(0); x_0(1) = x_0; x_0(2) = x_0$

$x_1: \{1, 2\} \rightarrow \Sigma_B(1); x_1(1) = \sigma_6; x_1(2) = \sigma_2$

$x_2: \{1, 2\} \rightarrow \Sigma_B(2); x_2(1) = \sigma_7; x_2(2) = \sigma_7$

One can see that $\tau_B = \tau_{\bar{a}_1} \circ \tau_{\bar{a}_2}$.

References


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