Key and superkey for a closure function

By LE VAN BAO and HO THUAN

INSTITUTE OF COMPUTER SCIENCE AND CYBERNETICS, HANOI
VIÊN TINH TOAN VA ĐIEU KHIEN
LIEU GIAI, BA ĐINH, HANOI VIETNAM

In this paper, we investigate some characteristic properties of keys and superkeys for a closure function, defined on the power set of a finite set U. In particular we give a necessary condition under which a subset X of U is a key and explicit formula to compute the intersection of all keys for f, a necessary and sufficient condition for which a closure function f has precisely one key.

Moreover, the translation of a closure function f which, in some sense, preserves the keys for f, as well as the relationship between the keys for a closure function and the keys for the corresponding relation scheme are also considered.

These results are closely related to those presented in [1].

1. Keys of closure function

In this section after proving some lemmas, we give a characteristic condition under which a subset K can be a key for a closure function.

Definition [2]. Let $U=\{a_1, a_2, \ldots, a_n\}$ be a set of n elements (attributes) and 2^U its power set. The function $F\colon 2^U \to 2^U$ is called a closure function or closure iff for every $X, Y \in 2^U$

a) $X \subseteq F(X)$,

b) $F(F(X)) = F(X)$,

c) $Y \subseteq X \Rightarrow F(Y) \subseteq F(X)$.

Let $K \in 2^U$. We say that K is a superkey of the closure function F if $F(K)=U$, and K is said to be a key of F if $F(K)=U$, but $F(X) \neq U$ for any proper subset X of K. We set $\overline{X}=F(X)\setminus X$.

We define two sets P and T for the closure function F as follows

a) $T = \bigcup\{X\colon X \in 2^U \text{ and } F(X) \neq X\}$,

b) $P = \bigcup\{\overline{X}\colon X \in 2^U \text{ and } F(X) \neq X\}$.

Lemma 1.1. Let F be a closure function and $X \subseteqq U$ then:

$$F(X) \subseteqq X \cup P.$$

Lemma 2.1. Let F be a closure function and $X \subseteqq U$ and $a \notin T$. Then

$$F(X \setminus a) = F(X) \quad \text{or} \quad F(X \setminus a) = F(X) \setminus a.$$

Proof. There are two possible cases:
a) If $a \notin X$ then $X = X \setminus a$. It follows that $F(X) = F(X \setminus a)$.
b) If $a \in X$ then from the definition of T we have $F(X) = X$, and $F(X) \setminus a = X \setminus a$. Thus $F(X \setminus a) = F(F(X) \setminus a)$. On the other hand we have

$$F(X) \setminus a \subseteqq F(F(X) \setminus a) = F(X \setminus a) \subseteqq F(X).$$

It is clear that $F(X \setminus a) = F(X)$ or $F(X \setminus a) = F(X) \setminus a$. The proof is complete.

Lemma 3.1. Let F be a closure function. Then:

$$P \subseteqq F(T).$$

Proof. Let $a \in P$. From the definition of P there would exist an $X \subseteqq U$ such that $F(X) \neq X$ and $a \in F(X) \setminus X$. Clearly $X \subseteqq T$. Hence $F(X) \subseteqq F(T)$, showing that $a \in F(T)$.

Lemma 4.1. Let F be a closure function. If $a \in F(X) \setminus P$ then

$$a \in X.$$

Proof. We have $F(X) = X \cup (F(X) \setminus X) \subseteqq X \cup P$. On the other hand $a \in F(X)$ and $a \notin P$. This implies $a \in X$.

Lemma 5.1. Let F be a closure function. If $a \notin T$ and $F(Y) \subseteqq F(X)$ then

$$F(Y \setminus a) \subseteqq F(X \setminus a).$$

Proof. Since $a \notin T$, taking account of Lemma 2.1 we get $F(X \setminus a) = F(X)$ or $F(X \setminus a) = F(X) \setminus a$.
a) If $F(X \setminus a) = F(X)$, it is obvious that

$$Y \setminus a \subseteqq F(Y) \setminus a \subseteqq F(X) \setminus a \subseteqq F(X) = F(X \setminus a)$$

implies

$$F(Y \setminus a) \subseteqq F(F(X \setminus a)) = F(x \setminus a).$$

b) If $F(X \setminus a) = F(X) \setminus a$ we have

$$Y \setminus a \subseteqq F(Y) \setminus a \subseteqq F(X) \setminus a = F(X \setminus a).$$

Clearly

$$F(Y \setminus a) \subseteqq F(F(X \setminus a)) = F(X \setminus a),$$

the proof is complete. As an immediate consequence of Lemma 5.1 we have the following.

Lemma 6.1. Let F be a closure function. If $a \notin T$ and $F(Y) = F(X)$ then

$$F(X \setminus a) = F(Y \setminus a).$$

Lemma 7.1. Let F be a closure function. If $a \in K \cap F(K \setminus a)$, then K is not a key of F.

Proof. From $a \in K$ we have $K \setminus a \subsetneqq K$. Thus $F(K \setminus a) \subseteqq F(K)$. On the other hand $a \in F(K \setminus a)$. Clearly $a \cup (K \setminus a) \subseteqq F(K \setminus a)$, thus $K \subseteqq F(K \setminus a)$. It is obvious that $F(K) \subseteqq F(F(K \setminus a)) = F(K \setminus a) \subseteqq F(K)$. Therefore $K \setminus a \subsetneqq K$ and $F(K) = F(K \setminus a)$. From this we get that K is not a key of the closure function F.

Theorem 1.1. Let F be a closure function and K a key of F then:

$$U \setminus P \subseteqq K \subseteqq (U \setminus P) \cup (P \cap T).$$

Proof. We shall begin with showing that $U \setminus P \subseteqq K$. Assume to the contrary, that is $U \setminus P \nsubseteqq K$. From this, there would exist an $a \in (U \setminus P) \setminus K$. Clearly $a \notin P$ and $a \notin K$. On the other hand, since K is a key of the closure function F, it is obvious that $a \in F(K)$. From $a \notin P$ and taking account of Lemma 4.1, we get $a \in K$ which conflicts with $a \notin K$.

To complete the proof it remains to show that $K \subseteqq (U \setminus P) \cup (P \cap T)$. We know that $U = (U \setminus P) \cup P = (U \setminus P) \cup (P \cap T) \cup (P \setminus T)$. Suppose to the contrary, i.e. $K \nsubseteqq (U \setminus P) \cup (T \cap P)$. Then there exists an $a \in K \cap (P \setminus T)$.

From this we have $a \in K$, $a \in P$ and $a \notin T$. Because K is a key of F, we have $F(K) = U = F(U)$. From $a \notin T$, taking account of Lemma 6.1, we get $F(K \setminus a) = F(U \setminus a)$. From $a \notin T$, evidently $T \subseteqq U \setminus a$. It is obvious that $F(T) \subseteqq F(U \setminus a)$. In view of the Lemma 3.1, $P \subseteqq F(T)$. Combining this with $a \in P$ we obtain

$$a \in P \subseteqq F(T) \subseteqq F(U \setminus a) = F(K \setminus a).$$

It is clear that $a \in K \cap F(K \setminus a)$ showing, by Lemma 7.1, that K is not a key of F. We thus arrive to a contradiction. The proof is complete.

2. Intersection of all keys for a closure function

We propose in this section to describe the intersection of all key of a closure function.

Lemma 8.2. Let F be a closure function and $a \in P$. Then there exists a key K of F such that $a \notin K$.

Proof. Becauese $a \in P$, there exists an $X \subseteqq U$ such that $a \in F(X) \setminus X$. Let $C \subseteqq U$ such that $F(X) \cup C = U$ and $F(X) \cap C = \emptyset$. Clearly, $U \subseteqq F(X) \cup C \subseteqq F(X \cup C) \subseteqq U$. Thus $F(X \cup C) = U$ and there exists a key $K \subseteqq X \cup C$. It is clear that $a \notin K$.

Theorem 2.2. Let F be a closure function and let I be the intersection of all keys of F. Then

$$I = U \setminus P.$$

Proof. From Theorem 1.1 we have $U \setminus P \subseteqq I$. To complete the proof, it remains to show that $I \subseteqq U \setminus P$. In view of Lemma 8.2 we obtain $I \cap P = \emptyset$ showing that $I \subseteqq U \setminus P$.

Hence $I = U \setminus P$. The proof is complete.

3. Sufficient and necessary condition under which a closure function has precisely one key

In this section we present a theorem which gives a sufficient and necessary condition for a closure function F to have precisely one key.

Theorem 3.3. Let F be a closure function. Then F has precisely one key iff $T\cap P\subseteq F(T\setminus P)$.

Proof. Sufficiency: Let $T\cap P\subseteq F(T\setminus P)$. From this we have $T=(T\setminus P)\cup \cup(T\cap P)\subseteq F(T\setminus P)$. It is clear that $F(T\setminus P)\subseteq F(T)\subseteq F(F(T\setminus P))=F(T\setminus P)$. Thus $F(T)=F(T\setminus P)$.

By Lemma 3.1, $P\subseteq F(T)$. It is clear that: $F(T\cup P)\subseteq F(T)$. From this, $F(T)=F(T\cup P)$. Taking account of Lemma 1.1 we find $F(T\cup P)\subseteq(T\cup P)\cup P= =T\cup P$. Thus $T\cup P=F(T\cup P)$. Consequently $F(T\setminus P)=F(T)=F(T\cup P)=T\cup P$. On the other hand we have $T\setminus P\subseteq U\setminus P$.

Thus $T\cup P=F(T\setminus P)\subseteq F(U\setminus P)$. From this we find $U=(U\setminus P)\cup(T\cup P)\subseteq \subseteq F(U\setminus P)\subseteq U$. Finally we have $U=F(U\setminus P)$.

Now we shall show that $U\setminus P$ is the unique key of F. If $U\setminus P$ is not a key of F then there exists a key X of F such that $X\subsetneqq U\setminus P$. By Theorem 1.1 we have $U\setminus P\subseteq X\subsetneqq U\setminus P$ showing that $U\setminus P$ is the unique key of the closure function F.

Necessity: Let F be a closure function that has precisely one key K. We invoke Theorem 2.2 to deduce that $I=U\setminus P=K$, showing that $U\setminus P$ is a key of F. Thus $F(U\setminus P)=U$. There are two possible cases.

a) If $U\setminus P\neq U$ then from the definition of T we have $U\setminus P\subseteq T$. Thus $U\setminus P\subseteq \subseteq T\setminus P$ and clearly $U=F(U\setminus P)\subseteq F(T\setminus P)\subseteq U$. This implies $U=F(T\setminus P)$. Consequently $T\cap P\subseteq F(T\setminus P)$.

b) If $U\setminus P=U$ then clearly $P=\emptyset$. From this we have $\emptyset=T\cap P\subseteq F(T\setminus P)$. The proof is complete.

Example. Let $U=\{a, b, c\}$.
F: $2^U\to 2^U$ is a closure function,

$F(\emptyset) = \emptyset$,

$F(a) = ab$,

$F(b) = b$,

$F(c) = abc$,

$F(ab) = ab$,

$F(ac) = abc$,

$F(cb) = cba$,

$F(abc) = abc$.

From this we have:

$F(a) = ab \neq a, \quad \bar{a} = b$,

$F(c) = abc \neq c, \quad \bar{c} = ab$,

$F(ac) = abc \neq ac, \quad \overline{ac} = b, \quad f(cb) \neq abc \Rightarrow \overline{cb} = a$.

We obtain:

$T = acb,$

$P = ab,$

$T \cap P = ab.$

a) If K is a key of the closure function F then:

$$U \setminus P \subseteq K \subseteq (U \setminus P) \cup (P \cap T).$$

Thus $c \subseteq K \subseteq cab.$

b) The intersection of all keys of F is

$$I = U \setminus P = c.$$

c) $T \cap P = ab$, $F(T \setminus P) = F(c) = abc \Rightarrow T \cap P \subseteq F(T \setminus P)$. From this, F has precisely one key $K = U \setminus P = c$.

4. Translations of closure functions

In this section we shall be concerned with a class of translations of closure functions. Starting from a given closure function, translations make it possible to obtain more simple closure functions so that the key — finding problem becomes less cumbersome, etc. On the other hand, from the set of key for the new, closure function obtained in this way the corresponding keys of the original closure function can be found by a single translation.

Let $C(F)$ denote the family of all keys for the closure F. We define two sets H and G as follows:

$$G = \cap \{K \mid K \in C(F)\},$$

$$H = \cup \{K \mid K \in C(F)\}.$$

Lemma 9.4. Let F be a closure function in U, and $A \subseteq U$. We define a new F_A by

$$F_A(E) = F(E \cup A) \setminus A \quad \text{for} \quad E \subseteq U \setminus A.$$

Then: F_A is a closure function in $U \setminus A$.

Proof.

a) Let $E \subseteq U \setminus A$. Since F is a closure function, $E \subseteq F(E \cup A)$ and $E \cap A = \emptyset$. Clearly $E \subseteq F(E \cup A) \setminus A$. Consequently $E \subseteq F_A(E)$.

b) Let $E_1 \subseteq E_2 \subseteq U \setminus A$. Clearly, $F(E_1 \cup A) \subseteq F(E_2 \cup A)$, which implies $F_A(E_1) = F(E_1 \cup A) \setminus A \subseteq F(E_2 \cup A) \setminus A = F_A(E_2)$.

c) Let $E \subseteq U \setminus A$. To complete the proof it remains to show that $F_A(E) = F_A(F_A(E))$. We have $F_A(F_A(E)) = F_A(F(E \cup A) \setminus A) = F(F(E \cup A) \setminus A) \cup A) \setminus A$. Since $A \subseteq F(E \cup A)$, $F(F(E \cup A) \setminus A) \cup A) \setminus A = F(F(E \cup A)) \setminus A = F(E \cup A) \setminus A = F_A(E)$. From a), b), and c), we conclude that F_A is a closure function.

Lemma 10.4. Let F be a closure function in X, $A \cap X = \emptyset$. We define a new F^A by:

$$F^A(E) = F(E \setminus A) \cup A \quad \text{for} \quad E \subseteq X \cup A.$$

Then F^A is a closure function in $X \cup A$.

Proof.

a) Let $E\subseteq X\cup A$. We have $E=(E\setminus A)\cup(E\cap A)$. On the other hand $E\setminus A\subseteq F(E\setminus A)$ and $E\cap A\subseteq A$, showing that $E\subseteq F(E\setminus A)\cup A=F^A(E)$.

b) Let $E_1\subseteq E_2\subseteq X\cup A$. This implies $F(E_1\setminus A)\subseteq F(E_2\setminus A)$ and $F^A(E_1)=F(E_1\setminus A)\cup A\subseteq F(E_2\setminus A)\cup A\subseteq F^A(E_2)$.

c) Let $E\subseteq X\cup A$. Since F is a closure function in X and $A\cap X=\emptyset$, we have $F(E\setminus A)\cap A=\emptyset$. It is clear that: $F^A(F^A(E))=F^A(F(E\setminus A)\cup A)=F((F(E\setminus A)\cup A)\setminus A)\cup A=F(F(E\setminus A))\cup A=F(E\setminus A)\cup A=F^A(E)$. Consequently $F^A(F^A(E))=F^A(E)$ and F is a closure function in $X\cup A$.

Lemma 11.4. Let F be a closure function in U, $A\subseteq U$. Then:

1. $F(X)\setminus A\subseteq F_A(X\setminus A)$ for all $X\subseteq U$, and
2. $F_A(X)\cup A=F(X\cup A)$ for all $X\subseteq U\setminus A$.

Proof.

1. From the definition of F_A we have $F_A(X\setminus A)=F((X\setminus A)\cup A)\setminus A=F(X\cup A)\setminus A$. On the other hand $F(X)\subseteq F(X\cup A)$. Thus $F(X)\setminus A\subseteq F(X\cup A)\setminus A$. Consequently $F(X)\setminus A\subseteq F_A(X\setminus A)$.

2. We have $F_A(X)=F(X\cup A)\setminus A$. Since $A\subseteq F(X\cup A)$, we get $F_A(X)\cup A=F(F(X\cup A)\setminus A)\cup A=F(X\cup A)$.

Theorem 4.4. Let F be a closure function in U, $A\subseteq G$. Then: K is a key of F_A if and only if $A\cap K=\emptyset$ and $K\cup A$ is a key of F.

Proof. We first prove the necessity: Suppose that K is a key of F_A. Obviously $F_A(K)=U\setminus A$ and $A\cap K=\emptyset$. Taking Lemma 11.4 into account we get:

$$U=(U\setminus A)\cup A=F_A(K)\cup A\subseteq F(K\cup A)\subseteq U,$$

showing that $K\cup A$ is a superkey of F. If $K\cup A$ were not a key of F then there would exist a key $\bar{K}$ of F such that $A\subseteq\bar{K}\subsetneqq K\cup A$. Consequently there would exist an $K_1\subsetneqq K$ such that: $\bar{K}=K_1\cup A$, $K_1\cap A=\emptyset$. Since $\bar{K}$ is a key for F, $F(K_1\cup A)=U$. Applying Lemma 11.4, clearly $U\setminus A=F(K_1\cup A)\setminus A\subseteq F_A(K_1\cup A\setminus A)=F_A(K_1)$. So we have $K_1\subsetneqq K$, $F_A(K_1)=U\setminus A$. This contradicts the hypothesis that K is a key of F_A.

We now turn to the proof of sufficiency. Suppose that $K\cap A=\emptyset$ and $K\cup A$ is a key for F_A. We have to show that K is a key for F_A. Since $K\cup A$ is a key for F, we have $F(A\cup K)=U$. By virtue of Lemma 11.4 and $K\cap A=\emptyset$, we get $U\setminus A=F(K\cup A)\setminus A\subseteq F_A(K\cup A\setminus A)=F_A(K)\subseteq U\setminus A$. Thus $U\setminus A=F_A(K)$, showing that K is a superkey for F_A. Assume that K is not a key of F_A, then there would exist a key $\bar{K}$ of F_A such that $\bar{K}\subsetneqq K$ and $F_A(\bar{K})=U\setminus A$. Applying Lemma 11.4, it follows $U=F_A(\bar{K})\cup A=F(\bar{K}\cup A)$ where $\bar{K}\cup A\subsetneqq K\cup A$. This contradicts the fact that $K\cup A$ is a key for F, that completes the proof.

Theorem 5.4. Let F be a closure function in U, $A\subseteq U$ and $A\cap H=\emptyset$. Then K is a key of F_A iff K is a key of F.

Proof.

1. The necessity: Suppose that K is a key for F_A. Obviously $F_A(K)=U\setminus A$. By virtue of Lemma 11.4 we have $F(K\cup A)=F_A(K)\cup A=(U\setminus A)\cup A=U$, showing

that $K \cup A$ is a superkey for F. Hence, there exists a key $\bar{K}$ of F such that $\bar{K} \subseteq K \cup A$. Since $A \cap H = \emptyset$ then $\bar{K} \cap A = \emptyset$. From this, it is easy to see that $\bar{K} \subseteq K$. There are two possible cases:

a) $\bar{K} = K$. Then obviously K is a key for F.

b) $\bar{K} \subsetneq K$. Since $\bar{K}$ is a key for F, $F(\bar{K}) = U$. Applying Lemma 11.4, we have $U \setminus A = F(\bar{K}) \setminus A \subseteq F_A(\bar{K} \setminus A) \subseteq U \setminus A$ and $\bar{K} \cap A = \emptyset$, that is $F_A(\bar{K}) = U \setminus A$. This contradicts the fact that K is a key for F_A.

2. The sufficiency: Suppose that K is a key for F. We have to prove that K is also a key for F_A. We have, by the definition of keys, $F(K) = U$. Applying Lemma 11.4, $U \setminus A = F(K) \setminus A \subseteq F_A(K \setminus A) \subseteq U \setminus A$. Thus $F_A(K \setminus A) = U \setminus A$. Since $A \cap H = \emptyset$, it follows $K \cap A = \emptyset$. Consequently $F_A(K) = U \setminus A$ showing that K is a superkey of F_A. Now assume to the contrary, that K is not a key for F_A. Then, there would exist a key $\bar{K}$ of F_A such that $\bar{K} \subsetneq K$. Obviously $F_A(\bar{K}) = U \setminus A$. We invoke Lemma 11.4 to deduce $F(\bar{K} \cup A) = F_A(\bar{K}) \cup A = (U \setminus A) \cup A = U$, showing that $\bar{K} \cup A$ is a superkey of F. Consequently, there exists a key $\bar{\bar{K}}$ of F such that $\bar{\bar{K}} \subseteq \bar{K} \cup A$, $\bar{\bar{K}} \cap A = \emptyset$. From this $\bar{\bar{K}} \subseteq \bar{K} \subsetneq K$. This contradicts the hypothesis that K is a key for F.

This completes the proof.

To continue let us recall a result from § 1. Let F be a closure function in U. Let us set

$$T = \bigcup \{X \mid X \in 2^U \text{ and } F(X) \neq X\},$$

$$P = \bigcup \{\bar{X} \mid X \in 2^U \text{ and } F(X) \neq X\}.$$

Then, the necessary condition under which K is a key for F is

1. $U \setminus P \subseteq K \subseteq (U \setminus P) \cup (T \cap P)$, and

2. the intersection I of all keys for F is $I = U \setminus P$. We have the following theorems.

Theorem 6.4. Let F be a closure function in U and $I = U \setminus \bigcup \{F(X) \setminus X \mid X \in Z^U$ and $F(X) \neq X\}$. Then K is a key of F_I if and only if $K \cap I = \emptyset$ and $K \cup I$ is a key of F.

Theorem 7.4. Let F be a closure function in U, and $N = P \setminus T$. Then K is a key of F_N if and only if K is a key of F.

Lemma 12.4. Let F be a closure function in U, $U \cap A = \emptyset$. Then

1. $F^A(X) \setminus A \subseteq F(X \setminus A)$, $X \subseteq U \cup A$,

2. $F(X) \cup A = F^A(X \cup A)$, $X \subseteq U$.

Proof. We first prove

1. Let $X \subseteq U \cup A$. From the definition of F^A we have:

$$F^A(X) \setminus A = (F(X \setminus A) \cup A) \setminus A = F(X \setminus A) \setminus A \subseteq F(X \setminus A).$$

2. Let $X \subseteq U$. We have $F^A(X \cup A) = F(X \cup A \setminus A) \cup A = F(X \setminus A) \cup A$. Since $A \cap U = \emptyset$, $A \cap X = \emptyset$. It is clear that $F^A(X \cup A) = F(X \setminus A) \cup A = F(X) \cup A$. This completes the proof.

Theorem 8.4. Let F be a closure function in U and $A \cap U = \emptyset$. Then $K \cap A = \emptyset$ and K is a key of F^A iff K is a key of F.

Proof. We first prove the necessity: Suppose that K is a key of F^A and $K \cap A = \emptyset$. Obviously $F^A(K) = U \cup A$. Taking Lemma 12.4 we get:

$$U = U \cup A \setminus A = F^A(K) \setminus A \subseteqq F(K \setminus A) \subseteqq U.$$

Obviously, $F(K) = F(K \setminus A) = U$, showing that K is a superkey of F. If K were not a key of F, then there would exist a key $\bar{K}$ such that $\bar{K} \subsetneqq K$ and $F(\bar{K}) = U$. From the definition of F^A we find: $F^A(\bar{K}) = F(\bar{K} \setminus A) \cup A = F(\bar{K}) \cup A = U \cup A$. This contradicts the hypothesis that K is a key for F. We now turn to the proof of the sufficiency. Suppose that K is a key for F. We have to show that $K \cap A = \emptyset$ and K is a key for F^A. Since K is a key of F, we have $F(K) = U$ and $K \subseteqq U$. Thus $K \cap A = \emptyset$. On the other hand $F^A(K) = F(K \setminus A) \cup A = F(K) \cup A = U \cup A$ showing that K is a superkey of F^A. If K is not a key of F^A, then there would exist a key $\bar{K}$ such that $\bar{K} \subsetneqq K$ and $F^A(\bar{K}) = U \cup A$. We have $U = F^A(\bar{K}) \setminus A \subseteqq F(\bar{K} \setminus A) = F(\bar{K}) \subseteqq U$. Thus $F(\bar{K}) = U$. This contradicts the hypothesis that K is a key of F. Hence K is a key of F^A. The proof is complete.

5. On a relationship between keys for relation scheme and keys for closure function

Let us recall some necessary notions and definitions. Definition of a closure function: Let $U = \{A_1, A_2, \ldots, A_n\}$ be a set of n elements (attributes) and 2^U its power set. The function f: $2^U \to 2^U$ is called a closure function or closure iff for every $X, Y \in 2^U$,

a) $X \subseteqq f(X)$,

b) $f(f(X)) = f(X)$,

c) if $X \subseteqq Y$ then $f(X) \subseteqq f(Y)$.

Let $K \subseteqq U$, K is said to be a superkey for the closure function f if $f(K) = U$. K is said to be a key for the closure function f if K is a superkey for f but $f(X) \neq U$ for any proper subset X of K. Let $C(f)$ denote the family of all keys for the closure function f.

Definition of a relation scheme: [3].

Armstrong's axioms [4]. Let $X, Y, Z \subseteqq U$;

Rule 1: (Reflexivity) if $Y \subseteqq X$ then $X \to Y$;

Rule 2: (Transitivity) if $X \to Y$ and $Y \to Z$ then $X \to Z$;

Rule 3: (Augmentation) if $X \to Y$ then $X \cup Z \to Y \cup Z$.

Relation scheme:

A relation scheme is a 2-tuple (U, F) where:

a) U is a finite set (of attributes),

b) F is a finite set of functional dependencies (FD).

Let F be a given set of FD's of a relation scheme. We can apply these rules to the FD's in F to derive new FD's.

The set of all FD's that are derivable from F by repeated applications of Armstrong's rules (including the FD's in F) is called the closure of F and is denoted by F^+.

Let $X \subseteq U$ be a given set of attributes. We define the closure of X (relative to F), denote by X^+, to be the set of all attributes that are functionally dependent on X:

$$X^+ = \{A | (X \to A) \in F^+\}.$$

Algorithm for finding X^+:

$$X^{(0)} = X,$$

$$X^{(i+1)} = X^{(i)} \cup \{R_j | L_j \to R_j \in F \text{ and } L_j \subseteq X^{(i)}\}.$$

There exists an N such that $X^{(N)} = X^{(N+1)}$. Then $X^+ = X^{(N)}$. We have $X \to Y \in F^+$ iff $Y \subseteq X^+$.

Let (U, F) be a relation scheme and let X be a subset of U. We say that X is a superkey of (U, F) if every attribute in U functionally depends on X. If the set X is a superkey and it does not properly contain any superkey then X is a key for (U, F).

$C(U, F)$ denotes the set of all keys of a relation scheme (U, F).

Theorem 9.5. Let (U, F) be a relation scheme. We define the function $f: 2^U \to 2^U$ as follows:

$$X \in 2^U: f(X) = X^+.$$

Then 1. f is a closure function;
2. $C(f) = C(U, F)$.

Proof. We first prove 1.
a) $X \subseteq X^+$. Clearly $X \subseteq f(X)$.
b) $X = (X^+)^+$ implies $f(X) = f(f(X))$.
c) $X \subseteq Y \Rightarrow X^+ \subseteq Y^+$ implies $f(X) \subseteq f(Y)$.
Consequently f is a closure function.

2. Now let K be a key of the relation scheme (U, F). Obviously $K^+ = U$. Thus we have $f(K) = U$, showing that K is a superkey for f. Now assume to the contrary that, K is a not a key for f. Then there would exist a key $\bar{K}$ of f such that $\bar{K} \subsetneqq K$ and $f(\bar{K}) = U$. From the definition of f we have $\bar{K}^+ = U$. Thus $\bar{K} \to U$. This contradicts the hypothesis that K is a key of (U, F).

Now let K be a key of the closure function f. Obviously $f(K) = U$. Thus $K^+ = U, K$ is a superkey for (U, F). Now assume to the contrary, that K is not a key for (U, F). Then there would exist a key $\bar{K}$ of (U, F) such that $\bar{K} \subsetneqq K$ and $\bar{K} \to U$. We have $K^+ = U$. Thus $f(\bar{K}) = U$. This contradicts the hypothesis that K is a key for f.

Theorem 10.5. Let f be a closure function in U. We define the relation scheme (U, F) as follows:

$$F = \{X \to f(X) | X \in 2^U\}.$$

Then

$$C(f) = C(U, F).$$

Proof. From the definition of F we have $X \to f(X) \in F$. Thus $f(X) \subseteq X^+$. Now we have to prove $X^+ \subseteq f(X)$.

We proceed by induction on n. If $n=0$ we have $X^{(0)}=X\subseteq f(X)$. Assume it is true for n i.e. $x^{(n)}\subseteq f(X)$. In fact we have $X^{(n+1)}=X^{(n)}\cup\{\cup Y|Z\to Y\in F, Y=F(Z)$ and $Z\subseteq X^{(n)}\}$. From $Z\in X^{(n)}$ we have $f(Z)\subseteq f(X^{(n)})\subseteq f(f(X))=f(X)$. Obviously $X^{(n+1)}\subseteq f(X)$. Finally, we find $f(X)=X^{+}$. Applying Theorem 9.5, we have $C(f)=$ $=C(Z, F)$. The proof is complete.

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