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On Some Cyclic Connectivity Properties of Directed Graphs (Examples and Problems) *

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To Professor Ferenc Gécseg on his sixtieth birthday

Introduction

The essence of the paper consists in ten properties (each defining a class of finite directed graphs) listed in § 2 and in open questions (relating to dependencies among the properties) raised in §§ 8–10.

A number of dependence and independence assertions can be deduced easily or follow trivially from the ten properties. The originality of the statements in §§ 3, 8, 9 and of the examples in §§ 6, 7 does not exceed the level of routine consequences of the definitions.

Since the number of properties is ten, one can think "a priori" that the class of graphs which possess at least one property is partitioned into $1023(= 2^{10} - 1)$ subclasses (called types). In fact, the dependency statements imply that there are not more than twenty-one types; on the other hand, examples are got for ten types. The 21 imaginable types correspond in a natural manner to the 21 independent vertex sets of the hierarchy diagram shown in Figure 1. One of the types consists of some connected graphs which are not strongly connected, the remaining ≤ 20 types constitute a partition of the class of the strongly connected graphs.

A part of the open problems concerns to the existence of the eleven types whose non-emptiness is not decided in the article. In the last section an exciting topics is affected: the (fond?) hope for elaborating a structure theory of the strongly connected (directed) graphs.

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I Notions and facts

§ 1

By a graph, we mean a finite directed graph $G = (V, E)$ where $V(\neq \emptyset)$ is the set of vertices and $E(\neq \emptyset)$ is the set of edges of G. We suppose that G is simple (in detail: any edge joins two *different* vertices and there is at most one edge between any fixed vertex pair) and connected. The outdegree and indegree of $a(\in V)$ are denoted by $\delta^-(a)$ and $\delta^+(a)$, respectively.

Throughout the paper we study graphs whose vertices satisfy[1] $\delta^-(a) \cdot \delta^+(a) \geq 2$; hence transient vertices (i.e., vertices fulfilling $\delta^-(a) = \delta^+(a) = 1$) are excluded.

A graph G is said to be *strongly connected* if, for each ordered vertex pair (a, b), there exists a directed path from a to b.

As usual, we say that two edges $\mathfrak{e} = (a, b)$ and $\mathfrak{f} = (c, d)$ of G are *adjacent* if the number of different elements of the vertex set $\{a, b, c, d\}$ is three. We say that $\mathfrak{e}$ and $\mathfrak{f}$ are *consecutively* adjacent if $b = c$ or $a = d$ holds; $\mathfrak{e}$ and $\mathfrak{f}$ are said to be *oppositely* adjacent if either $a = c$ or $b = d$.

Remark 1 The assumption that transient vertices cannot occur is useful, and it is not a serious restriction. The structure of cycles of a graph is not altered essentially if a transient vertex and the two edges incident to it are replaced by a new edge.

§ 2

By a *cycle* Z of a graph G, we understand a sequence

$$a_1, a_2, \ldots, a_t \tag{2.1}$$

of pairwise different vertices such that $t \geq 3$ and there exist the t (directed) edges

$$(a_1, a_2), (a_2, a_3), \ldots, (a_{t-1}, a_t), (a_t, a_1) \tag{2.2}$$

in G. We say that the vertices in (2.1) and the edges in (2.2) are contained in Z (or, equivalently, that they belong to Z). The cycles (2.1) and $a_2, a_3, \ldots, a_t, a_1$ are considered to equal. We say that the *length* of the cycle (2.1) is t. A cycle of length 3 is also called a *cyclical triangle.*[2]

Let x be an element of either V or E. We say that x is *cyclic* if there is a cycle Z which contains x. Two different elements x, y of $V \cup E$ are called *cyclically completetable* if there exists a cycle Z such that both x and y belong to Z.

Next ten conditions (A), (B), ..., (K) will be introduced for a directed graph $G = (V, E)$.

(A) Any vertex of G is cyclic.

(B) Any edge of G is cyclic.

[1] This condition implies that we do not deal with graphs being sheer cycles.

[2] If a_1, a_2, a_3 are pairwise joined and they do not constitute a cyclical triangle, then they form clearly a transitive triangle.

(C) Any pair a, b of vertices of G is cyclically completable.

(D) Any pair $a(\in V), \mathfrak{e}(\in E)$ is cyclically completable.

(E) Any pair $\mathfrak{e}, \mathfrak{f}$ of edges of G is cyclically completable when $\mathfrak{e}$ and $\mathfrak{f}$ are non-adjacent or consecutively adjacent edges.

(F) There exists an $a(\in V)$ such that any pair $a, b(\in V - \{a\})$ is cyclically completable.

(G) There exists an $a(\in V)$ such that any pair $a, \mathfrak{e}(\in E)$ is cyclically completable.

(H) There exists an $\mathfrak{e}(\in E)$ such that any pair $a(\in V), \mathfrak{e}$ is cyclically completable.

(J) There exists an $\mathfrak{e}(\in E)$ such that any pair $\mathfrak{e}, \mathfrak{f}(\in E - \{\mathfrak{e}\})$ is cyclically completable.

(K) There exists an $\mathfrak{e}(\in E)$ such that any pair $\mathfrak{e}, \mathfrak{f}(\in E - \{\mathfrak{e}\})$ is cyclically completable when $\mathfrak{e}$ and $\mathfrak{f}$ are non-adjacent or consecutively adjacent edges.

The class of all (connected simple) graphs G which fulfil (A) is denoted by $\mathbf{A}$. The notations $\mathbf{B}, \mathbf{C}, \ldots, \mathbf{K}$ are used in an analogous sense.

Remark 2 It is well known that Condition (B) is equivalent to the strong connectedness of G.

Remark 3 The condition "any pair of edges is cyclically completable" does not occur among (A)–(K). This condition is not fulfilled by a connected simple graph unless it is a single cycle (cf. Footnote 1).

§ 3

Our aim in the present section is to state thirteen inclusions for the graph classes introduced above. The next ten inclusions follow immediately from how Conditions (A)–(K) have been defined:[3]

$$\mathbf{B} \subseteq \mathbf{A}, \mathbf{C} \subseteq \mathbf{F}, \mathbf{G} \subseteq \mathbf{F}, \mathbf{H} \subseteq \mathbf{F},$$

$$\mathbf{J} \subseteq \mathbf{K}, \mathbf{J} \subseteq \mathbf{G}, \mathbf{E} \subseteq \mathbf{K}, \mathbf{D} \subseteq \mathbf{C}, \mathbf{D} \subseteq \mathbf{G}, \mathbf{D} \subseteq \mathbf{H} \tag{3.1}$$

Lemma 1 *Let $\mathfrak{e} = (b, c)$ be an edge of G such that any pair $\mathfrak{e}, \mathfrak{f}(\in E - \{\mathfrak{e}\})$ is cyclically completable when $\mathfrak{e}, \mathfrak{f}$ are non-adjacent or consecutively adjacent edges. Choose an arbitrary element a of $V - \{b\}$. Then the pair $a, \mathfrak{e}$ is cyclically completable.*

[3]The evident formulae $\mathbf{F} \subseteq \mathbf{A}, \mathbf{D} \subseteq \mathbf{B}, \mathbf{J} \subseteq \mathbf{H}$ are omitted from (3.1). These are consequences of other assertions in this section (by the transitivity of inclusion).

Proof. From among the two possibilities $a = c$ and $a \neq c$ it suffices to treat the second one. There are at least three edges incident to a, we can select an edge $\mathfrak{g}$ out of them such that $\mathfrak{g}$ is either non-adjacent or consecutively adjacent to $\mathfrak{e}$. The pair $\mathfrak{e}, \mathfrak{g}$ is cyclically completable, hence the same holds for the pair $\mathfrak{e}, a$. □

Lemma 1 implies at once

Corollary 1 *We have* $\mathbf{E} \subseteq \mathbf{D}$ *and* $\mathbf{K} \subseteq \mathbf{H}$. □

Proposition 1 *We have* $\mathbf{F} \subseteq \mathbf{B}$.

Proof. Suppose $G \in \mathbf{F}$. Choose an arbitrary edge $\mathfrak{e} = (c, d)$ in G. By (F), a is accessible from d and c is accessible from a. It follows the accessibility of c from d, that is, the cyclicity of $\mathfrak{e}$. (Our idea remains valid even if $a \in \{c, d\}$.) □

II Hierarchy and examples

§ 4

The assertions contained in § 3 (Corollary 1, Proposition 1 and the formulae in (3.1)) determine a hierarchy of the graph classes $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{K}$. This hierarchy is shown in Figure 1.

One can now pose the general problem whether there exists any further interrelation among the ten graph classes or not. A more particular question is whether each of the thirteen inclusions is proper.

In what follows, the general problem is broken up into a number of subproblems. We do not succeed in carrying out a full discussion, a part of the subproblems will be left open. It will be shown, however, that eleven of the thirteen inclusions mentioned above are proper (Proposition 2).

§ 5

Figure 1 shows a cycle-free (directed) graph. There are nine maximal independent[4] vertex sets in this graph:

$$\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{F}\}, \{\mathbf{C}, \mathbf{J}\}, \{\mathbf{D}, \mathbf{K}\},$$

$$\{\mathbf{D}, \mathbf{J}\}, \{\mathbf{E}, \mathbf{J}\}, \{\mathbf{C}, \mathbf{G}, \mathbf{H}\}, \{\mathbf{C}, \mathbf{G}, \mathbf{K}\}.$$

Thus there are twenty-one non-empty independent vertex sets:

$$\left.\begin{array}{ccccc} \{\mathbf{A}\}, & \{\mathbf{B}\}, & \{\mathbf{C}\}, & \{\mathbf{D}\}, & \{\mathbf{E}\}, \\ \{\mathbf{F}\}, & \{\mathbf{G}\}, & \{\mathbf{H}\}, & \{\mathbf{J}\}, & \{\mathbf{K}\}, \\ \{\mathbf{C}, \mathbf{G}\}, & \{\mathbf{C}, \mathbf{H}\}, & \{\mathbf{C}, \mathbf{J}\}, & \{\mathbf{C}, \mathbf{K}\}, & \\ \{\mathbf{D}, \mathbf{J}\}, & \{\mathbf{D}, \mathbf{K}\}, & \{\mathbf{E}, \mathbf{J}\}, & \{\mathbf{G}, \mathbf{H}\}, & \\ \{\mathbf{G}, \mathbf{K}\}, & \{\mathbf{C}, \mathbf{G}, \mathbf{H}\}, & \{\mathbf{C}, \mathbf{G}, \mathbf{K}\}. & & \end{array}\right\} \tag{5.1}$$

[4]Two vertices of a directed graph $G = (V, E)$ are called independent if they are mutually inaccessible (by directed paths). A subset of V is said to be independent if its elements are pairwise independent. Maximality is understood with respect to set inclusion. It is clear that the independent vertex sets are exactly the subsets of the maximal independent sets. Although the empty set is independent, we shall disregard it.

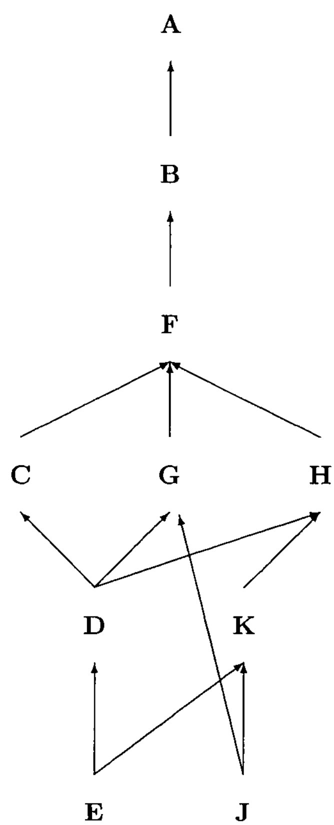


Figure 1

Whenever $\{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_g\}$ is one of these twenty-one sets (where $1 \le g \le 3$), then we define a type of graphs in the following manner:

G is of $(X_1 X_2 \dots X_g)$-type precisely if

(α) G belongs to $\mathbf{X}_1 \cap \mathbf{X}_2 \cap \dots \cap \mathbf{X}_g$ and

(β) G does not belong to any of the classes (out of $\mathbf{A}, \mathbf{B}, \dots, \mathbf{K}$) which are inaccessible from the vertices $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_g$ in the graph of Figure 1.

For example, G is contained in the (CJ)-type exactly when

G belongs to $\mathbf{C} \cap \mathbf{J}$ (hence to $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{K}$ also), and

G does not belong to $\mathbf{D}$ (thus also $G \notin \mathbf{E}$).

In sense of this definition, the class of (connected) graphs fulfilling Condition (A) is partitioned into at most twenty-one types.[5] The remaining two sections of Chapter II are devoted to giving examples which show the non-emptiness of ten types.

§ 6

We examine in this section some strongly connected graphs (without transient vertices) which have four or five vertices. The outdegrees and indegrees of the vertices of a graph are expressed by a matrix of form

$$\begin{pmatrix} \delta^-(a_1) & \delta^-(a_2) & \dots & \delta^-(a_v) \\ \delta^+(a_1) & \delta^+(a_2) & \dots & \delta^+(a_v) \end{pmatrix}$$

where the vertices are numbered in such a manner that

(i) $\delta^-(a_1) \ge \delta^-(a_2) \ge \dots \ge \delta^-(a_v)$ (where $v = |V|$), and

(ii) $1 \le i < v$, $\delta^-(a_i) = \delta^-(a_{i+1})$ imply $\delta^+(a_i) \ge \delta^+(a_{i+1})$.

Example 1 There is only one strongly connected tournament with four vertices (apart from isomorphy), see Figure 2/a. The degree matrix of this graph is

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}.$$

There is one cycle of length four: *abcd* and there are two cyclical triangles: *abd* and *acd*.

The fulfilment of Condition (C) is clear. (J) is satisfied with the edge (da). (D) does not hold since the vertex b is not cyclically completable with the edge (ac).

We have got that the type of this graph is (CJ).

In the following five examples, graphs with five vertices and eight edges are considered. The degree matrix of Examples 2–5 is

$$\begin{pmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

[5] By (β), the subclasses called types are pairwise disjoint. We used the words "at most" because it is not sure that all the types are non-empty.

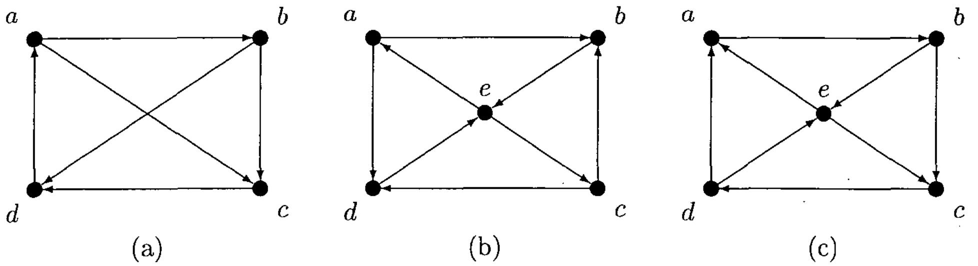


Figure 2

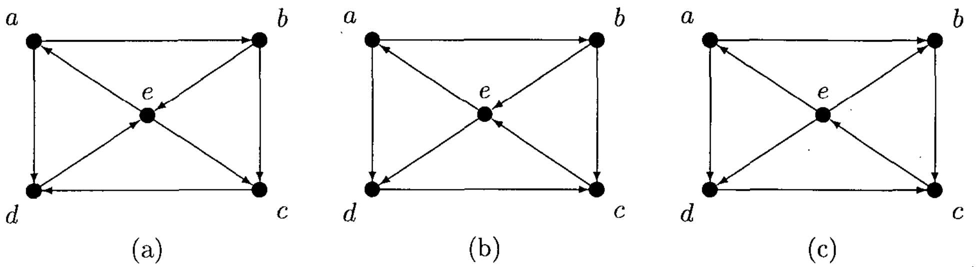


Figure 3

Example 2 In the graph of Figure 2/b there are four cyclic triangles: abe, cbe, cde and ade. There is no cycle of length four or five. Condition (G) is satisfied with e. The graph fulfils neither (C) nor (H). ((C) is false since with a and c form a pair which is not cyclically completable.)

The type of this graph is (G).

Example 3 The graph of Figure 2/c has five cycles; in detail, $abcde$ and $abecd$ are ones of length five, the length of $abcd$ is four, furthermore, abe and cde are cyclical triangles. (J) is satisfied with the edge (ab), the fulfilment of (E) can be checked easily.

The type of this graph is (EJ). It is contained in each of the classes $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{K}$.

Example 4 The length of any cycle of the graph of Figure 3/a is five or three. There is one cycle containing all vertices: $abcde$, and there are three cyclical triangles: abe, ade and cde. Conditions (G), (K) are satisfied with $e, (de)$, respectively; the fulfilment of (C) is clear. (D) is false for this graph (for example, a and (ec) are not cyclically completable). The falsity of (J) follows from the fact that, for an arbitrary edge $\mathfrak{f}$, there is an edge $\mathfrak{g}$ such that $\mathfrak{f}, \mathfrak{g}$ are oppositely adjacent edges.

This graph is of type (CGK).

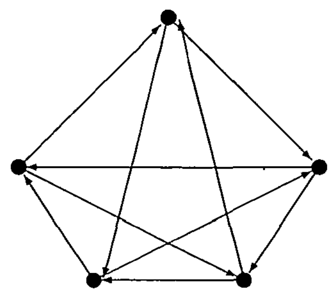


Figure 4

Example 5 Any cycle of the graph of Figure 3/b is of length three or four. There are two cycles whose length is four: *abce* and *adce*; there are two cyclical triangles: *abe* and *ced*. (G) is satisfied with e, (K) is fulfilled with (ce). (C) is not valid because b and d are not cyclically completable. (J) does not hold (by the same reason as in the preceding example).

The type of this graph is (GK).

Example 6 The degree matrix of the graph of Figure 3/c is

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

There is no cycle of length five. Two cycles have length four: *abce* and *adce*; there are two cyclical triangles: *bce* and *ced*. Condition (J) holds with (ce). Condition (C) is not fulfilled since b and d are not cyclically completable.

The type of this graph is (J).

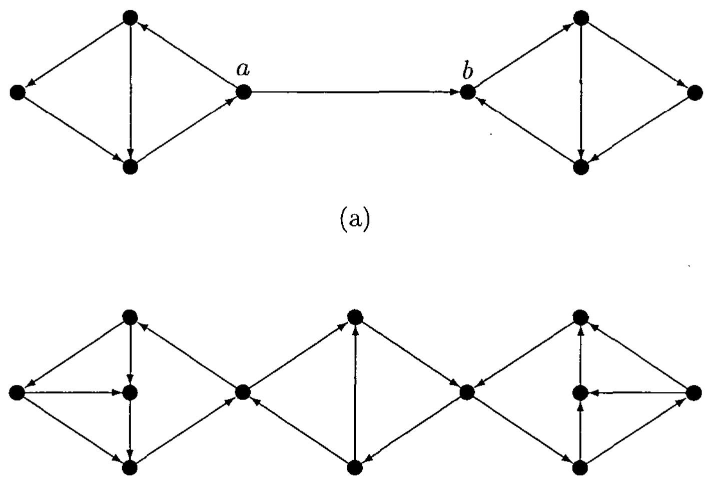
Example 7 Consider the graph of Figure 4. Each outdegree and indegree equals 2 in it. It is easy to see (without a detailed examination of the cycles) that this highly symmetric tournament satisfies (E). Condition (J) is not fulfilled (similarly to Examples 4 and 5).

We have got that this graph belongs to the type (E).

Remark 4 It seems that the remaining strongly connected graphs having five vertices (each non-transient) do not represent any other type than the types to which Examples 1–7 belong. For the reader who wants to recapitulate the investigation of these graphs, the following method can be advised:

first to get an overview of the possible degree matrices,

for any degree matrix, to construct all of its graph realizations,



(a)

(b)

Figure 5

if two graphs are dual[6] to each other, then to examine only one of them (because duality does not alter the structure of cycles essentially).

§ 7

We have seen in § 6 some examples which were obtained by scanning all graphs with a very small number of vertices, they show the non-emptiness of seven types. In the present section examples for three additional types will be given.[7]

The fact that the type (A) of graphs is not empty is clear (it is well known that the class of strongly connected graphs does not exhaust the class of connected graphs in which every vertex is cyclic). For the sake of completeness of the treatment, we put first an instance for this type.

Example 8 Every vertex of the graph in Figure 5/a is cyclic, and (ab) is a non-cyclic edge. Thus the graph belongs to the type (A).

Example 9 Figure 5/b shows a graph for which (B) is valid (or, equivalently, it is strongly connected), but (F) does not hold. This means that the graph is of type (B).

[6]Two directed graphs are said to be dual if one is obtained from the other by reversing the orientation of all the edges. It may happen that a graph is isomorphic to its dual.

[7]In the course of constructing these graphs, the restriction that transient vertices are forbidden needed to be regarded.

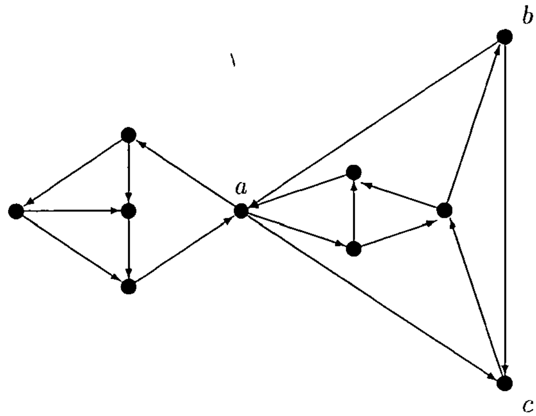


Figure 6

Example 10 The graph in Figure 6 satisfies (F) (with the vertex a). It is easy to see that (C) and (H) are not valid. (G) is not fulfilled also (observe that a and the edge (bc) are not cyclically completable). Therefore this graph is of type (F).

III Overview and open questions

§ 8

Remember that twenty-one graph types have been introduced in § 5. We have seen in Sections 6–7 examples for ten types. The question of the existence of the remaining types is open:

Problem 1 *Decide for any of the eleven types (C), (D), (H), (K), (CG), (CH), (CK), (DJ), (DK), (GH), (CGH) – consisting of directed graphs – whether the type is empty or not.*

Problem 1 is a comprehensive question in the sense that the subsequent Problems 2 and 3 are (essentially) particular cases of it.

Recall the hierarchy shown in Figure 1. We are going to suggest some possibilities for improving this hierarchy. Next we consider the pairs of graph classes (out of $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{K}$) for which the corresponding vertices are adjacent in Figure 1.

Proposition 2 *We have the proper inclusions*

$$\mathbf{B} \subset \mathbf{A}, \mathbf{F} \subset \mathbf{B}, \mathbf{C} \subset \mathbf{F}, \mathbf{G} \subset \mathbf{F},$$

$$\mathbf{H} \subset \mathbf{F}, \mathbf{D} \subset \mathbf{C}, \mathbf{D} \subset \mathbf{G}, \mathbf{D} \subset \mathbf{H},$$

$$\mathbf{E} \subset \mathbf{K}, \mathbf{J} \subset \mathbf{G}, \mathbf{J} \subset \mathbf{K}.$$

Proof. Examples 8, 9 show that $\mathbf{B}$ is properly included in $\mathbf{A}$ and $\mathbf{F}$ is properly included in $\mathbf{B}$, respectively. The inclusion $\mathbf{C} \subset \mathbf{F}$ is guaranteed by[8] Example 2. The remaining eight inclusions are ensured by Examples 10, 2, 1, 1, 1, 1, 2, 4, respectively. □

We do not have examples for the strictness of the inclusions $\mathbf{K} \subseteq \mathbf{H}$ and $\mathbf{E} \subseteq \mathbf{D}$, hence we can raise:

Problem 2 *Decide the validity of the equalities* $\mathbf{K} = \mathbf{H}$ *and* $\mathbf{E} = \mathbf{D}$.

§ 9

There are nine independent vertex pairs in the graph of Figure 1. Our present aim is to discuss the pairs of graph classes which correspond to these vertex pairs.

Examples 3, 6 and 7 guarantee the truth of the following assertions:

Proposition 3 *The intersection* $\mathbf{E} \cap \mathbf{J}$ *is not empty and it is properly included both by* $\mathbf{E}$ *and by* $\mathbf{J}$. *The intersections* $\mathbf{D} \cap \mathbf{J}$ *and* $\mathbf{C} \cap \mathbf{J}$ *are proper in the same sense.* □

Any of the formulae

$$\mathbf{G} \subseteq \mathbf{K}, \mathbf{G} \subseteq \mathbf{H}, \mathbf{G} \subseteq \mathbf{C}, \mathbf{H} \subseteq \mathbf{C}, \mathbf{K} \subseteq \mathbf{C}, \mathbf{K} \subseteq \mathbf{D},$$

is disproved either by Example 2 or by Example 6; our examples leave open the truth of the (strict) inclusions in the opposite sense. Thus we can pose:

Problem 3 *Decide the validity of the inclusions*

$$\mathbf{K} \subset \mathbf{G}, \mathbf{H} \subset \mathbf{G}, \mathbf{C} \subset \mathbf{G}, \mathbf{C} \subset \mathbf{H}, \mathbf{C} \subset \mathbf{K}, \mathbf{D} \subset \mathbf{K}.$$

§ 10

We have throughout adopted in the above considerations that transient vertices are forbidden. After determining the hierarchy completely (i.e., after solving Problem 1), the question may arise how the hierarchy changes when transient vertices are allowed. The solution of this question does not seem to be difficult.

Now we turn to another possibility of varying the subject. In our former analysis, graphs having cut vertices[9] were not excluded. Cut vertices have occured in Examples 8, 9, 10 actually. It is evident that Example 8 can be replaced by a twofold connected graph; the same is, however, not trivial for Examples 9 and 10. Consequently we formulate

Problem 4 *Is the hierarchy of the graph classes* $\mathbf{B}, \mathbf{C}, \ldots, \mathbf{K}$ *modified if we restrict ourselves to graphs without cut vertices? Especially, do the formulae* $\mathbf{F} \subset \mathbf{B}$ *and* $\mathbf{G} \subset \mathbf{F}$ *remain valid after this restriction?*

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[8] Instead of Example 2, any of Examples 5, 6, 10 is suitable for this purpose. In what follows, we mention only one instance in analogous situations, and the search for other appropriate examples is left to the reader.

[9] In other words, exactly onefold connected graphs. The orientation of the edges is indifferent in this notion.

§ 11

In the hierarchy of graphs studied by us, the intersection $\mathbf{E} \cap \mathbf{J}$ is the common part of all classes. It can be hoped that a structural description of this relatively narrow graph class will be elaborated:

Problem 5 *Characterize the structure of the graphs which belong to the type (EJ).*

A bold challenge is initiated in the last open question of the paper. A complete elucidation of this question would imply a systematization of the structure of all finite directed graphs. Even if this goal will not prove to be successful, partial solutions are of importance also:

Problem 6 *Consider a graph class* $\mathbf{X}$ *from among the nine classes* $\mathbf{B}, \mathbf{C}, \ldots, \mathbf{K}$. *Let a constructive procedure be obtained how the members of the class* $\mathbf{X}$ *can be produced from graphs belonging to classes which are proper subclasses of* $\mathbf{X}$ *(in the hierarchy).*

Remark 5 It is at once clear that the solution of Problem 6 for all the nine classes yields the structural overview of the *strongly connected* graphs. This observation can be supplemented by two well-known facts:

(i) any directed graph G admits a unique decomposition into a cycle-free graph so that each maximal strongly connected subgraph of G is contracted into a single vertex;

(ii) it is possible to get a good survey of the structure of cycle-free (directed) graphs.

Details may be found in Chapters 3 and 10 of the book [1] of Harary, Norman, and Cartwright.

References

[1] Harary, F., Norman, R. Z., and Cartwright, D., *Structural models: An introduction to the theory of directed graphs,* Wiley, New York, 1965. (French translation: Dunod, Paris, 1968.)