Affine matching of two sets of points in arbitrary dimensions

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Abstract

In many applications of computer vision, image processing, and remotely sensed data processing, an appropriate matching of two sets of points is required. Our approach assumes one-to-one correspondence between these sets and finds the optimal global affine transformation that matches them. The suggested method can be used in arbitrary dimensions. A sufficient existence condition for a unique transformation is given and proven.

1 Introduction

Many applications lead to the following mathematical problem: Two corresponding sets of points \( \{p_i\} \) and \( \{q_i\} \) \( (i = 1, 2, \ldots, n) \) are given in the \( k \)-dimensional Euclidean space \( \mathbb{R}^k \), and the transformation \( T : \mathbb{R}^k \rightarrow \mathbb{R}^k \) is to be found that gives the minimal mean squared error

\[
\sum_{i=1}^{n} \|T(q_i) - p_i\|^2.
\]

The dimension \( k \) is usually 2 or 3. Some solutions have been proposed for this problem assuming rigid–body transformation (i.e., where only rotations and translations are allowed) [1, 3, 6, 7, 13], affine transformation (i.e., which maps straight lines to straight lines, parallelism is preserved, but angles can be altered) [8], and non-linear transformation (i.e., which can map straight lines to curves) [2, 5, 8]. In [10], a solution is proposed when the correspondence between the

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point sets is unknown, assuming affine transformation. It is mentioned, that if the correspondence was known, a simpler solution is possible e.g., using least squares method, but neither such a method nor a sufficient existence condition for unique solution is given or referenced.

In this paper, we present a method for solving the problem assuming affine transformation, which can be used in arbitrary dimensions. The method is described in Section 2. We state and prove a sufficient existence condition for a unique solution in Section 3. A related open problem concerning degeneracy is presented in Section 4.

2 Method for affine matching of two sets of points

Given a matrix

\[
\mathcal{T} = \begin{pmatrix}
    t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\
    t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\
    0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

it determines an affine transformation \( \mathcal{T} : \mathbb{R}^k \to \mathbb{R}^k \) as follows: For \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \) in \( \mathbb{R}^k \) we have \( y = \mathcal{T}(x) \) if and only if

\[
\begin{pmatrix}
    y_{i1} \\
    y_{i2} \\
    \vdots \\
    y_{ik} \\
    1
\end{pmatrix} = \begin{pmatrix}
    t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\
    t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\
    0 & 0 & \cdots & 0 & 1
\end{pmatrix} \begin{pmatrix}
    x_{i1} \\
    x_{i2} \\
    \vdots \\
    x_{ik} \\
    1
\end{pmatrix}.
\]

Note that homogeneous coordinates are used. Each affine transformation \( \mathcal{T} \) can uniquely be represented in this form [4]. The transformation has \( k \cdot (k + 1) \) degrees of freedom according to the non-constant matrix elements.

Let us fix an affine transformation \( \mathcal{T} : \mathbb{R}^k \to \mathbb{R}^k \) and the corresponding \( \mathcal{T} \) as above. Let \( \{p_i\} \) and \( \{q_i\} \) be two sets of \( n \) points, where

\[
p_i = (p_{i1}, p_{i2}, \ldots, p_{ik}) \in \mathbb{R}^k \quad \text{and} \quad q_i = (q_{i1}, q_{i2}, \ldots, q_{ik}) \in \mathbb{R}^k \quad (i = 1, 2, \ldots, n).
\]

Let \( \{p'_i\} \) be a set of \( n \) points in \( \mathbb{R}^k \), where \( p'_i = \mathcal{T}(q_i) \) \( (i = 1, 2, \ldots, n) \). Define the merit function \( S \) of \( k \cdot (k + 1) \) variables as follows:

\[
S(t_{11}, \ldots, t_{k,k+1}) = \sum_{i=1}^{n} \|p'_i - p_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} (t_{j1} \cdot q_{i1} + \ldots + t_{jk} \cdot q_{ik} + t_{j,k+1} - p_{ij})^2.
\]

which is generally regarded as the matching error.
The least square solution of matrix $T$ is determined by minimizing the function $S$. Function $S$ may be minimal if all of the partial derivatives $\frac{\partial S}{\partial t_{uv}}, \ldots, \frac{\partial S}{\partial t_{k,k+1}}$ are equal to zero. The required $k \cdot (k + 1)$ equations:

$$\frac{\partial S}{\partial t_{uv}} = 2 \cdot \sum_{i=1}^{n} q_{iv} \cdot (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0$$

$$(u = 1, 2, \ldots, k, \ v = 1, 2, \ldots, k),$$

$$\frac{\partial S}{\partial t_{u,k+1}} = 2 \cdot \sum_{i=1}^{n} (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0$$

$$(u = 1, 2, \ldots, k).$$

We get the following system of linear equations:

$$\left(\begin{array}{ccc}
    a_{11} & \ldots & a_{1k} & b_1 \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{k1} & \ldots & a_{kk} & b_k \\
    b_1 & \ldots & b_k & n
\end{array}\right)
\left(\begin{array}{c}
    t_{11} \\
    t_{1,k+1} \\
    t_{21} \\
    t_{2,k+1} \\
    \vdots \\
    t_{k1} \\
    t_{k,k+1}
\end{array}\right)
= \left(\begin{array}{c}
    c_{11} \\
    \vdots \\
    c_{1k} \\
    c_1 \\
    \vdots \\
    c_{kk} \\
    d_k
\end{array}\right),$$

where

$$a_{uv} = a_{vu} = \sum_{i=1}^{n} q_{iu} \cdot q_{iv},$$

$$b_u = \sum_{i=1}^{n} q_{iu},$$

$$c_{uv} = \sum_{i=1}^{n} p_{iu} \cdot q_{iv}.$$
\[ d_u = \sum_{i=1}^{n} p_{iu} \]

\[(u = 1, 2, \ldots, k, v = 1, 2, \ldots, k).\]

The above system of linear equations can be solved by using an appropriate numerical method [9]. There exists a unique solution if and only if \( \det(M) \neq 0 \), where

\[
M = \begin{pmatrix}
  a_{11} & \ldots & a_{1k} & b_1 \\
  \vdots & \vdots & \vdots & \vdots \\
  a_{k1} & \ldots & a_{kk} & b_k \\
  b_1 & \ldots & b_k & n
\end{pmatrix}
\]

Note that if a problem is close to singular (i.e., \( \det(M) \) is close to 0), the method can become unstable.

3 Discussion

In this section we state and prove a sufficient existence condition for a unique solution for the system of linear equations.

By a hyperplane of the Euclidean space \( \mathbb{R}^k \) we mean a subset of the form \( \{a + x : x \in S\} \) where \( S \) is a \((k - 1)\)-dimensional linear subspace. Given some points \( q_1, \ldots, q_n \) in \( \mathbb{R}^k \), we say that these points span \( \mathbb{R}^k \) if no hyperplane of \( \mathbb{R}^k \) contains them. If any \( k + 1 \) points from \( q_1, \ldots, q_n \) span \( \mathbb{R}^k \) then we say that \( q_1, \ldots, q_n \) are in general position.

**Theorem 1.** If \( q_1, \ldots, q_n \) span \( \mathbb{R}^k \) then \( \det(M) \neq 0 \).

**Proof.** Suppose \( \det(M) = 0 \). Consider the vectors \( v_j = (q_{1j}, q_{2j}, \ldots, q_{nj}) \) (1 \( \leq j \leq k \)) in \( \mathbb{R}^n \), and let \( v_{k+1} = (1, 1, \ldots, 1) \in \mathbb{R}^n \). With the notation \( m = k + 1 \) observe that \( M = \left( \langle v_i, v_j \rangle \right)_{m \times m} \) where \( \langle , \rangle \) stands for the scalar multiplication.

Since the columns of \( M \) are linearly dependent, we can fix a \((\beta_1, \ldots, \beta_m) \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\}\) such that \( \sum_{j=1}^{m} \beta_j \langle v_i, v_j \rangle = 0 \) holds for \( i = 1, \ldots, m \). Then

\[
0 = \sum_{i=1}^{m} \beta_i \cdot 0 = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{m} \beta_j \langle v_i, v_j \rangle = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{m} \beta_j \langle v_i, v_j \rangle = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{m} \beta_j v_j
\]

whence \( \sum_{i=1}^{m} \beta_i v_i = 0 \). Therefore all the \( q_j, 1 \leq j \leq n \), are solutions of the following (one element) system of linear equations:

\[
\beta_1 x_1 + \cdots + \beta_k x_k = -\beta_m.
\]

Since the system has solutions and \( (\beta_1, \ldots, \beta_m) \neq (0, \ldots, 0) \), there is an \( i \in \{1, \ldots, k\} \) with \( \beta_i \neq 0 \). Hence the solutions of (1) form a hyperplane of \( \mathbb{R}^k \). This hyperplane contains \( q_1, \ldots, q_n \). Now it follows that if \( q_1, \ldots, q_n \) span \( \mathbb{R}^k \) then \( \det(M) \neq 0 \). Q.e.d.
4 Conclusions

In real applications, it is assumed that both \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \) span \( \mathbb{R}^k \). Then, if the matching error is zero (i.e., \( p'_i = T(q_i) = p_i \) for \( i = 1, 2, \ldots, n \)), the transformation is necessarily non-degenerate, i.e., \( \det(T) \neq 0 \). Moreover, in this case the following property is fulfilled:

**Observation 2.** For all \( I \subseteq \{1, \ldots, n\} \) with \( k + 1 \) elements, the \( p_i, i \in I \), span \( \mathbb{R}^k \) if and only if the \( q_i, i \in I \), span \( \mathbb{R}^k \).

This raises the question whether the transformation is necessarily non-degenerate in general or when Observation 2 holds or at least when Observation 2 "strongly" holds in the following computational sense: each simplex with vertices in \( \{p_1, \ldots, p_n\} \) or with vertices in \( \{q_1, \ldots, q_n\} \) has a large volume (\( k \)-dimensional measure) compared with its edges.

Surprisingly, all these questions have a negative answer, for we have the following three dimensional example.

**Example 3.** With \( n = 5 \) and \( k = 3 \) let \( q_1 = (0,0,24), q_2 = (24,0,0), q_3 = (0,24,0), q_4 = (0,0,0), \) and \( q_5 = (-24,-48,16) \). These five points determine five tetrahedra with reasonably large volumes, the smallest of them being 1536, the volume of the tetrahedron \( (q_2, q_3, q_4, q_5) \). Let \( p_1 = (0,0,0), p_2 = (3,0,0), p_3 = (0,3,0), p_4 = (0,0,3), p_5 = (3,3,3) \), these are some vertices of a cube, so the tetrahedra they determine are at least of volume 9/2. Yet,

\[
T = \begin{pmatrix}
2 & -6 & -6 & 12 \\
-9 & -1 & -9 & 18 \\
0 & 0 & 0 & 8 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which is degenerate.

Experience shows that in real applications the choice of points always guarantees that the transformation is non-degenerate \([11, 12]\). However, from theoretical point of view the following open problem is worth raising: Find a meaningful sufficient condition to ensure non-degeneracy.

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References


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