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Boolean-type retractable automata with traps

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Abstract

As in other branches of the algebra, it is a natural idea to find connections between automata and their congruence lattices. For example, describe all automata whose congruence lattices are Boolean algebras. Although this problem will not be solved in this paper, we give a necessary condition for automata to be automata whose congruence lattices are Boolean algebras.

The main object of this paper is to describe a special class of automata with this (necessary) condition. More precisely, we describe all Boolean-type retractable automata (Definition 4.) with traps.

By an automaton we shall mean a system $\mathbf{A} = (A, X, \delta)$ consisting of a state set A, an input set X and a transition function $\delta : A \times X \longrightarrow A$ $(A \neq \emptyset, X \neq \emptyset)$.

Denote X^* the free monoid over X and e the empty word of X. The transition function δ can be extended to $A \times X^*$ such that

$$\delta(a,p) = \begin{cases} a & \text{if } p = e \\ \delta(\delta(a,q),x) & \text{if } p = qx \ (q \in X^*, x \in X) \end{cases}$$

for all $a \in A, p \in X^*$. As known, an equivalence relation α on the state set A is called a congruence on the automaton $\mathbf{A} = (A, X, \delta)$ if $(a,b) \in \alpha$ implies $(\delta(a,x), \delta(b,x)) \in \alpha$ for all $a, b \in A$ and $x \in X$. The set of all congurences of an automaton $\mathbf{A}$ forms a lattice. This lattice will be denoted by $\mathcal{L}(\mathbf{A})$. The least element and the greatest element of $\mathcal{L}(\mathbf{A})$ will be denoted by ι and ω, respectively.

If ρ is a congruence on an automaton $\mathbf{A} = (A, X, \delta)$ and A/ρ denotes the set of all ρ-classes $[a]\rho$ of $\mathbf{A}, a \in A$, then $\mathbf{A}/\rho = (A/\rho, X, \delta_\rho)$ is an automaton, where δ_ρ is defined by letting $\delta_\rho([a]\rho, x) = [\delta(a,x)]\rho$, for all $a \in A$ and $x \in X$. The automaton $\mathbf{A}/\rho$ is called the factor automaton $\mathbf{A}$ modulo ρ.

If $\mathbf{R} = (R, X, \delta_R)$ is a subautomaton of an automaton $\mathbf{A} = (A, X, \delta)$ (here δ_R is the restriction of δ to R),then the subset R of A will be called a right ideal of $\mathbf{A}$ (see [2]). It can be easily verified that, for every right ideal R of $\mathbf{A}$,

$$\rho_R = \{(a,b) \in A \times A : a = b \quad \text{or} \quad a, b \in R\}$$

is a congruence on $\mathbf{A}$. This congurence is called the Rees congurence determined by R. The factor automaton $\mathbf{A}/\rho_R$ is called the Rees factor automaton of $\mathbf{A}$ modulo ρ_R (or modulo R).

A mapping φ of the state set A of an automaton $\mathbf{A} = (A, X, \alpha)$ into the state set B of an automaton $\mathbf{B} = (B, X, \beta)$ is called a homomrphism of $\mathbf{A}$ into $\mathbf{B}$ if $\lambda(\alpha(a,x)) = \beta(\lambda(a), x)$ for all $a \in A$ and $x \in X$. The congruence on $\mathbf{A}$ determined by the homomorphism λ will be denoted by con λ.

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Definition 1 *A right ideal R of an automaton $\mathbf{A} = (A, X, \delta)$ will be called a retract right ideal if there is a homomorphism λ of $\mathbf{A}$ onto $\mathbf{R}$ which leaves the elements of R fixed. λ will be called a retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$.*

Definition 2 *We shall say that an automaton $\mathbf{A}$ is a retractable automaton if every right ideal of $\mathbf{A}$ is a retract right ideal.*

Theorem 3 *If $\mathbf{A}$ is an automaton such that $\mathcal{L}(\mathbf{A})$ is complemented [5], then $\mathbf{A}$ is a retractable automaton.*

Proof. Let R be a right ideal of an automaton $\mathbf{A} = (A, X, \delta)$. If $\mathcal{L}(\mathbf{A})$ is complemented, then, for the Rees congurence ρ_R, there is an element η_R in $\mathcal{L}(\mathbf{A})$ such that $\rho_R \wedge \eta_R = \iota$ and $\rho_R \vee \eta_R = \omega$. Then $\mathbf{A}/\eta_R = (A/\eta_R, X, \delta\eta_R)$ is isomorphic to $\mathbf{R} = (R, X, \delta_R)$.

Let λ_R denote the canonical homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\eta_R$, that is $\eta_R = \text{con } \lambda_R$. Identifying $\mathbf{A}/\eta_R$ with $\mathbf{R}$ it can be easily verfied that λ_R is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$.

Definition 4 *An automaton $\mathbf{A} = (A, X, \delta)$ will be called a Boolean- type retractable automaton if, for every right ideal R of $\mathbf{A}$, there is a retract homomorphism λ_R of $\mathbf{A}$ onto $\mathbf{R}$ such that $R \subseteq S$ implies $\text{con } \lambda_S \subseteq \text{con } \lambda_R$ in $\mathcal{L}(\mathbf{A})$, for all right ideals R and S of $\mathbf{A}$.*

Theorem 5 *If $\mathbf{A}$ is an automaton such that $\mathcal{L}(\mathbf{A})$ is a Boolean algebra, then $\mathbf{A}$ is a Boolean-type retractable automaton.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be an automaton such that $\mathcal{L}(\mathbf{A})$ is a Boolean algebra. As a Boolean algebra is a complemented lattice, it follows, by Theorem 3, that $\mathbf{A}$ is a retractable automaton. Let R and S be arbitrary right ideals of $\mathbf{A}$ with $R \subseteq S$. Then $\rho_R \subseteq \rho_S$, that is $\rho_R \wedge \rho_S = \rho_R$. From this equality it follows that $\eta_R \vee \eta_S = \eta_R$, that is $\eta_S \subseteq \eta_R$ which means that $\text{con } \lambda_S \subseteq \text{con } \lambda_R$. Thus $\mathbf{A}$ is a Boolean-type retractable automaton.

Following [4], an element a_0 of the state set A is called a trap of the automaton $\mathbf{A} = (A, X, \delta)$ if $\delta(a_0, x) = a_0$, for all $x \in X$.

Theorem 6 *Every right ideal of a retractable automaton having traps contains a trap.*

Proof. Let R be a right ideal of a retractable automaton $\mathbf{A} = (A, X, \delta)$ with traps. Let a_0 be an arbitrary trap of $\mathbf{A}$ and λ_R a retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Then $\delta(\lambda_R(a_0), x) = \lambda_R(\delta(a_0, x)) = \lambda_R(a_0)$, for all $x \in X$. So $\lambda_R(a_0)$ is a trap of $\mathbf{A}$. As $\lambda_R(a_0) \in R$, the theorem is proved.

Definition 7 *An automaton will be called a one-trap-automaton (or an OT-automaton) if it has exactly one trap. If $\mathbf{A} = (A, X, \delta)$ is an OT-automaton with the trap a_0, then it will be denoted by $\mathbf{A} = (A, X, \delta; a_0)$.*

Theorem 8 *Every Boolean-type retractable automaton with traps has a homomorphic image which is a Boolean-type retractable OT-automaton.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a Boolean-type retractable automaton with traps. Let R_t denote the set of all traps of $\mathbf{A}$. Then R_t is a right ideal of $\mathbf{A}$. It is evident that the factor automaton $\mathbf{A}/\rho_{R_t} = (A/\rho_{R_t}, X, \delta_{r_{R_t}})$ is an OT-autoamton. We show that $\mathbf{A}/\rho_{R_t}$ is also a Boolean-type retractable automaton. Let α denote the canonical homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\rho_{R_t}$. Let R be an arbitrary right ideal of $\mathbf{A}/\rho_{R_t}$. Then $R\alpha^{-1} = \{a \in A : \alpha(a) \in R\}$ is a right ideal of $\mathbf{A}$. By Theorem 6, $R\alpha^{-1} \cap R_t \neq \emptyset$. So R contains the trap of $\mathbf{A}/\rho_{R_t}$. As $\mathbf{A}$ is a Boolean-type retractable automaton, there is a retract homomorphism $\lambda_{R\alpha^{-1}}$ of $\mathbf{A}$ onto $R\alpha^{-1}$. We define a mapping λ_R of $\mathbf{A}/\rho_{R_t}$ onto $\mathbf{R}$ as follows

$$\lambda_R(\alpha(a)) = \alpha(\lambda_{R\alpha^{-1}}(a)),$$

for all $a \in A$. We show that λ_R is a homomorphism of $\mathbf{A}/\rho_{R_t}$ onto $\mathbf{R}$. Let $a \in A, x \in X$ be arbitrary elements. Then

$$\delta_{\rho_{R_t}}(\lambda_R(\alpha(a)), x) = \lambda_{\rho_{R_t}}(\alpha(\lambda_{R\alpha^{-1}}(a)), x) = \alpha(\delta(\lambda_{R\alpha^{-1}}(a), x))$$

$$= \alpha(\lambda_{R\alpha^{-1}}(\delta(a, x))) = \lambda_R(\alpha(\delta(a, x))) = \lambda_R(\delta_{\rho_{R_t}}(\alpha(a), x)).$$

So λ_R is a homomorhism of $\mathbf{A}/\rho_{R_t}$ onto $\mathbf{R}$. It is evident that λ_R leaves the elements of R fixed. So λ_R is a retract homomorphism of $\mathbf{A}/\rho_{R_t}$ onto $\mathbf{R}$. Next we show that $R_1 \subseteq R_2$ implies con $\lambda_{R_2} \subseteq$ con λ_{R_1} in $\mathcal{L}(\mathbf{A}/\rho_{R_t})$, for every right ideals R_1 and R_2 of $\mathbf{A}/\rho_{R_t}$. Let R_1 and R_2 be right ideals of $\mathbf{A}/\rho_{R_t}$ with $R_1 \subseteq R_2$. Then $R_1\alpha^{-1} \subseteq R_2\alpha^{-1}$ and con $\lambda_{R_2\alpha^{-1}} \subseteq$ con $\lambda_{R_1\alpha^{-1}}$. Let a, b be arbitrary elements of $\mathbf{A}$ with $(\alpha(a), \alpha(b)) \in$ con λ_{R_2}, that is $\lambda_{R_2}(\alpha(a)) = \lambda_{R_2}(\alpha(b))$. Then $\alpha(\lambda_{R_2\alpha^{-1}}(a)) = \alpha(\lambda_{R_2\alpha^{-1}}(b))$ and so $\lambda_{R_2\alpha^{-1}}(a), \lambda_{R_2\alpha^{-1}}(b) \in R_t$ or $\lambda_{R_2\alpha^{-1}}(a), \lambda_{R_2\alpha^{-1}}(b) \notin R_t$ and $\lambda_{R_2\alpha^{-1}}(a) = \lambda_{R_2\alpha^{-1}}(b)$. Assume $\lambda_{R_2\alpha^{-1}}(a), \lambda_{R_2\alpha^{-1}}(b) \in R_t \subseteq R_1\alpha^{-1} \subseteq R_2\alpha^{-1}$. Then $\emptyset \neq [a]$ con $\lambda_{R_2\alpha^{-1}} \cap R_t = [b]$ con $\lambda_{R_1\alpha^{-1}} \cap R_t$. As $[a]$ con $\lambda_{R_2\alpha^{-1}} \cap R_t \subseteq [a]$ con $\lambda_{R_1\alpha^{-1}} \cap R_t$ and $[b]$ con $\lambda_{R_1\alpha^{-1}} \cap R_t \subseteq [b]$ con $\lambda_{R_1\alpha^{-1}} \cap R_t$, we get $\lambda_{R_1\alpha^{-1}}(a) \in R_t$ and $\lambda_{R_1\alpha^{-1}}(b) \in R_t$. Then $\alpha(\lambda_{R_1\alpha^{-1}}(a)) = \alpha(\lambda_{R_1\alpha^{-1}}(b))$, that is $\lambda_{R_1}(\alpha(a)) = \lambda_{R_1}(\alpha(b))$. So $(\alpha(a), \alpha(b)) \in$ con λ_{R_1}. Assume $\lambda_{R_2\alpha^{-1}}(a), \lambda_{R_2\alpha^{-1}}(b) \notin R_t, \lambda_{R_2\alpha^{-1}}(a) = \lambda_{R_2\alpha^{-1}}(b)$. Then $(a, b) \in$ con $\lambda_{R_2\alpha^{-1}} \subseteq$ con $\lambda_{R_1\alpha^{-1}}$, that is $\lambda_{R_1\alpha^{-1}}(a) = \lambda_{R_1\alpha^{-1}}(b)$. So $\alpha(\lambda_{R_1\alpha^{-1}}(a)) = \alpha(\lambda_{R_1\alpha^{-1}}(b))$ that is $\lambda_{R_1}(\alpha(a)) = \lambda_{R_1}(\alpha(b))$. Thus $(\alpha(a), \alpha(b)) \in$ con λ_{R_1}. Consequently con $\lambda_{R_2} \subseteq$ con λ_{R_1}. So $\mathbf{A}/\rho_{R_t}$ is a Boolean-type retractable OT-automaton.

By Theorem 8, we can concetrate our attention to only Boolean-type retractable OT-automata.

By [1], if $\underline{a}$ is a state of an automaton $\mathbf{A} = (A, X, \delta)$, then the intersection of all right ideals of $\mathbf{A}$ containing $\underline{a}$ is called the principal right ideals of $\mathbf{A}$ generated by a. This right ideal will be denoted by $R(a)$. It can be easily verfied that $R(a) = \delta(a, X^*) = \{\delta(a, p) : p \in X^*\}$.

The relation $\mathcal{R}$ on an automaton $\mathbf{A} = (A, X, \delta)$ defined as follows

$$\mathcal{R} = \{(a, b) \in A \times A : R(a) = R(b)\}$$

is an equivalence relation on A. The $\mathcal{R}$-class of A containing the elements a of A will be denoted by R_a. Let $R(a) - R_a$ be denoted by $R[a]$.

Theorem 9 *If a is an arbitrary element of an OT-automaton $\mathbf{A}$, then $R[a]$ is either empty (if a is the trap of $\mathbf{A}$) or a right ideal of $\mathbf{A}$ (if a is not the trap of $\mathbf{A}$).*

Proof. See, for example, [1].

Let $\mathbb{A} = (A, X, \delta)$ be an automaton. The factor automata $\mathbb{R}(a)/\rho_{R[a]}$ will be called the principal r-factors of $\mathbb{A}$ and they will be denoted by $\mathbb{R}\{a\}$. The state set and the transition function of $\mathbb{R}\{a\}$ will be denoted by $R\{a\}$ and $\delta_{\mathbb{R}\{a\}}$, respectively.

Let T be a set with a partially ordering $\leq$ such that every two-element subset of T has a lower bound in T and every non-empty subset of T having an upper bound in T contains a greates element. Then T is a semilattice under multiplication "." by letting $a \cdot b (a, b \in T)$ be the (necessarily unique) greatest lower bound of a and b in T. Following [6], a semilattice which can be constructed as above is called a tree. It is easy to see that the ideals of a tree T are those non-empty subsets I of T for which $b \in I$ and $a \leq b$ together imply $a \in I$ for all $a, b \in T$. If I is an ideal of a tree T, then the mapping π of T onto I letting $\pi(a)$ be the greatest element in the set $\{x \in I : x \leq a\}$ is a retract homomprhism of T onto I (see [6]). So evry ideal of a tree is a retract ideal [6].

Theorem 10 *The set $Prf(\mathbb{A})$ of all principal r-factors of a retractable OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$ is a tree with the least element $\mathbb{R}\{a_0\}$ under ordering $\leq$ defined as follows: $\mathbb{R}\{a\} \leq \mathbb{R}\{b\}$ if and only if $R(a) \subseteq R(b)$.*

Proof. Let $\mathbb{A} = (A, X, \delta; a_0)$ be a retractable OT-automaton. It is evident that $\leq$ is a partially ordering on Prf($\mathbb{A}$). Let $\{\mathbb{R}\{a_j\} : j \in J\}$ be a non-empty subset of Prf($\mathbb{A}$). Assume that $\mathbb{R}\{a_j\} \leq R\{a\}$ for some $a \in A$. We shall prove that there is an element j_0 in J such that $\mathbb{R}\{a_j\} \leq R\{a_{j_0}\}$ for all $j \in J$. By the assumption that $\mathbb{R}\{a_j\} \leq \mathbb{R}\{a\}$ for all $j \in J$, we have $R(a_j) \subseteq R(a)$ for all $j \in J$. As $\mathbb{A}$ is a retractable automaton, there is a retract homomorphism λ of $\mathbb{A}$ onto $\mathbb{B} = (\cup R(a_j), X, \delta)$. So $\lambda(a) \in \{\cup R(a_j) : j \in J\}$ that is $\lambda(a) \in R(a_{j_0})$ for some $j_0 \in J$. Thus $R(\lambda(a)) \subseteq R(a_{j_0})$. It can be easily verified that $R(a_j) \subseteq R(a)$ implies $R(\lambda(a_j)) \subseteq R(\lambda(a))$ for all $j \in J$. So $R(\lambda(a)) = R(a_{j_0})$ that is $\mathbb{R}\{a_{j_0}\}$ is the greatest element of $\{\mathbb{R}\{a_j\} : j \in J\}$. Let $\mathbb{R}\{a\}$ and $\mathbb{R}\{b\}$ be arbitrary elements in Prf($\mathbb{A}$). Let K denote the set of all principal r-factors $\mathbb{R}\{c\}$ of $\mathbb{A}$ for which $\mathbb{R}\{c\} \leq \mathbb{R}\{a\}$ and $\mathbb{R}\{c\} \subseteq \mathbb{R}\{b\}$. As a_0 is in every right ideal of $\mathbb{A}$, it follows that K is not empty. So $\mathbb{R}\{a\}$ and $\mathbb{R}\{b\}$ have a common lower bound. Consequently the set of all principal r-factors of $\mathbb{A}$ forms a tree under ordering $\leq$. It is evident that $\mathbb{R}\{a_0\}$ is the least element of Prf($\mathbb{A}$).

Definition 11 *We shall say that an OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$ is trapped if $\delta(a, x) = a_0$ for all $a \in A$ and $x \in X$.*

We note that a trivial automaton (when the state set has only one element) is trapped.

Definition 12 *An OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$ will be called an r-simple OT-automaton if it is not rapped and $R = A$ or $R = \{a_0\}$, for all right ideals R of $\mathbb{A}$.*

Theorem 13 *Every principal r-factor of an OT-automaton is either r-simple or trapped.*

Proof. Let a be an arbitrary element of an OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$. It is easy to see that $\mathbb{R}\{a\}$ is an OT-automaton. If $a = a_0$, then $\mathbb{R}\{a\}$ is trivial. Assume $a \neq a_0$. If $\delta(b, x) \in R[a]$ for all $b \in R_a$ and $x \in X$ such that $\delta(b, x) \notin R[a]$, then $\mathbb{R}\{a\}$ is r-simple.

Definition 14 *An OT-automaton is called an r-semisimple OT-automaton if its every principal r-factor is either trivial or r-simple.*

Next we characterize the r-semisimple OT-automata. Let $X^+ = X^* - \{e\}$, where e is the empty word.

Theorem 15 *An OT-automaton* $\mathbb{A} = (A, X, \delta; a_0)$ *is r-semisimple if and only if every right ideal R of* $\mathbb{A}$ *satisfies the following:*
(i) for every $a \in R$ *there are elements* $b \in R$ *and* $p \in X^+$ *such that* $a = \delta(b, p)$.

Proof. Let $\mathbb{A} = (A, X, \delta; a_0)$ be an r-semisimple OT-automaton. Let R be a right ideal of $\mathbb{A}$. If $a \in R$, then $R(a) \subseteq R$ and $\mathbb{R}\{a\}$ is either trivial (if $a = a_0$) or r-simple. We may assume $a \neq a_0$. Then $\mathbb{R}\{a\}$ is r-simple. So there is an element b in $R(a) - R[a]$, such that $\delta(b, x) \notin R[a]$ for some $x \in X$. So $(R[a]) \cup \{\delta(b, p) : p \in X^+\} = R(a)$ which implies $a = \delta(b, p)$ for some $p \in X^+$.

Conversely, assum that an OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$ satifies (i). We prove that $\mathbb{A}$ is r-semisimple. Let c be an arbitrary element of A. We may assume $c \neq a_0$. Then $\mathbb{R}\{c\}$ is a non-trivial OT-automaton. We must show that $\mathbb{R}\{c\}$ is not trapped. Let a be an arbitrary element of $R(c)$ with $a \neq a_0$. Then, applying condition (i) for $R = R(c)$, there are elements b in $R(c)$ and p in X^+ such that $a = \delta(b, p)$. So $\mathbb{R}\{c\}$ is not trapped. Consequently $\mathbb{A}$ is $r-$semisimple.

We remark that condition (i) can be exchanged by condition
(ii) for every $a \in R$ there are elements $b \in R$ and $x \in X$ such that $a = \delta(b, x)$.

Definition 16 *Let* $\mathbb{A} = (A, X, \delta_A)$ *be a subautomaton of an automaton* $\mathbb{B} = (B, X, \delta)$. *We shall say that* $\mathbb{B}$ *is a dilation of* $\mathbb{A}$ *if there is a mapping* φ *of* B *onto* A *which leaves the elements of* A *fixed and* $\delta(b, x) = \delta_A(\varphi(b), x)$ *for all* $b \in B$ *and* $x \in X$.

Theorem 17 *An automaton is a Boolean-type retractable OT-automaton if and only if it is a dilation of an r-semisimple Boolean-type retractable OT-automaton.*

Proof. Assume that $\mathbb{B} = (B, X, \eta; a_0)$ is a Boolean-type retractable OT-automaton. Let $A = \eta(B, X) = \{\eta(b, x) : b \in B, x \in X\}$ and δ be the restriction of η to A. As $\mathbb{B}$ is a Boolean-type retractable automaton and A is a right ideal of $\mathbb{B}$, there is a retract homomorphism φ of $\mathbb{B}$ onto $\mathbb{A}$. Let $B_a = \{b \in B - A : \varphi(b) = a\}, a \in A$. If $b \in B_a$, then $A \ni \eta(b, x) = \varphi(\eta(b, x)) = \delta(\varphi(b), x)$. This implies that $\mathbb{B}$ is a dilation of $\mathbb{A}$.

It is evident that $\mathbb{A}$ is an r-semisimple OT-automaton (the r-semisimplicity follows from Theorem 15).

We show that $\mathbb{A}$ is a Boolean-type retractable automaton. Let R be an arbitrary right ideal of $\mathbb{A}$. Then R is also a right ideal of $\mathbb{B}$. So there is a retract homomorhism of $\mathbb{B}$ onto $\mathbb{R}$. The restriction of φ to $\mathbb{A}$ is a retract homomorphism of $\mathbb{A}$ onto $\mathbb{R}$. As $\mathbb{B}$ is a Boolean-type retractable automaton, it follows that $\mathbb{A}$ is a Boolean-type retractable one. Thus the first part of the theorem is proved.

Conversely, assume that an automaton $\mathbb{B} = (B, X, \eta)$ is a dilation of an r-semisimple Boolean-type retractable OT-automaton $\mathbb{A} = (A, X, \delta; a_0)$. Then there is a mapping φ of B into A which leaves the elements of A fixed and $\eta(b, x) = \delta(\varphi(b), x)$ for all $b \in B$ and $x \in X$.

It can be easily verfied that $\mathbb{B}$ is an OT-automaton with the trap a_0.

We prove that $\mathbb{B}$ is a Boolean-type retractable automaton. Let I be a right ideal of $\mathbb{B}$. Then $R = I \cap A$ is not empty and a right ideal of $\mathbb{A}$. As $\mathbb{A}$ is a Boolean-type retractable automaton, there is a retract homomorphism λ_R of $\mathbf{A}$ onto $\mathbf{R}$. Let $\wedge_I$ be defined on B as follows

$$\wedge_I(b) = \begin{cases} b & \text{if } b \in I \\ \lambda_R(\varphi(b)) & \text{if } b \notin I. \end{cases}$$

It is evident that $\wedge_I$ leaves the elements of I fixed and the restriction of $\wedge_I$ to A equals λ_R. We show that $\wedge_I$ is a homomorphism of $\mathbf{B}$ onto $\mathbf{I}$. Let $b \in B$ and $x \in X$ be arbitrary elements. If $b \in A$, then $\eta(\wedge_I(b), x) = \eta(\lambda_R(b), x) = \delta(\lambda_R(b), x) = \lambda_R(\delta(b, x)) = \wedge_I(\delta(b, x))$.

If $b \in (B - A) \cap I$, then $\eta(\wedge_I(b), x) = \eta(b, x) = \wedge_I(\eta(b, x))$, because $\eta(b, x) \in I$. If $b \in (B - A) - I$, then $\eta(\wedge_I(b), x) = \eta(\lambda_R(\varphi(b)), x) = \delta(\lambda_R(\varphi(b)), x) = \lambda_R(\delta(\varphi(b), x)) = \wedge_I(\delta(\varphi(b), x)) = \wedge_I(\eta(b, x))$.

Thus $\wedge_I$ is a retract homomorphism of $\mathbf{B}$ onto $\mathbf{I}$.

Assume that I and J are right ideals of $\mathbf{B}$ with $I \subseteq J$. Let $R_1 = I \cap A$ and $R_2 = J \cap A$. Then $R_1 \subseteq R_2$ and so con $\lambda_{R_2} \subseteq$ con λ_{R_1}. We show that con $\wedge_J \subseteq$ con $\wedge_I$. Assume $(a, b) \in$ con $\lambda_J; a, b \in B$. Then $\wedge_J(a) = \wedge_J(b)$. If $a, b \in J$, then, by the definition of $\wedge_J$, we have $a = b$. In this case $(a, b) \in$ con $\wedge_I$. If $a \in J$ and $b \notin J$, then $a = \wedge_J(a) = \wedge_J(b) = \lambda_{R_2}(\varphi(b)) \in A$. So $a \in A \cap J = R_2$ from which we get $a = \lambda_{R_2}(a)$. Thus $\lambda_{R_2}(a) = \lambda_{R_2}(\varphi(b))$ and so $\lambda_{R_1}(a) = \lambda_{R_1}(\varphi(b)) = \wedge_I(b)$. If $a \in I$, then $\lambda_{R_1}(a) = \wedge_I(a)$. If $a \notin I$, then, using $a = \varphi(a)$, we get $\lambda_{R_1}(a) = \lambda_{R_1}(\varphi(a)) = \wedge_I(a)$. So $\wedge_I(a) = \wedge_I(b)$. In the case $a \notin J$, the proof is similar.

If $a, b \notin J$, then $\wedge_J(a) = \wedge_J(b)$ implies $\lambda_{R_2}(\varphi(a)) = \lambda_{R_2}(\varphi(b))$. Then, by con $\lambda_{R_2} \subseteq$ con λ_{R_1}, we get $\lambda_{R_1}(\varphi(a)) = \lambda_{R_1}(\varphi(b))$. So $\wedge_I(a) = \wedge_I(b)$, because $a, b \notin I$.

Thus con $\wedge_J \subseteq$ con $\wedge_I$ has been proved. Consequently $\mathbf{B}$ is a Boolean-type retractable OT-automaton. Thus the theorem is proved.

Let $\mathbf{A} = (A, X, \delta; a_0)$ be an OT-automaton. Consider the set

$$A^0 = \begin{cases} A - \{a_0\} & \text{if } |A| > 1 \\ \{a_0\} & \text{if } A = \{a_0\} \end{cases}$$

and define the transition function $\delta^0 : A^0 \times X \longrightarrow A^0$ as follows

$$\delta^0(a, x) = \begin{cases} \delta(a, x) & \text{if } a, \delta(a, x) \in A^0 \\ \text{not defined} & \text{if } a \notin A^0 \text{ or } \delta(a, x) \notin A^0. \end{cases}$$

(A^0, X, δ^0) is a partial automaton which will be denoted by $\mathbf{A}^0$.

We note that if $\mathbf{A}$ is a trivial automaton then $\mathbf{A}^0$ equals $\mathbf{A}$.

A mapping φ of A_1^0 into A_2^0 will be called a partial homomorphism of a partial automaton $\mathbf{A}_1^0 = (A_1^0, X, \delta_1^0)$ into a partial automaton $\mathbf{A}_2^0 = (A_2^0, X, \delta_2^0)$ if $\delta_1(a, x) \in A_1^0$ implies $\delta_2(\varphi(a), x) \in A_2^0$ and $\delta_2(\varphi(a), x) = \varphi(\delta_1(a, x))$ for all $a \in A_1^0$ and $x \in X$.

Consider the following construction.

Let T be a tree with a least element ν. For every $\alpha \in T - \{\nu\}$, let $\mathbf{A}_\alpha = (A_\alpha, X, \delta_\alpha; a_\alpha)$ be an r-simple OT-automaton and let $\mathbf{A}_\nu = (\{a_0\}, X, \delta_\nu)$ be a trivial automaton.

Assume $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

For all $\alpha, \beta \in T$ with $\alpha \geq \beta$, let $f_{\alpha,\beta}$ be a partial homomorphism of A_α^0 into A_β^0 such that

(i) $\varphi_{\alpha\alpha} = \mathrm{id}A^0_\alpha$ (the identical mapping of A^0_α),
(ii) $\varphi_{\beta,\gamma}(\varphi_{\alpha,\beta}(a)) = \varphi_{\alpha,\gamma}(a)$, for all $a \in A^0_\alpha$ and $\alpha \geq \beta \geq \gamma, (\alpha, \beta, \gamma \in T)$.

For every $a \in A^0_\alpha$ and $x \in X$, let $\overline{\alpha}[a, x]$ denote the greatest element of the set $\{\beta \in T : \delta_\beta(\varphi_{\alpha,\beta}(a), x) \in A^0_\beta\}$.

Assume that
(iii) for every $\alpha > \beta$ and $b \in A^0_\beta$ there are elements $a \in A^0_\alpha$,
$p = x_1x_2 \ldots x_n \in X^+ (x_1, x_2, \ldots x_n \in X)$ and
$\alpha_0, \alpha_1, \ldots, \alpha_n \in T$ such that $\alpha = \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_n = \beta$ and
$\overline{\alpha}_0[a, x_1] = \alpha_1$,
$\overline{\alpha}_1[\delta_{\alpha_1}(\varphi_{\alpha,\alpha_1}(a), x_1), x_2] = \alpha_2$

$\vdots$

$\overline{\alpha}_i[\delta_{\alpha_i}(\varphi_{\alpha,\alpha_i}(a), x_1x_2 \ldots x_i), x_{i+1}] = \alpha_{i+1}$,

$\vdots$

$\overline{\alpha}_{n-1}[\delta_{\alpha_{n-1}}((\varphi_{\alpha,\alpha_{n-1}}(a), x_1x_2 \ldots x_{n-1}), x_n] = \alpha_n$,
$\delta_{\alpha_n}(\varphi_{\alpha,\alpha_n}(a), x_1x_2 \ldots x_n) = b$.

Let $A = \{\cup A^0_\alpha : \alpha \in T\}$.

Define a transition function $\delta : A \times X \longrightarrow X$ as follows. If $a \in A^0_\alpha$ and $x \in X$, then let $\delta(a, x) = \delta_{\overline{\alpha}[a,x]}(\varphi_{\alpha,\overline{\alpha}[a,x]}(a), x)$.

It can be easily verified that $\mathbf{A} = (A, X, \delta; a_0)$ is an automaton which will be denoted by $[\mathbf{A}_\alpha, X, \delta_\alpha; \varphi_{\alpha,\beta}, T, a_0]$.

Next, we describe the r-semisimple Boolean-type retractable OT-automata.

Theorem 18 *An automaton is an r-semisimple Boolean-type retractable OT-automaton if and only if it is isomorphic with an automaton $[A_\alpha, X, \delta_\alpha; \varphi_{\alpha,\beta}, T, a_0]$ constructed as above.*

Proof. In the first part of the proof, we show that $\mathbf{A} = [A_\alpha, X, \delta_\alpha; \varphi_{\alpha,\beta}, T, a_0]$ is an r-semisimple Boolean-type retractable OT-automaton. It is evident that $\mathbf{A}$ is OT-automaton.

We show that $\mathbf{A}$ is retractable. Let I be a right ideal of $\mathbf{A}$. By the assumption that $\mathbf{A}_\alpha, \alpha \in T - \{\nu\}$ are r-simple and $\mathbf{A}_\nu$ is a trivial automaton, it follows that I is of the form $\{\cup A^0_\alpha : \alpha \in \Gamma\}$ where Γ is a non-empty subset of T. We show that Γ is an ideal of T. Let $\alpha \neq \nu$ be an arbitrary element of Γ. We show that $\beta < \alpha$ implies $\beta \in \Gamma$ for all $\beta \in T$. Let $\beta \in T$ and $b \in A^0_\beta$ be arbitrary elements such that $\beta < \alpha$. By (iii), there are elements a in $A^0_\alpha, p = x_1x_2 \ldots x_n$ in X^+ and $\alpha_1, \alpha_2, \ldots a_n$ in T with $\alpha \geq \alpha_1 \geq \ldots \geq \alpha_n = \beta$ such that
$\delta(a, x_1) \in A^0_{\alpha_1}$
$\delta(a, x_1x_2) = \delta(\delta(a, x_1), x_2) \in A^0_{\alpha_2}$

$\vdots$

$\delta(a, x_1x_2 \ldots x_n) = b = \delta(\delta(\ldots\delta(\delta(a, x_1), x_2)\ldots), x_n) \in A^0_\beta$.

As $a \in A^0_\alpha$ and $A^0_\alpha \subseteq I$, we have $\delta(a, x_1x_2 \ldots x_n) \in I$. So $A^0_\beta \cap I \neq \emptyset$ which implies $A^0_\beta \subseteq I$, that is $\beta \in \Gamma$. Thus Γ is an ideal of T.

Let π denote the retract homomorphism of T onto Γ. We define a retract homomorphism λ_I of $\mathbf{A}$ onto $\mathbf{I}$ as follows. For an arbitrary element a in A, let

$$\lambda_I(a) = \varphi_{\alpha,\pi(\alpha)}(a), \quad a \in A^0_\alpha. \tag{1}$$

By (i) and the fact that π is a retract homomorphism of T onto Γ, we can see that λ_I leaves the elements of I fixed. We prove that λ_I is a homomorphism. Let $a \in A$ and $x \in X$ be arbitrary elements. We may assume $a \neq a_0$. Let $a \in A^0_\alpha, \alpha \neq \nu$. Then

$$\lambda_I(\delta(a,x)) = \lambda_I(\delta_{\overline{\alpha}[a,x]}(\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x)) =$$

$$= \varphi_{\overline{\alpha}[a,x],\pi(\overline{\alpha}[a,x])}(\delta_{\overline{\alpha}[a,x]}(\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x)) =$$

$$= \delta_{\pi(\overline{\alpha}[a,x])}(\varphi_{\alpha,\pi(\overline{\alpha}[a,x])}(a),x) \in A^0_{\pi(\overline{\alpha}[a,x])}, \tag{2}$$

using (ii) and the fact that $\delta_{\overline{\alpha}[a,x]}\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x) \in A^0_{\overline{\alpha}[a,x]}$ and so $\varphi_{\overline{\alpha}[a,x],\pi(\overline{\alpha}[a,x])}$ maps $\delta\overline{\alpha}[a,x](\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x)$ into $A^0_{\pi(\overline{\alpha}[a,x])}$.

On the other hand, using (ii),

$$\delta(\lambda_I(a),x) = \delta(\varphi_{\alpha,\pi(\alpha)}(a),x) =$$

$$= \delta_{\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x]}(\varphi_{\pi(\alpha),\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x]}(\varphi_{\alpha,\pi(\alpha)}(a)),x)$$

$$= \delta_{\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x]}(\varphi_{\alpha,\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(\alpha),x]}(a),x) \in \tag{3}$$

$$\in A^0_{\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x]}.$$

To prove that $\lambda_I(\delta(a,x)) = \delta(\lambda_I(a),x)$, we show that (2) and (3) are equal to each other.

First consider the case when $\overline{\alpha}[a,x] \geq \pi(\alpha)$. Then $\alpha \geq \overline{\alpha}[a,x] \geq \pi(\alpha)$, and so $\pi(\overline{\alpha}[a,x]) = \pi(\alpha)$. Thus (2) is equal to $\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x)$ which is in $A^0_{\pi(\alpha)}$. This also implies that $\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x] = \pi(\alpha)$, because $\varphi_{\alpha,\pi(\alpha)}(a) \in A^0_{\pi(\alpha)}$ and $\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x)$ is not equal to the trap of $A_{\pi(\alpha)}$. Thus (3) is equal to $\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x)$ which means that (2) and (3) are equal to each other.

Consider the case $\overline{\alpha}[a,x] < \pi(\alpha)$. As Γ is an ideal of T and $\pi(\alpha) \in \Gamma$, we have $\overline{\alpha}[a,x] \in \Gamma$. So $\pi(\overline{\alpha}[a,x]) = \overline{\alpha}[a,x]$. Thus (2) is equal $\delta_{\overline{\alpha}[a,x]}(\delta_{\alpha,\overline{\alpha}[a,x]}(a),x)$. As $\varphi_{\pi(\alpha),\overline{\alpha}[a,x]}(\varphi_{\alpha,\pi(\alpha)}(a)) = \varphi_{\alpha\overline{\alpha}[a,x]}(a)$ (see (ii)), we have

$$\delta_{\overline{\alpha}[a,x]}(\varphi_{\pi(\alpha),\overline{\alpha}[a,x]}(\varphi_{\alpha,\pi(\alpha)}(a)),x) = \delta_{\overline{\alpha}[a,x]}(\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x) \in A^0_{\overline{\alpha}[a,x]}.$$

So $\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x] \geq \overline{\alpha}[a,x]$.

Let β be an arbitrary element of T with $\pi(\alpha) \geq \beta > \overline{\alpha}[a,x]$. Then $\delta_\beta(\varphi_{\pi(\alpha),\beta}(\varphi_{\alpha,\pi(\alpha)}(a)),x) = \delta_\beta(\varphi_{\alpha,\beta}(a),x)$. As $\beta > \overline{\alpha}[a,x]$, we get that $\delta_\beta(\varphi_{\alpha,\beta}(a),x)$ is the trap of A_β.

We note that this also implies that $\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x)$ is the trap of $A_{\pi(\alpha)}$, because $\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x) \in A^0_{\pi(\alpha)}$ would imply that $\delta_\beta(\varphi_{\pi(\alpha),\beta}$ $(\varphi_{\alpha,\pi(\alpha)}(a)),x) = \varphi_{\pi(\alpha),\beta}$ $(\delta_{\pi(\alpha)}(\varphi_{\alpha,\pi(\alpha)}(a),x) \in A^0_\beta$, contradicting that an automata $\delta_\beta(\varphi_{\alpha,\beta}(a),x) = \delta_\beta(\varphi_{\pi(\alpha),\beta}(\varphi_{\alpha,\pi(\alpha)}(a)),x)$ is the trap of A_β. Consequently $\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x] \leq \overline{\alpha}[a,x]$. This and $\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x] \geq \overline{\alpha}[a,x]$, proved above, together imply that $\overline{\pi(\alpha)}[\varphi_{\alpha,\pi(\alpha)}(a),x] = \overline{\alpha}[a,x]$. So (3) is equal to $\delta_{\overline{\alpha}[a,x]}(\varphi_{\alpha,\overline{\alpha}[a,x]}(a),x)$ which equals (2). Consequenlty λ_I is a homomorphism of **A** onto **I**.

To show that $\mathbb{A}$ is a Boolean-type retractable automaton, we prove that $I \subseteq J$ implies con $\lambda_J \subseteq$ con λ_I for all right ideals I and J of $\mathbb{A}$ where λ_I and λ_J constructed as in (1). Let $I \subseteq J$ be right ideals of $\mathbb{A}$. Then $I = \{\cup A^0_\alpha : \alpha \in \Gamma_I\}$ and $J = \{\cup A^0_\beta : \beta \in \Gamma_J\}$, Γ_I and Γ_J are ideals of T. Let π_I and π_J be the retract homomorphism of T onto I and J, respectively. Let λ_I and λ_J denote the retract homomorphism of $\mathbb{A}$ onto $\mathbb{A}/\rho_I$ and $\mathbb{A}/\rho_J$, respectivel. We must show that con $\lambda_J \subseteq$ con λ_I (that is $\lambda_J(a) = \lambda_J(b)$ implies $\lambda_I(a) = \lambda_I(b)$ for all $a, b \in A$). Let a and b be arbitrary elements in A with $a \in A^0_\alpha$ and $b \in A^0_\beta$, for some $\alpha, \beta \in T$. If $\lambda_J(a) = \lambda_J(b)$, then, by (1), $\varphi_{\alpha,\pi_J(\alpha)}(a) = \varphi_{\beta,\pi_J(\beta)}(b)$. So $\pi_J(\alpha) = \pi_J(\beta)$. As $I \subseteq J$, we get $\pi_I(\alpha) = \pi_I(\beta)$. As $\pi_J(\alpha) \geq \pi_I(\alpha)$ and $\pi_J(\beta) \geq \pi_I(\beta)$, we have $\varphi_{\alpha,\pi_I(\alpha)}(a) = \varphi_{\pi_J(\alpha),\pi_I(\alpha)}(\varphi_{\alpha,\pi_J(\alpha)}(a)) = \varphi_{\pi_J(\beta),\pi_I(\beta)}(\varphi_{\beta,\pi_J(\beta)}(b)) = \varphi_{\beta,\pi_I(\beta)}(b)$, that is $\lambda_I(a) = \lambda_I(b)$.

Consequently con $\lambda_J \subseteq$ con λ_I. Thus $\mathbb{A}$ is a Boolean-type retractable automaton.

We show that $\mathbb{A}$ is r-semisimple. Let I be a right ideal of $\mathbb{A}$. Then $I = \{\cup A^0_\alpha : \alpha \in \Gamma\}$ where Γ is an ideal of T. If $I = \{a_0\}$, then let $\lambda_I(a) = a_0$ for all $a \in A$. It is evident that λ_I is a homomorphism of $\mathbb{A}$ onto $\mathbb{I}$. Assume $I \neq \{a_0\}$. Let a be an arbitrary element of I. Then $a \in A^0_\alpha$ for some $\alpha \in \Gamma$. We show that there are elements b in I and p in X^+ such that $a = \delta(b, p)$. We may assume $a \neq a_0$. Let b be an arbitrary element in A^0_α. As A_α is r-simple, $R(b)$ (in A_α) equals A_α. So $a = \delta_\alpha(b, p)$ for some $p \in X^*$. If $|A^0_\alpha| > 1$ then b can be choosen such that $b \neq a$. In this case $p \in X^+$. If $A^0_\alpha = \{a\}$ then, by the r-simplicity of A, there is an element q in X^+ with $a = \delta_\alpha(a, q)$ (in the other case A_α must be trapped). Consequently $a = \delta(b, p)$ for some $b \in I$ and $p \in X^+$. Thus $\mathbb{A}$ is an r-semisimple automaton.

To prove the converse, let $\mathbb{A} = (A, X, \delta, a_0)$ be an r-semsimple Boolean-type retractable OT-automaton. Then there is a family Φ of retract homomorphisms φ_R of $\mathbb{A}$ onto $\mathbb{R}$, R are right ideals of $\mathbb{A}$, such that $R_1 \subseteq R_2$ implies con $\lambda_{R_2} \subseteq$ con λ_{R_1} for all right ideal R_1, R_2 of $\mathbb{A}$. It is evident that $A = \cup_{a \in A} R_a (= \cup_{a \in A} R^0\{a\})$.

By Theorem 10, the set Prf($\mathbb{A}$) of all principal r-factors of $\mathbb{A}$ is a tree under ordering $\leq$ defined as follows: $\mathbb{R}\{a\} \leq \mathbb{R}\{b\}$ if and only if $R(a) \subseteq R(b)$. The least element of Prf($\mathbb{A}$) is $\mathbb{R}\{a_0\}$, which is a trivial automaton. As $\mathbb{A}$ is r-semisimple, the automata $\mathbb{R}\{a\}, a \in A$ are r-simple OT-automata and $|\mathbb{R}\{a\}| = 1$ if and only if $a = a_0$. It is evident that $\mathbb{R}\{a\} \cap \mathbb{R}\{b\} = \emptyset$ if $\mathbb{R}\{a\} \neq \mathbb{R}\{b\}$. Let $\mathbb{R}\{a\}, \mathbb{R}\{b\}$ be arbitrary elements of Prf($\mathbb{A}$) with $\mathbb{R}\{a\} \geq \mathbb{R}\{b\}$ (that is $R(a) \supseteq R(b)$). Let $\varphi_{R(a),R(b)}$ denote the restriction of the retract homomorphism $\varphi_{R(b)} \in \Phi$ to $R(a)$. We show that $\varphi_{R(a),R(b)}$ maps R_a into R_b. Let z be an arbitrary element of R_a. If $b = a_0$, then $\varphi_{R(a),R(b)}(z) = a_0 \in R_b$. We show that $\varphi_{R(a),R(b)}(z) \in R_b$ also holds for all $b \neq a_0$. Assume, in an indirect way, that $\varphi_{R(a),R(b)}(z) \notin R_b$ for some $z \in R_a, b \neq a_0$. As $R[b]$ is a right ideal of $\mathbb{A}$, we get $R_b \not\ni \delta(\varphi_{R(a),R(b)}(z), x) = \varphi_{R(a),R(b)}(\delta(z, x))$ for all $x \in X$. As $\delta(z, X) = R(a)$, we get

$$\varphi_{R(a),R(b)}(R(a)) \subseteq R[b]. \tag{4}$$

As $\varphi_{R(b)}$ maps $\mathbb{A}$ onto $R(b)$ and leaves the elements of $R(b)$ fixed, we get that $\varphi_{R(a),R(b)}$ maps $R(a)$ onto $R(b)$ and leaves the elements of $R(b)$ fixed. Consequently

$$\varphi_{R(a),R(b)}(R(a)) = R(b),$$

contradicting (4). So $\varphi_{R(a),R(b)}(R_a) \subseteq R_b$. Thus $\varphi_{R(a),R(b)}$ determines a partial homomorphism $\varphi_{\mathbb{R}\{a\},\mathbb{R}\{b\}}$ of the partial automaton $\mathbb{R}^0\{a\}$ into the partial automaton $\mathbf{R}^0\{b\}$ as follows:

$$\varphi_{\mathbf{R}\{a\},\mathbf{R}\{b\}} : z \in R^0\{a\} \longrightarrow \varphi_{R(a),R(b)}(z).$$

We show that the family Φ^* of all partial homomorphisms $\varphi_{\mathbf{R}\{a\},\mathbf{R}\{b\}}$ ($\mathbf{R}\{a\}$, $\mathbf{R}\{b\} \in$ Prf ($\mathbf{A}$)) satisfies conditions (i), (ii) and (iii).

It is evident that $\varphi_{\mathbf{R}\{a\},\mathbf{R}\{b\}} = \mathrm{id}_{\mathbf{R}^0\{a\}}$ (see (i)).

To show (ii), let $\mathbf{R}\{a\} \geq \mathbf{R}\{b\} \geq \mathbf{R}\{c\}$ be arbitrary elements of Prf($\mathbf{A}$). Let e be an arbitrary element of R_a. As $\varphi_{R(a)}, \varphi_{R(b)}, \varphi_{R(c)} \in \Phi$, we have con $\varphi_{R(a)} \subseteq$ con $\varphi_{R(b)} \subseteq$ con $\varphi_{R(c)}$, that is $\varphi_{R(c)}(\varphi_{R(b)}(e)) = \varphi_{R(c)}(e)$. From this equality we get

$$\varphi_{R(b),R(c)}(\varphi_{R(a),R(b)}(e)) = \varphi_{R(a),R(c)}(e).$$

So the elements of Φ^* satisfy condition (ii).

To prove condition (iii), let $\mathbf{R}\{a\} > \mathbf{R}\{b\}$. Let $f \in R_b$. As $R(a) = \{\delta(a,p) : p \in X^*\}$, there is an element p in X^+ such that $f = \delta(a,p)$. If $p = x_1x_2 \ldots x_n$, then there are elements $a_1, a_2, \ldots, a_n = b$ in A such that

$\delta(a, x_1) \in R_{a_1}$
$\delta(a, x_1x_2) \in R_{a_2}$
$\vdots$
$f = \delta(a, x_1x_2 \ldots x_n) \in R_b$.

The proof will be complete if we show that $\delta(a,x) = \delta_{\overline{\mathbf{R}\{a\}}[a,x]}(\varphi_{\mathbf{R}\{a\},\overline{\mathbf{R}\{a\}}[a,x]}(a), x), a \in \mathbf{R}_a$, where $\overline{\mathbf{R}\{a\}}[a,x]$ is the greatest element of the set $\{\mathbf{R}\{b\} \in \mathrm{Prf}(\mathbf{A}) : \delta_{\mathbf{R}\{b\}}(\varphi_{\mathbf{R}\{a\},\mathbf{R}\{b\}}(a), x) \in R^0\{b\}$. Let $a \in R_a$ and $x \in X$ be arbitrary elements. Then there is an element b in A such that $\mathbf{R}\{b\} \leq \mathbf{R}\{a\}$ and $\delta(a,x) \in R_b$. If c is an element of A such that $\mathbf{R}\{c\} \leq \mathbf{R}\{b\}$, we have $\delta(\varphi_{R(a),R(c)}(a), x) = \delta(\varphi_{R(b),R(c)}(\varphi_{R(a),R(b)}(a)), x) = \varphi_{R(b),R(c)}\delta(\varphi_{R(a),R(b)}(a), x) \in R_c$, because $\varphi_{R(b),R(c)}$ maps R_b into R_c.

If c is an element of A such that $\mathbf{R}\{c\} > \mathbf{R}\{b\}$, that is $R(c) \supset R(b)$, then $\delta(a,x) \notin R_c$ and so $\delta(a,x) = \varphi_{R(a),R(c)}\delta(a,x) = \varphi_{R(a),R(c)}(a), x) \notin R_c$. Consequently $\delta(\varphi_{R(a),R(c)}(a), x) \notin R_c$ for all $\mathbf{R}\{c\} > \mathbf{R}\{b\}$ and $\delta(\varphi_{R(a),R(d)}(a), x) \in R_d$ for all $\mathbf{R}\{d\} \leq \mathbf{R}\{b\}$. Thus $\mathbf{R}\{b\} = \overline{\mathbf{R}\{a\}}[a,x]$ and so $\delta(a,x) = \delta_{\overline{\mathbf{R}\{a\}}[a,x]}(\varphi_{\mathbf{R}\{a\},\overline{\mathbf{R}\{a\}}[a,x]}(a), x)$ for all $a \in R_a$ and $x \in X$. Then (iii) is satisfied and $\mathbf{A} \cong [\mathbf{R}\{a\}, X, \delta_{\mathbf{R}\{a\}}, \varphi_{\mathbf{R}\{a\},\mathbf{R}\{b\}}, \mathrm{Prf}(\mathbf{A}), a_0]$.

Thus the theorem is proved.

Example 1 *Let* $\mathbf{A} = (A, X, \delta)$ *be an automaton such that*

$$A = \{a_0, a_1, a_2, a_3, a_4\}, \quad X = \{x, y\}$$

and

δ	a_0	a_1	a_2	a_3	a_4
x	a_0	a_0	a_0	a_0	a_0
y	a_0	a_2	a_1	a_4	a_3

The right ideals of **A** are $I_0 = \{a_0\}$, $I_1 = \{a_0, a_1, a_2\}$, $I_2 = \{a_0, a_3, a_4\}$ and $I_3 = A$.

Consider the following mappings:

$$\begin{array}{lllll}
\lambda_0 : A & \longrightarrow & \{a_0\} & \text{such that} & \lambda_0(a) = a_0 \ \text{ for all } \ a \in A, \\
\lambda_1 : A & \longrightarrow & I_1 & \text{such that} & \lambda_1(a) = a \ \text{ for all } \ a \in I_1 \ \text{ and} \\
 & & & & \lambda_1(a_3) = a_1, \lambda_1(a_4) = a_2, \\
\lambda_2 : A & \longrightarrow & I_2 & \text{such that} & \lambda_2(a) = a \ \text{ for all } \ a \in I_2 \ \text{ and} \\
 & & & & \lambda_2(a_1) = a_3, \lambda_2(a_2) = a_4,. \\
\lambda_3 : A & \longrightarrow & A & \text{such that} & \lambda_3(a) = a \ \text{ for all } \ a \in A.
\end{array}$$

It can be easily verfied that λ_i is a retract homomorphism of **A** onto $\mathbf{I}_i, i = 0, 1, 2, 3$, and that **A** is an r-semisimple Boolean-type retractable OT-automaton (with the trap a_0).

Consider the following automata

$$\mathbf{A}_0 = (\{a_0\}, X, \delta_0), \quad \mathbf{A}_1 = (\{a_0, a_1, a_2\}, X, \delta_1), \quad \mathbf{A}_2 = (\{a_0, a_3, a_4\}, X, \delta_2),$$

where

δ_0	a_0
x	a_0
y	a_0

δ_1	a_0	a_1	a_2
x	a_0	a_0	a_0
y	a_0	a_1	a_1

δ_2	a_0	a_3	a_4
x	a_0	a_0	a_0
y	a_0	a_2	a_3

$\mathbf{A}_0$ is a trivial automaton, $\mathbf{A}_1$ and $\mathbf{A}_2$ are r-simple OT-automata. Let $T = \{0, 1, 2\}$, a subset of the set of the non-negative integers with the usual ordering. T is a tree. Let

$\varphi_{i,i}$ be the identical mapping of $A_i^0, i = 0, 1, 2,$

$\varphi_{1,0} : A_1^0 \longrightarrow \{a_0\}$ such that $\varphi_{1,0}(a) = a_0$ for all $a \in A_1^0$,

$\varphi_{2,0} : A_2^0 \longrightarrow \{a_0\}$ such that $\varphi_{2,0}(a) = a_0$ for all $a \in A_2^0$,

$\varphi_{2,1} : A_2^0 \longrightarrow A_1^0$ such that $\varphi_{2,1}(a_3) = a_1, \varphi_{2,1}(a_4) = a_2$.

It can be verified that $\varphi_{i,j}, i, j \in T$ with $i \geq j$, satisfy conditions (i) (ii) and (iii). Moreover

$$\mathbf{A} \cong [\mathbf{A}_i, X, \delta_i; \varphi_{i,j}, T, a_0].$$

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