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Reconstruction of Unique Binary Matrices with Prescribed Elements*

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Summary

The reconstruction of a binary matrix from its row and column sum vectors is considered when some elements of the matrix may be prescribed and the matrix is uniquely determined from these data. It is shown that the uniqueness of such a matrix is equivalent to the impossibility of selecting certain sequences from the matrix elements. The unique matrices are characterized by several properties. Among others it is proved that their rows and columns can be permutated such that the 1's are above and left to the (non-prescribed) 0's. Furthermore, an algorithm is given to decide if the given projections and prescribed elements determine a binary matrix uniquely, and, if the answer is yes, to reconstruct it.

1 Introduction

Let $A = (a_{ij})$ be a binary matrix of size $m \times n$. Let its *row sum vector* be denoted by $R(A) = R = (r_1, r_2, \ldots, r_m)$,

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad (i = 1, 2, \ldots, m),$$

and let its *column sum vector* be denoted by $S(A) = S = (s_1, s_2, \ldots, s_n)$,

$$s_j = \sum_{i=1}^{m} a_{ij}, \qquad (j = 1, 2, \ldots, n).$$

The vectors R and S are also called the *projections* of A. Denote the *class* of binary matrices with row sum vector R and column sum vector S by $\mathcal{A}(R, S)$.

The problem of reconstruction of binary matrices from their projections has an extensive literature (for surveys, see e.g. [14] and [4]). Gale [9] and Ryser [13] have proved existence conditions. A necessary and sufficient condition of uniqueness is, for example, in [15].

In this paper, a generalization of the mentioned reconstruction problem will be considered. Let P and Q be binary matrices with size $m \times n$. We say $Q \geq P$ or Q

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covers P if $q_{ij} \geq p_{ij}$ for all positions $(i,j) \in \{1,2,\ldots,m\} \times \{1,2,\ldots,n\}$. The class $\mathcal{A}_P^Q(R,S)$ is then defined as

$$\mathcal{A}_P^Q(R,S) = \{A \mid A \in \mathcal{A}(R,S), \quad P \leq A \leq Q\}.$$

According to this definition, $\mathcal{A}_P^Q(R,S)$ can be regarded as the sub-class of $\mathcal{A}(R,S)$ having the prescribed value 1 in the positions where $p_{ij} = q_{ij} = 1$, and the prescribed value 0 where $p_{ij} = q_{ij} = 0$. It is clear that, if $P = O$ (zero matrix) and $Q = E(= (1)_{m \times n})$, then $\mathcal{A}_O^E(R,S) = \mathcal{A}(R,S)$.

Now, we show that this reconstruction problem can be simplified. It is clear that, if $A \in \mathcal{A}_P^Q(R,S)$ then $A \geq P$, so their *difference*, $A - P = (a_{ij} - p_{ij})_{n \times m}$, is a binary matrix with projections $R(A-P) = R(A) - R(P) = R - R(P)$ and $S(A-P) = S(A) - S(P) = S - S(P)$. Therefore, $A - P \in \mathcal{A}_O^{Q-P}(R - R(P), S - S(P))$. The reverse statement is also true in the sense that, if $B \in \mathcal{A}_O^Q(R,S)$ for some binary matrix Q, then $B + P \in \mathcal{A}_P^{Q+P}(R + R(P), S + S(P))$, where P is a binary matrix such that for all positions, if $p_{ij} = 1$, then $q_{ij} = 0$. This means that it is enough to study the class $\mathcal{A}_O^Q(R,S)$, or in short $\mathcal{A}^Q(R,S)$ or $\mathcal{A}^Q$.

It is interesting to note that the network flows [7] can also be used in the study of the class $\mathcal{A}^Q(R,S)$. To each class $\mathcal{A}^Q(R,S)$ there is a bipartite network with source s, sink t and nodes $\{R_1, R_2, \ldots, R_m\}$, $\{S_1, S_2, \ldots, S_n\}$ and arcs (s, R_i), (S_j, t) and (R_i, S_j) with capacity r_i, s_j and q_{ij}, respectively, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Then each matrix $A \in \mathcal{A}^Q(R,S)$ corresponds to a flow in this network (see [6]). In this way, the results in this paper have a reformulation in network flows.

Considering the connected literature, Kellerer published a necessary and sufficient condition [11] for the existence of measurable functions with given "marginals" which is applicable also to the matrices in the class $\mathcal{A}^Q$. Recently W.Y.C. Chen has published theorems about integral matrices with given row and column sums satisfying a so-called main condition [6]. However, this main condition restricts the validity of the results only to a part of the prescribed binary matrices. As we shall see, there is unique binary matrix not satisfying Chen's main condition (e.g., the only binary matrix of the so-called normalized class corresponding to Fig. 5.1). There are papers dealing with special $\mathcal{A}^Q$ classes: Fulkerson gave a necessary and sufficient condition for the existence of (0,1)-matrices with zero trace [8] and Anstee published results on matrices having at most one prescribed position in their columns [1],[3] and having a triangular block of 0's [2].

Henceforth, consider the class $\mathcal{A}^Q(R,S)$ where $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots s_n)$ are non-negative integer vectors and Q is a binary matrix of size $m \times n$. The position (i,j) is said to be *free* if the corresponding matrix element is not prescribed by Q, i.e. $q_{ij} = 1$.

In this paper, the aim is to generalize the uniqueness results of $\mathcal{A}$ to $\mathcal{A}^Q$ (and thus, to $\mathcal{A}_P^Q$). (The reconstruction problems of non-uniquely determined binary matrices is the subject of [10].) In Section 2 we reconsider the known results of uniqueness in certain classes $\mathcal{A}^Q(R,S)$, where Q has some special property. Then the general uniqueness problem is considered, when Q is an arbitrary binary matrix. Section 3 contains a definition of a switching chain, whose existence turns out to be a necessary and sufficient condition of the non-uniqueness of a binary matrix. Thus, a switching chain has the same role in the class $\mathcal{A}^Q$ as a switching component has in $\mathcal{A}$. In Section 4 a reconstruction algorithm is given to decide if the given projections and prescribed elements determine a binary matrix uniquely, and, if the answer is yes, to reconstruct it. The unique matrices can be characterized in different ways. Some of these properties are discussed in Section 5. It is proved that the 1's of these matrices can be covered by certain rectagles, and that their rows and columns can be permuted so that the 1's are above and to the left of the (non-prescribed) 0's.

2 Uniqueness in special classes

In this section we reconsider the uniqueness results in different special classes proving that none of them is sufficient to characterise the uniqueness in the class $\mathcal{A}^Q$.

We say that $A \in \mathcal{A}^Q(R,S)$ is a *non-unique* (or *ambiguous*) binary matrix (in $\mathcal{A}^Q$) if there is a matrix $A' \in \mathcal{A}^Q(R,S)$ such that $A \neq A'$. In the other case, A is *unique* (or *unambiguous*). Accordingly, the *reconstruction data*, the projections (R,S) and the prescribed values Q together, is *non-unique* or *unique* if the number of elements of the class $\mathcal{A}^Q$ is greater than one or exactly one, respectively. If $\mathcal{A}^Q(R,S) = \emptyset$ then the reconstruction data is *inconsistent*.

There are results connected with the uniqueness in the class $\mathcal{A}(R,S)$, i.e. when $Q = E$: Consider the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An *interchange* is a transformation of the (free) elements of A that changes a minor of type A_1 into type A_2 or vice versa, and leaves all other elements of A unaltered. (The word minor is used here in the sense of submatrix.) We say that the four elements of the minor form a *switching component.*

Theorem 2.1 [13,15]. The binary matrix $A \in \mathcal{A}(R,S)$ is ambiguous (in $\mathcal{A}(R,S)$) if and only if it has a switching component.

In the more general class of $\mathcal{A}^Q(R,S)$, the extension of this result is not trivial. Consider, for example, the class $\mathcal{A}^Q((1,1,1),(1,1,1))$, where

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

that is, the diagonal elements are prescribed. The matrices $A_3, A_4 \in \mathcal{A}^Q$ (see Fig. 2.1), but they have no switching components.

$$A_3 = \begin{pmatrix} \mathrm{x} & 0 & 1 \\ 1 & \mathrm{x} & 0 \\ 0 & 1 & \mathrm{x} \end{pmatrix} \qquad A_4 = \begin{pmatrix} \mathrm{x} & 1 & 0 \\ 0 & \mathrm{x} & 1 \\ 1 & 0 & \mathrm{x} \end{pmatrix}$$

Figure 2.1. Ambiguous matrices A_3 and A_4 having no switching components (x's denote the positions of the prescribed 0 elements).

The matrices A_3 and A_4 play a similar role in the classes of binary matrices having at most one prescribed element in each column as A_1 and A_2 do in $\mathcal{A}$ (classes having no prescribed element). Replacing a submatrix A_3 by A_4 or vice versa leaves the row and column sums unchanged. A *triangle interchange* is a replacement of any version of A_3 and A_4 obtained by applying the same row and column permutations to both A_3 and A_4 [1]. Anstee proved an analogous theorem [1 Corollary 3.2] in the case of prescribed 1's:

Theorem 2.2. Given a pair $A, B \in \mathcal{A}^Q(R,S)$, where Q has at most one 0 in each column, one can get from A to B by a series of interchanges and triangle interchanges without leaving $\mathcal{A}^Q(R,S)$.

However, if there is more than one prescribed element in the columns and rows, then the minors A_1, A_2, A_3 and A_4 are not enough to characterize uniqueness. For example, the matrices of Figure 2.2 are in the same class, but they have no such minors of free elements.

$$\begin{pmatrix} 0 & 1 & x & x \\ x & 0 & 1 & x \\ x & x & 0 & 1 \\ 1 & x & x & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & x & x \\ x & 1 & 0 & x \\ x & x & 1 & 0 \\ 0 & x & x & 1 \end{pmatrix}$$

Figure 2.2. Ambiguous binary matrices having two prescribed elements in each row and column, and having no minors A_1, A_2, A_3, A_4, or any minors obtained from them by permuting rows and columns.

3 Switching chain

Our most important new concept is a generalization of the concept of a switching component. We say that the binary matrix $A \in \mathcal{A}^Q$ has a *switching chain* if there is a series of different free positions of A, $< (i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots, (i_p, j_p), (i_p, j_1) >$, such that

$$a_{i_1 j_1} = a_{i_2 j_2} = \ldots = a_{i_p j_p} =$$
$$= 1 - a_{i_1 j_2} = 1 - a_{i_2 j_3} = \ldots = 1 - a_{i_p j_1}$$

($p \geq 2$). It follows from the definition that if $< (i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots, (i_p, j_p), (i_p, j_1) >$ is a switching chain of A and $a_{i_1 j_1} = a_{i_2 j_2} = \ldots = a_{i_p j_p} = 1$, then $a_{i_1 j_2} = a_{i_2 j_3} = \ldots = a_{i_p j_1} = 0$. This statement remains true if we switch the 1's and 0's of the chain. As examples of switching chain see A_1, A_2, A_3, A_4 and the matrices of Figure 2.2. Each of them contains switching chains. (In fact a switching component is a switching chain with $p = 2$.)

An important property is that by switching the 1's and 0's of a switching chain in a matrix, another matrix is obtained that has the same projections. Therefore, the non-existence of a switching chain in a matrix is a necessary condition for uniqueness. In fact, it is also sufficient.

Theorem 3.1. The binary matrix $A \in \mathcal{A}^Q(R,S)$ is unique if and only if A has no switching chain.

Proof. One direction is obvious. For the other direction, let us suppose that there is another binary matrix $A' \in \mathcal{A}^Q(R,S)$ $(A' \neq A)$. Then, there is a position (i_1, j_1) such that

$$a_{i_1 j_1} = 1, \quad a'_{i_1 j_1} = 0$$

(or $a_{i_1 j_1} = 0$, $a'_{i_1 j_1} = 1$, in which case we can use a similar proof). Since $r_{i_1} = r'_{i_1}$, there is a column j_2 $(\neq j_1)$ such that

$$a_{i_1 j_2} = 0, \quad a'_{i_1 j_2} = 1.$$

Then, since $s_{j_2} = s'_{j_2}$, there is a row i_2 such that

$$a_{i_2 j_2} = 1, \quad a'_{i_2 j_2} = 0,$$

and so on. After a finite number of steps the sequence will terminate, i.e., it follows from

$$a_{i_p j_p} = 1, \quad a'_{i_p j_p} = 0$$

that there is a column among (the up-to-now all different) $j_1, j_2, \ldots, j_p$, say j_k, such that

$$a_{i_p j_k} = 0, \quad a'_{i_p j_k} = 1.$$

That is, $< (i_k, j_k), (i_k, j_{k+1}), (i_{k+1}, j_{k+1}), (i_{k+1}, j_{k+2}), \ldots, (i_p, j_p), (i_p, j_k) >$ is a switching chain in A.

Remark. The proof is almost the same in the case of switching components in class $\mathcal{A}$ (see [13] and [15]), but in $\mathcal{A}$ it is also shown that this switching chain can be used to find a switching component. In the class $\mathcal{A}^Q$, this is not necessarily true.

4 Reconstruction of unique matrices

Now, we give the characterization that can be used to decide the uniqueness and to reconstruct unique matrices efficiently. We say that a minor is *mixed* if each of its rows and columns contains both a free 1 and a free 0.

Theorem 4.1. The binary matrix A is unique if and only if it has no mixed minor.

Proof. If there is a switching chain in a binary matrix, then the rows and the columns of the switching chain determine a minor consisting of rows and columns each containing free 1's and 0's.

To prove the other direction, let us suppose that A has a mixed minor. Then let $a_{i_1j_1} = 1$ be an element of the mixed minor. There is a column j_2 such that $a_{i_1j_2} = 0$ is an element of the mixed minor. Then, there is a row i_2 and a column j_3 such that $a_{i_2j_2} = 1$, $a_{i_2j_3} = 0$ and they are in the minor. We have to continue the procedure until there is a row i_p and a column j_p such that $a_{i_pj_p} = 1$, $a_{i_pj_k} = 0$ (both in the minor), where $j_k \in \{j_1, j_2, \ldots, j_{p-1}\}$. Then $< (i_k, j_k), (i_k, j_{k+1}), (i_{k+1}, j_{k+1}), (i_{k+1}, j_{k+2}), \ldots, (i_p, j_p), (i_p, j_k) >$ is a switching chain.

From Theorem 4.1 it follows for each minor of a unique matrix that there is a row or column of the minor such that in that row or column either there are only 1's in the free positions, or there are only 0's in the free positions or there are no free positions at all. These rows/columns are called *primitive rows/columns* of the minor. A primitive row/column can be recognised from the number of the free positions and the projection values of the minor in the following way. A primitive row contains 0's in the free positions (if there is free position) if and only if the sum of that row/column of the minor is 0. A primitive row contains 1's in the free positions if and only if the sum of that row/column is equal to the number of the free positions in that row/column of the minor.

Similarly, we say that i is a *primitive row of* $\mathcal{A}^Q(R, S)$ if $0 = r_i$ or $r_i = \sum_{j=1}^{n} q_{ij}$ and that j is a *primitive column of* $\mathcal{A}^Q(R, S)$ if $0 = s_j$ or $s_j = \sum_{i=1}^{m} q_{ij}$.

If the class $\mathcal{A}^Q(R, S)$ has only one matrix, then it has a primitive row or column. By reducing R and S by the projection of a primitive row or column and setting Q to 0 in this row or column, the new class $\mathcal{A}^{Q'}(R', S')$, has also only one matrix having the same elements as the original one in the positions $q'_{ij} = 1$. Trivially, if $\mathcal{A}^Q(R, S)$ is non- unique or empty, the $\mathcal{A}^{Q'}(R', S')$ is also non-unique or empty, respectively.

From this property of the unique binary matrices a reconstruction algorithm follows:

Algorithm 4.1 (to decide the uniqueness of the reconstruction data and to reconstruct a unique matrix $A \in \mathcal{A}^Q(R, S)$ from given projections R and S, and prescribed positions of Q):

Step 1. Let $A := O$, $R' := R$, $S' := S$, $Q' := Q$.

Step 2. If $0 \le r'_i \le \sum_{j'=1}^{n} q'_{ij'}$ and $0 \le s'_j \le \sum_{i'=1}^{m} q'_{i'j}$ is not fulfilled for all i and j, then the reconstruction data is inconsistent; stop.

Step 3. If $Q' = O$, then output A; stop.

Step 4. If no row and no column of $\mathcal{A}^{Q'}(R', S')$ is primitive, then the reconstruction data is non-unique or inconsistent; stop.

Step 5. Select a primitive row or column of $\mathcal{A}^{Q'}(R', S')$. For every (i, j) in this row or column such that $q'_{ij} = 1$,
i. set a_{ij} equal to 0 or 1, appropriately;
ii. reduce r'_i and s'_j by a_{ij};
iii. set q'_{ij} to 0.
Go to Step 2.

Remarks.

a. It is supposed that m and n are positive integers and R and S are vectors of m and n non-negative integers, respectively.

b. During the iterations the number of 0-rows or the number of 0-columns of Q' is increases at least by one. Thus, the algorithm will terminate after at most $m+n-1$ number of iterations, when all rows or columns of Q' contain only 0's (Step 3).

c. Step 2 is to test two conditions: The first is that vectors R' and S' contain only non-negative elements, and the second that the number of free positions in each row and column of the reduced class are enough to place r'_i and s'_j number of 1's, respectively. Both conditions are necessary for the existence.

d. Step 4 is to test if there is a primitive row or column in the class $\mathcal{A}^{Q'}(R', S')$. If no, then the matrix to be reconstructed has a mixed minor (see Theorem 4.1) consisting of the non-0-rows and non-0-columns of Q' (if the matrix exists at all).

e. It is not difficult to prove that the matrix A reconstructed by Algorithm 4.1 as an output in Step 3 is unique. It follows from the fact that primitive rows and columns do not contain any element of any switching chain.

f. Clearly, if a matrix A is constructed by Algorithm 4.1 then $A < Q$, because we assign 1's only into free positions (Step 5).

g. If the number of 1's in row i of A increases during the iterations, then r'_i decreases by the same number. This means that $r'_i + \sum_{j=1}^{n} a_{ij}$ remains constant in each iteration. In the first iteration this constant is

$$r'_i + \sum_{j=1}^{n} a_{ij} = r_i \tag{4.1}$$

(because $\sum_{j=1}^{n} a_{ij} = 0$ now). If we arrive Step 3 such that $Q' = O$ then $r'_i = 0$ and $s'_j = 0$ for each i and j (Step 2), and so again $R(A) = R$. Similarly, it can be shown that $S(A) = S$. That is, if a matrix A is constructed by Algorithm 4.1 then $A \in \mathcal{A}(R, S)$.

h. Algorithm 4.1 can be considered as a generalization of the assign and update algorithm [5] for reconstructing unique matrices without prescribed elements.

Therefore, Algorithm 4.1 is correct in the sense that it is terminated after a finite number of steps (Remarks b. and c.), the output matrix A is unique (Remark e.) and it is from the class $\mathcal{A}^{Q}(R, S)$ (Remarks f. and g.).

As an example see Figure 4.1.

2	x	.	.	.	.
2	.	x	.	.	x
4	.	.	.	x	.
1	.	.	.	.	.
2	.	x	.	.	x
	1	2	5	2	1

a.

1	x	.	1	.	.
1	.	x	1	.	x
0	1	1	1	x	1
0	.	.	1	.	.
1	.	x	1	.	x
	0	1	0	2	0

b.

1	x	.	1	.	0
1	0	x	1	.	x
0	1	1	1	x	1
0	0	0	1	0	0
1	0	x	1	.	x
	0	1	0	2	0

c.

0	x	1	1	.	0
0	0	x	1	1	x
0	1	1	1	x	1
0	0	0	1	0	0
0	0	x	1	1	x
	0	0	0	0	0

d.

0	x	1	1	0	0
0	0	x	1	1	x
0	1	1	1	x	1
0	0	0	1	0	0
0	0	x	1	1	x
	0	0	0	0	0

e.

Figure 4.1. Reconstruction of a unique binary matrix by Algorithm 4.1 showing matrix A and projections R' and S' during the iterations. The free elements of the minor to be reconstructed are denoted by '.'. The reconstructed elements of A are denoted by 0 and 1. The matrix Q' has a 1 at the positions where there is a '.'

- a. Starting configuration.
- b. Configuration after finding the primitive column 3 and primitive row 3.
- c. Configuration after finding the primitive columns 1, 5 and primitive row 4.
- d. Configuration after finding the primitive column 2 and primitive rows 2, 5.
- e. Configuration after finding the primitive column 4.

5 Characterization of unique matrices

Knowing Theorems 3.1 and 4.1 the unique matrices can be characterized by having no switching chain or having no mixed minor. Another possible characterizations are based on the comparison of the prescribed and free 1 and 0 positions of the rows. Let us introduce the following notations in connection with a matrix $A \in \mathcal{A}^Q$:

$$A^{(1)} = \{(i,j) \mid a_{ij} = 1\}, \quad A^{(0)} = \{(i,j) \mid a_{ij} = 0, q_{ij} = 1\}$$

and

$$Q^{(0)} = \{(i,j) \mid q_{ij} = 0\}.$$

In words, $A^{(1)}$ and $A^{(0)}$ denotes the sets of the free 1 and 0 positions of the binary matrix A, respectively, and $Q^{(0)}$ denotes the set of prescribed positions. Furthermore, let $A_i^{(1)}$ and $A_i^{(0)}$ denote the set of column indices of the free 1's and free 0's of A in row i $(1 \le i \le m)$, respectively.

Theorem 5.1. The binary matrix $A \in \mathcal{A}^Q(R,S)$ is unique if and only if for any subset I of the rows there is a row $i \in I$ such that

$$A_i^{(0)} \cap A_{i'}^{(1)} = \emptyset \tag{5.1}$$

for each $i' \in I$.

Remarks.

a. In another words, Theorem 5.1 says that, exactly in the case of uniqueness, from any subset of rows we can select at least one row such that in the columns of the free 0's of this row there is no 1 in any other row. This means that the 1's and prescribed elements of the selected row "cover" all the 1's of the other rows in the subset. In this sense the selected row is a longest row of the subset.

b. Specially, if there is no prescribed element, i.e. $Q = E$, then (5.1) means that a row having the greatest r_i covers every other row.

Proof. Suppose that A has a switching chain $SC = \langle (i_1,j_1),(i_1,j_2),(i_2,j_2),(i_2,j_3),\ldots,(i_p,j_p), (i_p,j_1) \rangle$ such that $a_{i_1j_1} = a_{i_2j_2} = \ldots = a_{i_pj_p} = 1$ and $a_{i_1j_2} = a_{i_2j_3} = \ldots = a_{i_pj_1} = 0$. Then let $I = \{i_1,i_2,\ldots,i_p\}$. If i_k is an arbitrary row of I $(1 \le k \le p)$, then i_{k+1} is another row of I such that $j_{k+1} \in A_{i_k}^{(0)} \cap A_{i_{k+1}}^{(1)}$ (if $k = p$ then instead of i_{k+1} let us select i_1). That is (5.1) is not fulfilled.

Suppose, now, that there is a subset of rows, I, such that for each row $i \in I$, there is a row $i' \in I$ such that $A_i^{(0)} \cap A_{i'}^{(1)} \ne \emptyset$. Let $i_1 \in I$ and i_2 another row index from I such that $j_2 \in A_{i_1}^{(0)} \cap A_{i_2}^{(1)}$ for some j_2, that is, $a_{i_1j_2} = 0$ and $a_{i_2j_2} = 1$. Applying the same condition to row i_2 we get a row i_3 from I and a column j_3 such that $a_{i_2j_3} = 0$ and $a_{i_3j_3} = 1$. And so on. After a finite number of steps the sequence will be ended, i.e. $a_{i_pj_k} = 0$ and $a_{i_kj_k} = 1$ for some $i_k \in \{i_1,i_2,\ldots,i_{p-1}\}$ and $j_k \in \{j_1,j_2,\ldots,j_{p-1}\}$. Then $\langle (i_k,j_k),(i_k,j_{k+1}),(i_{k+1},j_{k+1}),(i_{k+1},j_{k+2}),\ldots,(i_p,j_p),(i_p,j_k) \rangle$ is a switching chain in A.

Now, we give another characterization of the unique matrices by proving that their 1's can be covered by special rectangles. The construction of these covering rectangles can be done by

Procedure 5.1 (to construct special covering rectangles of 1's): This is an inductive procedure to find a sequence of rectangles having increasing number of rows and decreasing number of columns step by step. Applying Theorem 5.1 to the whole set of rows we know that if A is unique, then we can select at least one row i such that in the columns of $A_i^{(0)}$ A has no 1 element. Let the set of such rows be denoted by $I_1^{(1)} (\neq \emptyset)$, and let

$$J_1^{(1)} = \bigcap_{i \in I_1^{(1)}} \overline{A_i^{(0)}}$$

(overline denotes the complement set). Clearly, $A^{(1)} \supseteq (I_1^{(1)} \times J_1^{(1)}) \setminus Q^{(0)}$. If $A^{(1)} = (I_1^{(1)} \times J_1^{(1)}) \setminus Q^{(0)}$ then we have a rectangle (in a general sense that $I_1^{(1)} \times J_1^{(1)}$ consists of not necessarily consecutive rows and columns) covering the 1's of A and the Procedure is terminated. If

$$A^{(1)} \supset \bigcup_{t=1}^{p} (I_t^{(1)} \times J_t^{(1)}) \setminus Q^{(0)}$$

for some $p > 1$ (the symbols $\supset$ and $\subset$ are used only for strict containment) then we can select at least one row i from $\overline{I_p^{(1)}}$ such that A has no 1 element in $\overline{I_p^{(1)}} \times A_i^{(0)}$. ($A_i^{(0)} \neq \emptyset$, because in this case $i \in I_p^{(1)}$.) Let the union of the set of these rows and $I_p^{(1)}$ be denoted by $I_{p+1}^{(1)}$. Clearly, $I_p^{(1)} \subset I_{p+1}^{(1)}$. Let

$$J_{p+1}^{(1)} = \bigcap_{i \in I_{p+1}^{(1)}} \overline{A_i^{(0)}}.$$

Then $J_p^{(1)} \supset J_{p+1}^{(1)}$ (because $A_i^{(0)} \neq \emptyset$ in the new rows of $I_{p+1}^{(1)}$) and $A^{(1)} \supseteq (I_{p+1}^{(1)} \times J_{p+1}^{(1)}) \setminus Q^{(0)}$. After a finite number of steps (if p is big enough), we reach the situation

$$A^{(1)} = \bigcup_{t=1}^{p} (I_t^{(1)} \times J_t^{(1)}) \setminus Q^{(0)},$$

that is the 1's of matrix A are covered by the union of rectangles $I_t^{(1)} \times J_t^{(1)}$ $(1 \leq t \leq p)$.

As an example of the application of Procedure 5.1 see Figure 5.1(a), where $(\{1\}\times\{1,2,3,4,5,6\})\cup(\{1,2,3\}\times\{1,2,3,4\})\cup(\{1,2,3,4\}\times\{1,3,4\})\cup(\{1,2,3,4,5,6\}\times\{1\})$ is the set of covering rectangles constructed by Procedure 5.1.

4	1	1	x	x	1	1
1	1	x	x	x	0	0
3	1	1	1	x	0	x
2	1	0	1	x	x	x
1	1	x	0	0	0	x
1	1	x	0	x	x	0
	6	2	2	0	1	1

(*a*)

4	1	x	1	x	1	1
1	1	x	x	x	0	0
3	1	1	1	x	0	x
2	1	1	0	x	x	x
1	1	0	x	0	0	x
1	1	0	x	x	x	0
	6	2	2	0	1	1

(*b*)

Figure 5.1. (a) A unique binary matrix and its projections. (b) After changing columns 2 and 3 the matrix is ordered such that the 1's are to the left of the free 0's in each row, and the 1's are above the free 0's in each column.

Remark. Specially, if A has no 1 element (of course, in this case A is uniqe) then Procedure 5.1 gives $\{1,2,\ldots,m\}\times\emptyset$ as the only covering rectangle. In any other case the constructed rectangles are not degenerate.

Procedure 5.1 has proved a part of

Theorem 5.2. The binary matrix $A\in\mathcal{A}^Q(R,S)$ is unique if and only if there are subsets $I_1^{(1)}\subset I_2^{(1)}\subset\ldots\subset I_{p_1}^{(1)}$ of the row-indices $\{1,2,\ldots,m\}$ and subsets $J_1^{(1)}\supset J_2^{(1)}\supset\ldots\supset J_{p_1}^{(1)}$ of the column-indices $\{1,2,\ldots,n\}$ $(p_1\geq 1)$ such that

$$A^{(1)}=\bigcup_{t=1}^{p_1}(I_t^{(1)}\times J_t^{(1)})\setminus Q^{(0)}. \quad (5.2)$$

Proof. If A is unique then we can apply Procedure 5.1 to get the sequence of sets in (5.2).

To prove the other direction let us suppose that A is non-unique, but there are such covering rectangles. Then there is a switching chain $SC=<(i_1,j_1),(i_1,j_2),(i_2,j_2),(i_2,j_3),\ldots,(i_p,j_p),(i_p,j_1)>$ in A. Suppose that $a_{i_1j_1}=0$, $a_{i_1j_2}=1$, $a_{i_2j_2}=0$, $a_{i_2j_3}=1$ and so on. (Otherwise an analogous proof can be used.) The first two 1-valued elements of SC can not be covered by the same rectangle, because in this case (i_2, j_2) would be covered. Thus, there are two rectangles, say $I_{k_1}^{(1)} \times J_{k_1}^{(1)}$ and $I_{k_2}^{(1)} \times J_{k_2}^{(1)}$ $(1 \le k_1 < k_2 \le p_1)$, such that $I_{k_1}^{(1)} \subset I_{k_2}^{(1)}$ (because $i_2 \in I_{k_2}^{(1)} \setminus I_{k_1}^{(1)}$) and $J_{k_1}^{(1)} \supset J_{k_2}^{(1)}$ (because $j_2 \in J_{k_1}^{(1)} \setminus J_{k_2}^{(1)}$). To cover (i_3, j_4) we have another rectangle $I_{k_3}^{(1)} \times J_{k_3}^{(1)}$ such that $I_{k_2}^{(1)} \subset I_{k_3}^{(1)}$ and $J_{k_2}^{(1)} \supset J_{k_3}^{(1)}$. And so on. Finally, to cover (i_p, j_1) we have the rectangle $I_{k_p}^{(1)} \times J_{k_p}^{(1)}$ $(k_{p-1} < k_p \le p_1)$ such that $I_{k_{p-1}}^{(1)} \subset I_{k_p}^{(1)}$ and $J_{k_{p-1}}^{(1)} \supset J_{k_p}^{(1)}$. Furthermore, $I_{k_p}^{(1)} \subset I_{k_1}^{(1)}$ and $J_{k_p}^{(1)} \supset J_{k_1}^{(1)}$. But, here is the contradiction of $I_{k_1}^{(1)} \subset I_{k_2}^{(1)} \subset \ldots \subset I_{k_p}^{(1)} \subset I_{k_1}^{(1)}$ (and $J_{k_1}^{(1)} \supset J_{k_2}^{(1)} \supset \ldots \supset J_{k_p}^{(1)} \supset J_{k_1}^{(1)}$). That is, the uniqueness follows from (5.2).

The free 0 positions of the unique binary matrices can be characterized in a similar way: Consider a unique matrix $A \in \mathcal{A}^Q$. Then let us switch the free 1's and 0's in A. The new matrix is also unique (it has switching chain if and only if A has), and for its 1's, that is, for the free 0's of A, Theorems 5.1 and 5.2 can be applied. In this way we have analogous Theorems 5.3 and 5.4:

Theorem 5.3. The binary matrix $A \in \mathcal{A}^Q(R, S)$ is unique if and only if for any I subset of rows there is a row $i \in I$ such that

$$A_i^{(1)} \cap A_{i'}^{(0)} = \emptyset$$

for each $i' \in I$.

Theorem 5.4. The binary matrix $A \in \mathcal{A}^Q(R, S)$ is unique if and only if there are subsets $I_1^{(0)} \subset I_2^{(0)} \subset \ldots \subset I_{p_0}^{(0)}$ of the row-indices $\{1, 2, \ldots, m\}$ and subsets $J_1^{(0)} \supset J_2^{(0)} \supset \ldots \supset J_{p_0}^{(0)}$ of the column-indices $\{1, 2, \ldots, n\}$ $(p_0 \ge 1)$ such that

$$A^{(0)} = \bigcup_{t=1}^{p_0} (I_t^{(0)} \times J_t^{(0)}) \setminus Q^{(0)}.$$

For example, in the case of Figure 5.1(a)

$$(\{2,5,6\} \times \{2,3,4,5,6\}) \cup (\{2,4,5,6\} \times \{2,4,5,6\}) \cup (\{2,3,4,5,6\} \times \{4,5,6\})$$

is the set constructed by the Procedure 5.1 to cover the free 1's of the switched matrix (i.e. to cover the free 0's of the given matrix).

Remark. In the class $\mathcal{A}$ Theorems 5.2 and 5.4 give

$$A^{(1)} = \bigcup_{t=1}^{p_1} (I_t^{(1)} \times J_t^{(1)})$$

and

$$A^{(0)} = \bigcup_{t=1}^{p_0} (I_t^{(0)} \times J_t^{(0)}),$$

which is a special case of the structure results of [12].

Theorem 5.2 (and also 5.4) gives the possibility to "order" the rows and columns of the matrix such that the 1's are to the left of the free 0's in each row, and at the same time, the 1's are above the free 0's in each column of the ordered matrix. To get this matrix, we permute the rows and columns so that $I_t^{(1)}$ consists of the uppermost rows and $J_t^{(1)}$ consists of the leftmost columns for each $t \in \{1, 2, \ldots, p_1\}$. It is also true that if a matrix has this property then it has no switching chain. Thus, we have

Theorem 5.5. The binary matrix A is unique if and only if after eventual permutations the 1's are to the left of the free 0's in each row, and at the same time, the 1's are above the free 0's in each column.

For example, Fig. 5.1(b) shows the matrix ordered from the matrix Fig. 5.1(a).

Remark. In the class $\mathcal{A}$ (no prescribed elements) a unique matrix is easily transformed in such a form by ordering the rows and columns such that the projections are non-increasing vectors (see the normalized class in [14]).

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