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Directable nondeterministic automata*

B. Imreh † M. Steinby ‡

Dedicated to Professor Ferenc Gécseg on the occasion of his 60th birthday

Abstract

An automaton is directable if it has a directing word which takes it from every state to the same state. For nondeterministic (n.d.) automata directability can be defined in several meaningful ways. We consider three such notions. An input word w of an n.d. automaton $\mathcal{A}$ is

(1) D1-directing if the set of states aw in which $\mathcal{A}$ may be after reading w consists of the same single state c for all initial states a;

(2) D2-directing if the set aw is independent of the initial state a;

(3) D3-directing if some state c appears in all of the sets aw.

We consider the sets of D1-, D2- and D3-directing words of a given n.d. automaton, and compare the classes of D1-, D2- and D3-directable n.d. automata with each other. We also estimate the lengths of the longest possible minimum-length D1-, D2- and D3-directing words of an n-state n.d. automaton. All questions are studied separately for n.d. automata which have at least one next state for every input-state pair.

1 Introduction

An input word w is called a *directing* (or *synchronizing*) *word* of an automaton $\mathcal{A}$ if it takes $\mathcal{A}$ from every state to the same fixed state, *i.e.* if there is a state c such that $aw = c$ for all states a of $\mathcal{A}$. An automaton is *directable* if it has a directing word. Directable automata and directing words have been studied extensively in the literature from various points of view (*cf.* [1],[3], [4],[8],[9], [10],[11], for example). The main challenge from the very beginning has been Černý's Conjecture [3] which claims that any n-state ($n \geq 1$) directable automaton has a directing word of length $(n-1)^2$ or less. The bound suggested by the conjecture is the lowest possible, but the best known upper bounds are of order $\mathcal{O}(n^3)$, and the conjecture remains

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†Department of Informatics, József Attila University, Árpád tér 2, H-6720 Szeged, Hungary

‡TUCS and Department of Mathematics, University of Turku, FIN-20014 Turku, Finland

unsettled. On the other hand, for some special classes of automata even better and accurate bounds have been found (*cf.* [8],[9],[10],).

For nondeterministic (n.d.) automata directability can be defined in several meaningful ways three of which we will study here. An input word w of an n.d. automaton $\mathcal{A}$ is

(1) D1-directing if the set of states aw in which $\mathcal{A}$ may be after reading w consists of the same single state c whatever the initial state a is;

(2) D2-directing if the set aw is independent of the initial state a;

(3) D3-directing if there exists a state c which appears in all sets aw.

The D1-directability of complete n.d. automata was already studied by Burkhard [1]. In a paper [6] on games of composing relations on a finite set Goralčik *et al.,* in effect, considered D1- and D3-directability and also two special types of D2-directability for general n.d. automata. In [1] an exact exponential bound for the length of minimum-length D1-directing words of complete n.d. automata was given, and in [6] it was shown that neither for D1- nor for D3-directing words the bound can be polynomial for general n.d. automata, and that the same holds for the two special types of D2-directability considered. On the other hand, Carpi [2] has found $\mathcal{O}(n^3)$ bounds for D1-directing words of unambiguous automata and for synchronizing pairs of maximal rational codes recognized by such automata.

In this paper we consider the three types of directability from several points of view both for complete n.d. automata and for general n.d. automata. Section 2 contains the general preliminaries. In Section 3 we give the formal definitions of D1-, D2 and D3-directing words and study the sets of Di-directing words of a given automaton ($i = 1, 2, 3$). Moreover, a diagram showing the inclusion relationships between the various classes of directable n.d. automata is presented. In Section 4 we study the preservation of directability properties when one forms subautomata, epimorphic images or finite direct products of n.d. automata. Finally, in Section 5 we derive lower and upper bounds for the lengths of the shortest directing words of the three different types. For D1-directing words it only remained to note that Burkhard's exact value applies also to general n.d. automata. For D2- and D3-directing words exponential lower bounds are obtained by utilizing an idea used in [6], and by considering recognizers of the sets of D2- or D3-directing words of a given automaton we get also upper bounds for them. The gaps between the lower bounds and the upper bounds are, however, considerable. Here the D3-directing words of complete n.d. automata form a notable exception: for them we obtain a lower bound of order $\mathcal{O}(n^2)$ and an upper bound of order $\mathcal{O}(n^3)$.

2 Preliminaries

The cardinality of a set A is denoted by $|A|$. If $f : A \to B$ is a mapping, the image $f(a)$ of an element $a \in A$ is often denoted by af. Similarly, we may write Hf for $f(H) = \{af : a \in H\}$ when $H \subseteq A$. The composition of two mappings $f : A \to B$

and $g : B \to C$ is the mapping $fg : A \to C$, $a \mapsto (af)g$, and the product of two relations $\theta \subseteq A \times B$ and $\rho \subseteq B \times C$ is the relation

$$\theta\rho = \{(a, c) \in A \times C : (\exists b \in B)\ a\theta b,\ b\rho c\}$$

from A to C; that $(a, b) \in \theta$ holds is also expressed by writing $a\theta b$.

In what follows, X is always a finite nonempty alphabet. The set of all (finite) words over X is denoted by X^* and the empty word by ε. The length of a word w is denoted by $\lg(w)$.

An *automaton* is a system $\mathcal{A} = (A, X, \delta)$, where A is a finite nonempty set of *states*, X is the *input alphabet*, and $\delta : A \times X \to A$ is the *transition function*. The transition function is extended to $A \times X^*$ in the usual way. A *recognizer* is a system $\mathbf{A} = (A, X, \delta, a_0, F)$, where (A, X, δ) is an automaton, $a_0 (\in A)$ is the *initial state*, and $F (\subseteq A)$ is the set of *final states*. The *language recognized* by $\mathbf{A}$ is the set

$$L(\mathbf{A}) = \{w \in X^* : \delta(a_0, w) \in F\}.$$

A language is called *recognizable*, or *regular*, if it is recognized by some recognizer. The set of all recognizable languages over the alphabet X is denoted by $\mathrm{Rec}(X)$.

An automaton $\mathcal{A} = (A, X, \delta)$ can also be defined as an algebra $\mathcal{A} = (A, X)$ in which each input letter x is realized as the unary operation $x^{\mathcal{A}} : A \to A$, $a \mapsto \delta(a, x)$. Nondeterministic automata may then be introduced as generalized automata in which the unary operations are replaced by binary relations. Hence a *nondeterministic (n.d.) automaton* is a system $\mathcal{A} = (A, X)$ where A is a finite nonempty set of *states*, X is the *input alphabet*, and each letter $x (\in X)$ is realized as a binary relation $x^{\mathcal{A}} (\subseteq A \times A)$ on A. For any $a \in A$ and $x \in X$, $ax^{\mathcal{A}} = \{b \in A : (a, b) \in x^{\mathcal{A}}\}$ is the set of states into which $\mathcal{A}$ may enter from state a by reading the input letter x. For any $C \subseteq A$ and $x \in X$, we set $Cx^{\mathcal{A}} = \bigcup\{ax^{\mathcal{A}} : a \in C\}$. For $w \in X^*$ and $C \subseteq A$, $Cw^{\mathcal{A}}$ is obtained inductively thus:

(1) $C\varepsilon^{\mathcal{A}} = C$;

(2) $Cw^{\mathcal{A}} = (Cv^{\mathcal{A}})x^{\mathcal{A}}$ for $w = vx$, $v \in X^*$ and $x \in X$.

If $w \in X^*$ and $a \in A$, let $aw^{\mathcal{A}} = \{a\}w^{\mathcal{A}}$. This means that if $w = x_1x_2 \ldots x_k$, then $w^{\mathcal{A}} = x_1^{\mathcal{A}}x_2^{\mathcal{A}} \ldots x_k^{\mathcal{A}} (\subseteq A \times A)$. If C is the set in which $\mathcal{A}$ could be at a certain moment, then by the usual interpretation of nondeterminism $Cw^{\mathcal{A}}$ is the set of possible states after $\mathcal{A}$ has received the input word w. When $\mathcal{A}$ is known from the context, we usually write simply aw and Cw for $aw^{\mathcal{A}}$ and $Cw^{\mathcal{A}}$, respectively.

An n.d. automaton $\mathcal{A} = (A, X)$ is *complete* if $ax^{\mathcal{A}} \neq \emptyset$ for all $a \in A$ and $x \in X$. Complete n.d. automata are called c.n.d. automata for short. In what follows, we denote a deterministic automaton by $\mathcal{A} = (A, X, \delta)$ and a nondeterministic automaton by $\mathcal{A} = (A, X)$.

3 Directable nondeterministic automata

An automaton $\mathcal{A} = (A, X, \delta)$ is said to be *directable* if it has a *directing word* $w(\in X^*)$ such that $\delta(a, w) = \delta(b, w)$ for all $a, b \in A$ (cf. [3],[4], [8],[11], for example). Hence a directing word sends the automaton to a known state which is independent of the present state. This idea can be extended to n.d. automata in several nonequivalent ways. In the following definition three natural notions of directability of n.d. automata are introduced.

Definition 3.1. Let $\mathcal{A} = (A, X)$ be an n.d. automaton. For any word $w \in X^*$ we consider the following three conditions:

(D1) $(\exists c \in A)(\forall a \in A)(aw = \{c\})$;

(D2) $(\forall a, b \in A)(aw = bw)$;

(D3) $(\exists c \in A)(\forall a \in A)(c \in aw)$.

If w satisfies (Di), then w is a Di-*directing word* of $\mathcal{A}$ $(i = 1, 2, 3)$. For each $i = 1, 2, 3$, the set of Di-directing words of $\mathcal{A}$ is denoted by $\mathrm{D}_i(\mathcal{A})$, and $\mathcal{A}$ is called Di-*directable* if $\mathrm{D}_i(\mathcal{A}) \neq \emptyset$. The classes of D$i$-directable n.d. automata and c.n.d. automata are denoted by $\mathbf{Dir}(i)$ and $\mathbf{CDir}(i)$, respectively.

A D1-directing word drives the n.d. automaton from any state, or any nonempty set of states, to some fixed state. D2-directability generalizes the notion of directability in the sense that after reading a D2-directing word the set of possible states is independent of the starting state. In fact, if $w \in \mathrm{D}_2(\mathcal{A})$, then the set $Cw^{\mathcal{A}}$ is independent of $C(\subseteq A)$ as long as $C \neq \emptyset$. Finally, if w is a D3-directing word of $\mathcal{A} = (A, X)$, then at least one state in $Cw^{\mathcal{A}}$ is known even if the initial set C of possible states is unknown, but nonempty. Of course, if the n.d. automaton is complete, the current set of possible states is always nonempty.

In [1] Burkhard considered D1-directing words ("homogeneous experiments") for complete n.d. automata. The game of composing a constant relation from a set of relations studied in [6] amounts to the forming of a D1-directing word for an n.d. automaton, and other variants of such games correspond our D3-directability and two special types of D2-directability.

Let us begin the study of the relationships between the various notions of directibility by considering the directing words of a given n.d. automaton $\mathcal{A} = (A, X)$. For this purpose and future use we define the n.d. automata $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ by the following transition tables:

$\mathcal{A}_1$	x
1	2
2	1, 2

$\mathcal{A}_2$	x	y
1	2	–
2	1	–

$\mathcal{A}_3$	x
1	1, 2
2	2

$\mathcal{A}_4$	x	y
1	1, 2	–
2	2	–

It is clear that any D1-directing word is also D2- and D3-directing. Moreover, the inclusion $\mathrm{D}_1(\mathcal{A}_1) \subseteq \mathrm{D}_2(\mathcal{A}_1) \cap \mathrm{D}_3(\mathcal{A}_1)$ is proper since the word xx is D2- and D3-directing, but not D1-directing. That $\mathrm{D}_2(\mathcal{A})$ and $\mathrm{D}_3(\mathcal{A})$ may be incomparable can be seen by considering the n.d. automaton $\mathcal{A}_4$; obviously $x \in \mathrm{D}_3(\mathcal{A}_4) \setminus \mathrm{D}_2(\mathcal{A}_4)$ and $y \in \mathrm{D}_2(\mathcal{A}_4) \setminus \mathrm{D}_3(\mathcal{A}_4)$. On the other hand, it is clear that $\mathrm{D}_1(\mathcal{A}) \subseteq \mathrm{D}_2(\mathcal{A}) \subseteq \mathrm{D}_3(\mathcal{A})$ for any c.n.d. automaton $\mathcal{A}$. That both of these inclusions may be proper can be seen by considering the c.n.d. automaton $\mathcal{A}_1$. These observations may be summed up as follows.

Remark 3.2. For any n.d. automaton $\mathcal{A}$, $\mathrm{D}_1(\mathcal{A}) \subseteq \mathrm{D}_2(\mathcal{A}) \cap \mathrm{D}_3(\mathcal{A})$, and if $\mathcal{A}$ is complete, then $\mathrm{D}_1(\mathcal{A}) \subseteq \mathrm{D}_2(\mathcal{A}) \subseteq \mathrm{D}_3(\mathcal{A})$. Moreover, any one of the inclusions may be proper.

The following observations are also easily verified.

Remark 3.3. For any n.d. automaton $\mathcal{A} = (A, X)$, $\mathrm{D}_2(\mathcal{A})X^* = \mathrm{D}_2(\mathcal{A})$. If $\mathcal{A}$ is complete, then $X^*\mathrm{D}_1(\mathcal{A}) = \mathrm{D}_1(\mathcal{A})$, $X^*\mathrm{D}_2(\mathcal{A})X^* = \mathrm{D}_2(\mathcal{A})$, and $X^*\mathrm{D}_3(\mathcal{A})X^* = \mathrm{D}_3(\mathcal{A})$.

Next we note that the directing words of each type of any given n.d. automaton form a regular language.

Proposition 3.4. *For any n.d. automaton $\mathcal{A}$, the languages $\mathrm{D}_1(\mathcal{A})$, $\mathrm{D}_2(\mathcal{A})$ and $\mathrm{D}_3(\mathcal{A})$ are (effectively) recognizable.*

Proof. Let $\mathcal{A} = (A, X)$ be any n.d. automaton. Suppose that $\mathcal{A}$ has n states and let $A = \{a_1, \ldots, a_n\}$. First we define an automaton $\mathcal{B} = (B, X, \delta)$ so that $B = \{\{a_1u^{\mathcal{A}}, \ldots, a_nu^{\mathcal{A}}\} : u \in X^*\}$ and $\delta(\{C_1, \ldots, C_k\}, x) = \{C_1x^{\mathcal{A}}, \ldots, C_kx^{\mathcal{A}}\}$ for all $\{C_1, \ldots, C_k\} \in B$ and $x \in X$. Furthermore, let $b_0 = \{\{a_1\}, \ldots, \{a_n\}\} (\in B)$. It is clear that $\delta(b_0, u) = \{a_1u^{\mathcal{A}}, \ldots, a_nu^{\mathcal{A}}\}$ for every $u \in X^*$. Hence $L(\mathbf{B}_i) = \mathrm{D}_i(\mathcal{A})$ for $\mathbf{B}_i = (B, X, \delta, b_0, F_i)$, $i = 1, 2, 3$, when we set $F_1 = \{\{\{c\}\} : c \in A\} \cap B$, $F_2 = \{\{C\} : C \subseteq A\} \cap B$ and $F_3 = \{\{C_1, \ldots, C_k\} : C_1 \cap \ldots \cap C_k \neq \emptyset\} \cap B$. The constructions of the recognizers $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ are clearly effective.

Corollary 3.5. The D1-, D2- and D3-directability of an n.d. automaton are decidable properties.

Next we investigate the relationships between the various classes $\mathbf{Dir}(i)$ and $\mathbf{CDir}(i)$.

Proposition 3.6 *The pairwise inclusion relations between the classes $\mathbf{Dir}(i)$ and $\mathbf{CDir}(i)$, $i = 1, 2, 3$, are given by the Hasse diagram shown in Figure 1. All inclusions are proper and the pairwise intersections are as indicated by the diagram.*

Proof. Since $\mathcal{A}_2 \in \mathbf{Dir}(2) \setminus \mathbf{Dir}(3)$ and $\mathcal{A}_3 \in \mathbf{Dir}(3) \setminus \mathbf{Dir}(2)$, the classes $\mathbf{Dir}(2)$ and $\mathbf{Dir}(3)$ are incomparable and $\mathbf{Dir}(2) \cap \mathbf{Dir}(3)$ is contained properly in both of them. The inclusion $\mathbf{Dir}(1) \subseteq \mathbf{Dir}(2) \cap \mathbf{Dir}(3)$ follows from Remark 3.2, and its properness is witnessed by $\mathcal{A}_1$. The inclusions $\mathbf{CDir}(1) \subseteq \mathbf{CDir}(2) \subseteq \mathbf{CDir}(3)$, also implied by Remark 3.2, are proper since $\mathcal{A}_1 \in \mathbf{CDir}(2) \setminus \mathbf{CDir}(1)$ and $\mathcal{A}_3 \in \mathbf{CDir}(3) \setminus \mathbf{CDir}(2)$. It is clear that $\mathbf{CDir}(i) \subset \mathbf{Dir}(i)$ for every $i = 1, 2, 3$, and it follows now directly from the definitions that the intersections of all pairs of classes are correctly given by the diagram.

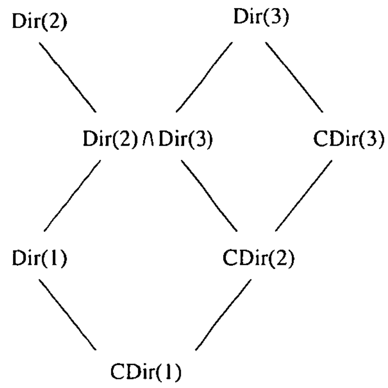

Figure 1:

4 Algebraic constructions and directability

In [8] it was noted that subautomata, epimorphic images and finite direct products of directable automata are directable. Here we consider these matters for nondeterministic automata. Throughout this section $\mathcal{A} = (A, X)$ and $\mathcal{B} = (B, X)$ are n.d. automata which have the same input alphabet.

Let us call $\mathcal{B}$ a *subautomaton* of $\mathcal{A}$ if $B \subseteq A$ and $bx^{\mathcal{B}} = bx^{\mathcal{A}}$ for all $b \in B$ and $x \in X$. It is easy to show that if $\mathcal{B}$ is a subautomaton of $\mathcal{A}$, then $bw^{\mathcal{B}} = bw^{\mathcal{A}}$ for all $b \in B$ and $w \in X^*$. This observation yields immediately the following facts.

Proposition 4.1. *If $\mathcal{B}$ is a subautomaton of an n.d. automaton $\mathcal{A}$, then $\mathrm{D}_i(\mathcal{A}) \subseteq \mathrm{D}_i(\mathcal{B})$, and hence every subautomaton of a* D*i-directable n.d. automaton is* D*i-directable, $i = 1, 2, 3$.*

In [5] a weaker notion of subautomaton was used which can be derived from the general notion of a substructure (cf. [7], for example). Let us say that $\mathcal{B}$ is a *weak subautomaton* of $\mathcal{A}$ if $B \subseteq A$ and $bx^{\mathcal{B}} = bx^{\mathcal{A}} \cap B$ for all $b \in B$ and $x \in X$. None of the claims of Proposition 4.1 holds for weak subautomata.

Example 4.2. The n.d. automaton $\mathcal{A} = (\{1,2,3\}, \{x\})$, where $x^{\mathcal{A}} = \{(1,2), (2,3), (3,3)\}$ is D1-, D2- and D3-directable, but the weak subautomaton corresponding to the subset $\{1,3\}$ has none of these properties. The c.n.d. automaton $\mathcal{B} = (\{1,2,3\}, \{x\})$, where $x^{\mathcal{B}} = \{(1,2), (2,1), (2,3), (3,2), (3,3)\}$ is D2- and D3-directable, but the weak subautomaton $(\{1,2\}, \{(1,2), (2,1)\})$ is neither D2- nor D3-directable although it is complete.

Also homomorphisms of n.d. automata can be defined in different ways. We consider a notion used in [5]: a mapping $\varphi : A \to B$ is a *morphism* from $\mathcal{A}$ to $\mathcal{B}$, and we express this by writing $\varphi : \mathcal{A} \to \mathcal{B}$, if $ax^{\mathcal{A}}\varphi = a\varphi x^{\mathcal{B}}$ for all $a \in A$ and $x \in X$. A surjective morphism is an *epimorphism*, and if there exists an epimorphism $\varphi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is an *image* of $\mathcal{A}$.

It is clear that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of n.d. automata, then $Cx^{\mathcal{A}}\varphi =$

$C\varphi x^{\mathcal{B}}$ whenever $C \subseteq A$ and $x \in X$, and hence $aw^{\mathcal{A}}\varphi = a\varphi w^{\mathcal{B}}$ for all $a \in A$ and $w \in X^*$. Using this observation one proves easily the following proposition.

Proposition 4.3. *If $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism of n.d. automata, then* $\mathrm{D}_i(\mathcal{A}) \subseteq \mathrm{D}_i(\mathcal{B})$, *and hence any image of a* D*i*-*directable n.d. automaton is* D*i*-*directable* $(i = 1, 2, 3)$.

The *direct product* of $\mathcal{A}$ and $\mathcal{B}$ is the n.d. automaton $\mathcal{A} \times \mathcal{B} = (A \times B, X)$ defined so that $(a, b)x^{\mathcal{A}\times\mathcal{B}} = ax^{\mathcal{A}} \times bx^{\mathcal{B}}$ for all $(a, b) \in A \times B$ and $x \in X$. Of course, this definition could be formulated more generally to give the direct product of n $(n \geq 0)$ n.d. automata. It is easy to show that

$$(a, b)w^{\mathcal{A}\times\mathcal{B}} = aw^{\mathcal{A}} \times bw^{\mathcal{B}}$$

for all $(a, b) \in A \times B$ and $w \in X^*$. It is also obvious that $\mathcal{A} \times \mathcal{B}$ is complete iff $\mathcal{A}$ and $\mathcal{B}$ both are complete.

For ordinary automata the catenation uv of a directing word u of $\mathcal{A}$ and a directing word v of $\mathcal{B}$ is a directing word of $\mathcal{A} \times \mathcal{B}$. In the case of D1- and D3-directability this construction does not always work since $bu^{\mathcal{B}}$ may be empty for some $b \in B$, and it may fail even for complete D1-directable n.d. automata because $(au^{\mathcal{A}})v^{\mathcal{A}}$ is not necessarily a singleton set. Indeed, it is easy to show by examples that the classes **Dir**(1), **CDir**(1), **Dir**(2), **Dir**(3) and **Dir**(2)$\cap$**Dir**(3) are not closed under direct products. For the two remaining classes the following positive results can be noted.

Proposition 4.4. *The direct product of two* D2-*directable c.n.d. automata is* D2-*directable, and the direct product of any two* D3-*directable c.n.d. automata is* D3-*directable.*

Proof. Let $\mathcal{A} = (A, X)$ and $\mathcal{B} = (B, X)$ be complete n.d. automata.

If $\mathcal{A}$ and $\mathcal{B}$ are D2-directable, then there are words $u, v \in X^*$ and subsets $C \subseteq A$ and $D \subseteq B$ such that $au^{\mathcal{A}} = C$ and $bv^{\mathcal{B}} = D$ for all $a \in A$ and $b \in B$. Then for all $(a, b) \in A \times B$,

$$(a, b)uv^{\mathcal{A}\times\mathcal{B}} = (C \times bu^{\mathcal{B}})v^{\mathcal{A}\times\mathcal{B}} = Cv^{\mathcal{A}} \times (bu^{\mathcal{B}})v^{\mathcal{B}} = Cv^{\mathcal{A}} \times D,$$

and hence $uv \in \mathrm{D}_2(\mathcal{A} \times \mathcal{B})$. Here we naturally need the fact that $bu^{\mathcal{B}} \neq \emptyset$ for all $b \in B$.

Assume now that $u \in \mathrm{D}_3(\mathcal{A})$ and $v \in \mathrm{D}_3(\mathcal{B})$, and that $c \in au^{\mathcal{A}}$ and $d \in bv^{\mathcal{B}}$ for all $a \in A$ and $b \in B$. Then for all $(a, b) \in A \times B$,

$$(a, b)uv^{\mathcal{A}\times\mathcal{B}} = (au^{\mathcal{A}})v^{\mathcal{A}} \times (bu^{\mathcal{B}})v^{\mathcal{B}}$$

contains the state (c', d), where c' is any given state from $cv^{\mathcal{A}}$.

5 Minimum-length directing words

If $\mathcal{A} = (A, X) \in \mathbf{Dir}(i)$ for some i, $1 \le i \le 3$, let

$$\mathrm{d}_i(\mathcal{A}) = \min\{\lg(w) : w \in \mathrm{D}_i(\mathcal{A})\}.$$

For all $i = 1, 2, 3$ and $n \ge 1$, we set

$$\mathrm{d}_i(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i),\ |A| = n\},$$

and

$$\mathrm{cd}_i(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{CDir}(i),\ |A| = n\},$$

Moreover, we denote by $\mathrm{d}(n)$ the usual maximal length of the minimum-length directing words of a deterministic n-state directable automaton as defined in [3], [4] or [8], for example.

It is clear that $\mathrm{d}(n) \le \mathrm{cd}_i(n) \le \mathrm{d}_i(n)$, for all $n \ge 1$ and $i = 1, 2, 3$. In [1] Burkhard proved that $\mathrm{cd}_1(n) = 2^n - n - 1$ for $n \ge 2$. To obtain this result he constructs for each $n \ge 2$ an n-state c.n.d. automaton for which the shortest D1-directing words are of length $2^n - n - 1$. On the other hand, he observes that if $w = x_1 \ldots x_m x_{m+1}$ is a minimum-length D1-directing word of a c.n.d. automaton $\mathcal{A} = (A, X)$, then Aw is a singleton set and the sequence $Ax_1, \ldots, Ax_1 \ldots x_m$ consists of pairwise different subsets of A with at least two elements. Hence, $\lg(w) \le 2^n - n - 1$. Since this observation is valid also for general n.d. automata, the functions $\mathrm{cd}_1(n)$ and $\mathrm{d}_1(n)$ are as follows. Moreover, the bound is accurate also for $n = 1$ since the empty word is a D1-directing word of any 1-state n.d. automaton.

Proposition 5.1. (Burkhard 1976) *For any* $n \ge 1$, $\mathrm{cd}_1(n) = \mathrm{d}_1(n) = 2^n - n - 1$.

In [6] Goralčik *et al.* proved that the number of factors needed to form a constant relation as a product of some given relations on an n-element set may grow exponentially with n. This result gives exponential lower bounds for $d_1(n)$ and $d_3(n)$. Since we already have an exact expression for $d_1(n)$, we use the example of [6], in a slightly modified form, to obtain a lower estimate for $d_3(n)$. By changing the construction suitably we obtain such lower bounds also for $d_2(n)$ and $cd_2(n)$.

For any $n \ge 2$, let $\omega(n)$ denote the maximal order of any permutation on the set $[n] = \{1, \ldots, n\}$. In [6] it was shown that $\omega(n) \ge \lfloor \sqrt[3]{n} \rfloor!$ when n is the sum of the first k primes for some k. From this it is easy to infer that $\omega(n) \ge \lfloor \sqrt[3]{n} - 1 \rfloor!$ for every $n \ge 2$.

Proposition 5.2. For any $n \ge 2$,

$$\text{(a)} \qquad \lfloor \sqrt[3]{n} - 1 \rfloor! < cd_2(n) \le \sum_{k=2}^{n} \binom{2^n - 1}{k},$$

and

$$\text{(b)} \qquad \lfloor \sqrt[3]{n} - 1 \rfloor ! < d_2(n) \leq \sum_{k=2}^{n} \binom{2^n}{k}.$$

Proof. First we establish both of the lower bounds. Since they are obviously valid for all small values of n, we may suppose that the permutations on $[n-1]$ of maximal order $\omega(n)$ consist of at least two cycles. Let σ be such a permutation and $C_1, \ldots, C_r$ its cycles. Obviously, we may also assume that the lengths $m_1, \ldots, m_r$ of these cycles are relative primes. Let us now define an n-state c.n.d. automaton $\mathcal{A} = ([n], \{x, y, z\})$ as follows.

Firstly, let $x^{\mathcal{A}} = \{(1, \sigma(1)), \ldots, (n-1, \sigma(n-1)), (n, n)\}$. In each cycle C_i we fix arbitrarily an element a_i and set $b_i = \sigma(a_i)$. Now $y^{\mathcal{A}}$ is defined so that $ay^{\mathcal{A}} = \{b_i\}$ if $a \in C_i$ $(1 \leq i \leq r)$, and $ny^{\mathcal{A}} = \{n\}$. Finally, $z^{\mathcal{A}}$ is defined so that

$$az^{\mathcal{A}} = \begin{cases} \{n\} & \text{if } a \in \{a_1, \ldots, a_r, n\}, \\ C_i & \text{if } a \in C_i \setminus \{a_i\}, \ (1 \leq i \leq r). \end{cases}$$

Clearly, $nw^{\mathcal{A}} = \{n\}$ for all words $w \in \{x, y, z\}^*$. On the other hand, $aw^{\mathcal{A}} = \{n\}$ also for all other states $a \in [n-1]$ only in case we may write $w = uzv$, where u is such a word that for all $1 \leq i \leq r$ and $a \in C_i$, $au^{\mathcal{A}} = a_i$. It should now be clear that $yx^{m_1 m_2 \ldots m_r - 1}z$ is the shortest D2-directing word of $\mathcal{A}$, and its length is $\omega(n) + 1 > \lfloor \sqrt[3]{n} - 1 \rfloor !$.

The upper bounds are obtained simply by estimating in each case the number of non-final states of the recognizer $\mathbf{B}_2$ defined in the proof of Propsition 3.4.

Remark. For values of n less than 1331=(11^3) the lower bounds of Proposition 5.2 are well below the bound $(n-1)^2$ given by Černý's well-known automata.

Proposition 5.3. *For any* $n \geq 1$, $(n-1)^2 \leq \mathrm{cd}_3(n) \leq \frac{1}{2}n(n-1)(n-2) + 1$.

Proof. Since the D3-directing words of an ordinary automaton are exactly its directing words, the lower bound is given by Černý's [3] well-known examples of n-state automata $(n \geq 1)$ for which the the shortest directing words are of length $(n-1)^2$.

A word w $(\in X^*)$ is said to D3-*merge* two distinct states a, b of an n.d. automaton $\mathcal{A} = (A, X)$ if $aw \cap bw \neq \emptyset$. We have the following lemmas.

Lemma 5.4. *A c.n.d.. automaton* $\mathcal{A} = (A, X)$ *is* D3-*directable if and only if there is a* D3-*merging word for every pair of distinct states* $a, b \in A$.

Proof. The condition is necessary since any D3-directing word D3-merges every pair of states of $\mathcal{A}$. Suppose now that for each pair $a, b \in A$, $a \neq b$, there is a D3-merging word $w_{a,b}$, and let $A = \{1, \ldots, n\}$. We define inductively a sequence $\nu_0, \nu_1, \ldots, \nu_{n-1}$ of words as follows. For each $i = 1, \ldots, n-1$, let

$$M(i) = 1\nu_{i-1} \cap 2\nu_{i-1} \cap \ldots \cap i\nu_{i-1}.$$

1. Let $\nu_0 = \varepsilon$. Then $M(1) \neq \emptyset$.

2. Suppose that for some i, $1 \le i \le n-2$, we have defined a word ν_{i-1} ($\in X^*$) such that $M(i) \neq \emptyset$. If $M(i) \cap (i+1)\nu_{i-1} \neq \emptyset$, then let $\nu_i = \nu_{i-1}$. Otherwise, choose any $a \in M(i)$ and any $b \in (i+1)\nu_{i-1}$ and set $\nu_i = \nu_{i-1}w_{a,b}$. In both cases $M(i+1) \neq \emptyset$, and hence $\nu_{n-1} \in D_3(\mathcal{A})$.

Lemma 5.5. *Let $\mathcal{A} = (A, X)$ be an n-state n.d. automaton. If a pair $a, b \in A$, $a \neq b$ of states has a* D3*-merging word, then it has a* D3*-merging word of length $\le \binom{n}{2}$.*

Proof. If $w = x_1 \ldots x_k$ is a D3-merging word for $a, b \in A$, then there are sequences of states $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots, b_k$ such that

(1) $a_0 = a$, $b_0 = b$,

(2) $a_i \in a_{i-1}x_i$ and $b_i \in b_{i-1}x_i$ for all $i = 1, \ldots, k$, and

(3) $a_k = b_k$.

If w is of a minimal length, the pairs $\{a_0, b_0\}, \ldots, \{a_{k-1}, b_{k-1}\}$ are all distinct and $k \le \binom{n}{2}$.

We may now complete the proof of Proposition 5.3. If a nontrivial c.n.d. automaton $\mathcal{A} = (A, X)$ is D3-directable, there must exist a pair of states $a, b \in A$, $a \neq b$ such that $ax \cap bx \neq \emptyset$ for some $x \in X$. By appending to such an x $n-2$ D3-merging words of length $\le \binom{n}{2}$ as in the proof of Lemma 5.4 we get a D3-directing word of length $\le 1 + (n-2)\binom{n}{2}$. It is clear that the bound is valid also for $n = 1$.

Proposition 5.6. *For any $n \ge 1$,*

$$\lfloor \sqrt[3]{n} - 1 \rfloor ! < \mathrm{d}_3(n) \le \sum_{k=2}^{n} \binom{2^n - 1}{k} - \sum_{k=2}^{n} \sum_{r=1}^{m(k)} (-1)^{r-1} \binom{n}{r} \binom{2^{n-r}}{k},$$

where $m(k) = \max\{i : k \le 2^{n-i}\}$.

Proof. For the lower bound it suffices to modify the construction of the automaton $\mathcal{A}$ used in the proof of Proposition 5.2 so that $az^{\mathcal{A}} = \emptyset$ if $a \in C_i \setminus \{a_i\}$ $(1 \le i \le r)$. The upper bound is obtained by considering any n-state D3-directable automaton $\mathcal{A} = (A, X)$ and estimating the number of possible non-final states of the recognizer $\mathbf{B}_3$ (defined in the proof of Proposition 3.4) from which a final state can be reached. First of all, we may discard all states containing the empty set. On the other hand, any state consisting of just one non-empty set is final. These two observations yield the first sum expression. From this number we should subtract the number of final states consisting of at least two subsets of A. Consider any k, $2 \le k \le n$. By the Principle of Inclusion and Exclusion the number of states $\{C_1, \ldots, C_k\}$ of $\mathbf{B}_3$ such that $C_1 \cap \ldots \cap C_k \neq \emptyset$ is given by

$$\binom{n}{1}\binom{2^{n-1}}{k} - \binom{n}{2}\binom{2^{n-2}}{k} + \cdots + (-1)^{m(k)-1}\binom{n}{m(k)}\binom{2^{n-m(k)}}{k}.$$

The double sum to be subtracted from the first sum is now obtained by forming the sum of theses sums for $k = 2, \ldots, n$.

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