

The Metric Dimension of Two-Dimensional Extended Meshes

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Abstract

We consider two-dimensional grids with diagonals, also called extended meshes or meshes. Such a graph consists of vertices of the form (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$, for given $m, n \geq 2$. Two vertices are defined to be adjacent if the ℓ_∞ distance between their vectors is equal to 1. A landmark set is a subset of vertices $L \subseteq V$, such that for any distinct pair of vertices $u, v \in V$, there exists a vertex of L with different distances to u and v . We analyze the metric dimension and show how to obtain a landmark set of minimum cardinality.

Keywords: metric dimension, mesh graph, grid graph, landmark set, resolving set

1 Introduction

Consider an undirected graph $G = (V, E)$. For $u, v \in V$, let $d(u, v)$ denote the edge distance between these two vertices. A vertex $x \in V$ separates u and v if $d(x, u) \neq d(x, v)$, and in this case, x is also called a separating vertex for u and v . A landmark set (LS) is a subset $L \subseteq V$ such that for any pair of vertices $u \neq v$, L has at least one vertex y that separates u and v . The vertices of a landmark set L are often referred to as *landmarks*. In the algorithmic metric dimension problem, the goal is to find a landmark set L of minimum cardinality. The cardinality of a minimum cardinality landmark set of G is called the *metric dimension* of G . In a variant of the metric dimension problem (called the weighted metric dimension problem) the goal is to find a landmark set L of minimum cost, where there is a non-negative cost function on G 's vertices.

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A two-dimensional extended mesh (or just mesh) $M_{m,n}$ (for integer parameters m and n) has $|V| = m \cdot n$ vertices of the form (i, j) , where $1 \leq i \leq m$ and $1 \leq j \leq n$. For vertices $(i_1, j_1), (i_2, j_2)$, let $((i_1, j_1), (i_2, j_2)) \in E$ if (and only if) $\max\{|i_1 - i_2|, |j_1 - j_2|\} = 1$. The resulting distance between two vertices is the ℓ_∞ distance between their vectors, that is, $d((i_1, j_1), (i_2, j_2)) = \max\{|i_1 - i_2|, |j_1 - j_2|\}$. This distance also called the chessboard distance. In accordance with the way we have defined the edges of the graph, it is the edge distance of $M_{m,n}$.

We number the rows of the mesh from top to bottom, and the columns from left to right. The rows are called R_1, R_2, \dots, R_m (R_i is also called row i), and the columns are called C_1, C_2, \dots, C_m (C_j is also called column j). The j th vertex of row i is denoted by (i, j) .

A graph related to the two-dimensional mesh is the two-dimensional lattice, $G_{m,n}$, where there is an edge between vertices $(i_1, j_1), (i_2, j_2)$, if (and only if) $|i_1 - i_2| + |j_1 - j_2| = 1$, and the resulting distance between two vertices is the ℓ_1 distance between their vectors, that is, $d((i_1, j_1), (i_2, j_2)) = |i_1 - i_2| + |j_1 - j_2|$. This distance also called the city block distance.

Former results involving mesh graph and lattice graph (both are also called grid graphs) can be found in [15, 14, 16, 13, 2, 1].

We survey some known results on mesh graphs. In the case $m = 1$ and $n > 1$ (or $n = 1$ and $m > 1$), $G_{m,n}$ is a path, and its metric dimension is known to be 1 [14]. In the case $m = n = 1$, the graph has a single vertex, so the metric dimension is zero by definition. In [15], it was shown that for a two-dimensional lattice the metric dimension is always 2 (for $n \geq m \geq 2$), and for a two-dimensional mesh and $m = n \geq 2$, the metric dimension is 3. In our previous work on two-dimensional lattice with a cost function on its vertices [1], we described a polynomial time algorithm for solving the weighted metric dimension problem.

In this work, we will extend the result in [15], and calculate the metric dimension of a two-dimensional mesh, for all values of $m, n \geq 2, m \neq n$. The metric dimension of $M_{m,n}$, denoted by $MD(M_{m,n})$ is the minimum cardinality of any LS for $M_{m,n}$.

In this work our main goal is to prove the following theorem.

Theorem 1. *The metric dimension of a mesh $M_{m,n}$ with $n > m \geq 2$ satisfies $MD(M_{m,n}) = \lceil \frac{n-1}{m-1} \rceil + 1$.*

We will prove a tight bound of $\lceil \frac{n-1}{m-1} \rceil + 1$ on $MD(M_{m,n})$. The lower bound is proved via a direct analysis of all possible column landmark sets (see formal definition on the next section), and for the upper bound we describe a specific landmark set of $M_{m,n}$, whose cardinality is $\lceil \frac{n-1}{m-1} \rceil + 1$.

In the case where $n-1$ is divisible by $m-1$, the mesh contains $\frac{n-1}{m-1}$ concatenated square sub-meshes with m rows and m columns (where every two such sub-meshes

share a single column). In this case we will show that each square sub-mesh (with m rows and m columns) has exactly two landmark vertices in a minimum cardinality LS. Note that substituting $m = n$ does not give the correct metric dimension for $m = n$ (which is 3 as stated above, and not 2), and this is a special case. Informally, the reason for this property is that once $M_{m,n}$ is not a square (i.e., $m \neq n$), given two vertices contained in a square sub-mesh of $M_{m,n}$, they can be separated by a vertex outside this sub-mesh, in some cases.

Interestingly, as a result of Theorem 1, one of the differences between a two-dimensional lattice and a two-dimensional mesh, is that while the metric dimension of two-dimensional lattice is always 2, as quoted above, the metric dimension of two-dimensional mesh (for $n > m \geq 2$) grows as a function of n .

The first articles on the metric dimension problem were by Harary and Melter [11] and by Slater [18]. The problem is NP-hard [14] and hard to approximate [4, 9] for general graphs, and it was studied for specific graph classes [11, 18, 14, 6, 3, 17, 7, 5, 10]. Applications can be found in [4, 11, 15, 8, 14, 6], where some of these applications are relevant for weighted graphs (see also [10]).

2 The column metric dimension and a lower bound

In this section we define an auxiliary concept, which we use in order to provide a lower bound on $MD(M_{m,n})$.

A *Column Landmark Set (CLS)* is a set $L \subseteq V$ such that for any pair of distinct vertices on one column, (i_1, j) and (i_2, j) (where $i_1 \neq i_2$), there exists $u \in L$ such that $d(u, (i_1, j)) \neq d(u, (i_2, j))$. The *Column Metric Dimension* of a graph $M_{m,n}$ is the minimum cardinality of any CLS for it, and it is denoted by $CMD(M_{m,n})$.

Since any LS is a CLS (as all vertices are separated by some vertex of LS, this clearly holds for pairs of vertices on the same column), we have for any mesh G that $MD(M_{m,n}) \geq CMD(M_{m,n})$. In this section we would like to prove a lower bound on $MD(M_{m,n})$, and for that we will prove the following lemma.

Lemma 1. *The column metric dimension of a mesh $M_{m,n}$ with $m, n \geq 2$ satisfies $CMD(M_{m,n}) \geq \lceil \frac{n-1}{m-1} \rceil + 1$. Thus, in the case $n > m$, we have $MD(M_{m,n}) \geq \lceil \frac{n-1}{m-1} \rceil + 1$.*

Note that the lower bound on $CMD(M_{m,n})$ is proved for any $m, n \geq 2$ and not only for the case $n > m$. In what follows we consider these more general cases. Before we prove the lemma, we state and prove several simple and useful claims. Consider specific values $m, n \geq 2$, and a specific CLS L for this graph. Let μ_j be the number of elements of L in column C_j (that is $\mu_j = |C_j \cap L|$),

and let $N_j = \sum_{\ell=1}^j \mu_\ell$ be the number of landmarks in the first j columns (i.e., $N_j = |(C_1 \cup C_2 \cup \dots \cup C_j) \cap L|$). We let $N_j = N_n$ for $j > n$.

The goal of the first claim is to provide a lower bound on the number of landmarks in the first few columns. We will use this claim for the last j columns as well (this follows from symmetry).

Claim 1. *Consider vertices $u_1 = (x_1, y')$ and $u_2 = (x_2, y')$ (on column y'). Let $v = (x, y)$ be a vertex that separates them, that is $d(v, u_1) \neq d(v, u_2)$. Then, $|y - y'| \leq m - 2$.*

Proof. Assume by contradiction that $|y - y'| \geq m - 1$. Then, for $k = 1, 2$, $d(v, u_k) = \max\{|x - x_k|, |y - y'|\} = |y - y'|$, as $|x - x_k| \leq m - 1$ while $|y - y'| \geq m - 1$, so the distances to v are determined by the columns. This contradicts $d(v, u_1) \neq d(v, u_2)$. \square

Claim 2. *The inequality $N_{\lceil \frac{m}{2} \rceil} \geq 1$ holds for any CLS L . Moreover, if $N_1 = 0$, then $N_{m-1} \geq 2$.*

Proof. If $N_1 > 0$, the first part holds, so we assume $N_1 = 0$ and prove both parts of the claim. Consider two vertices on column 1, $a_1 = (\lfloor \frac{m}{2} \rfloor, 1)$ and $a_2 = (\lfloor \frac{m}{2} \rfloor + 1, 1)$ (for even values of m they are exactly in the middle of column 1, for odd values of m they are approximately in the middle of column 1). Note that $\lfloor \frac{m}{2} \rfloor \geq 1$ and $\lfloor \frac{m}{2} \rfloor + 1 \leq m$, for $m \geq 2$, so these two vertices are well defined.

Let $b = (x_b, y_b)$ denote a vertex of L separating a_1 and a_2 . We will show $y_b \leq \lfloor \frac{m}{2} \rfloor$, proving $N_{\lceil \frac{m}{2} \rceil} \geq 1$. Assume by contradiction $y_b \geq \lfloor \frac{m}{2} \rfloor + 1$. We will show that the distance of b to a_1 and a_2 is defined by the difference between column indices and therefore the distances are equal. We get $d(a_1, b) = \max\{|x_b - \lfloor \frac{m}{2} \rfloor|, |y_b - 1|\}$ and $d(a_2, b) = \max\{|x_b - (\lfloor \frac{m}{2} \rfloor + 1)|, |y_b - 1|\}$. Since $1 \leq x_b \leq m$, we get $|x_b - \lfloor \frac{m}{2} \rfloor| \leq \lfloor \frac{m}{2} \rfloor \leq y_b - 1$ and $|x_b - \lfloor \frac{m}{2} \rfloor - 1| \leq \lfloor \frac{m}{2} \rfloor \leq y_b - 1$, so $d(a_1, b) = d(a_2, b) = y_b - 1$, a contradiction.

Next, we analyze N_{m-1} for the case $N_1 = 0$. Since $N_{m-1} > 0$ (as $m - 1 \geq \lfloor \frac{m}{2} \rfloor$ for $m \geq 2$), there is at least one vertex $c \in L$ in this case, as we have shown. It remains to show that there is a pair of vertices in the first $m - 1$ columns not separated by c , implying $|L| \geq 2$. Let $c = (x_c, y_c)$ be a vertex of L such that $2 \leq y_c \leq m - 1$. Once again we consider two vertices of the first column, and show that c does not separate them. Let $a'_1 = (x_c, 1)$, i.e., the vertex of the first column on the same row as c . If $x_c = 1$, let $a'_2 = (x_c + 1, 1)$ and otherwise $a'_2 = (x_c - 1, 1)$. Thus, a'_2 is well defined, as it is either the vertex just above a'_1 or just below it (since $m \geq 2$, at least one of these vertices exists). We have $d(a'_1, c) = \max\{0, |y_c - 1|\} = y_c - 1$ and $d(a'_2, c) = \max\{1, |y_c - 1|\} = y_c - 1$, since $y_c \geq 2$. Thus, as L contains a vertex separating a'_1 and a'_2 , and by Claim 1 this

vertex is on one of the first $m - 1$ columns (or of the n columns, if $n < m - 1$), we get $N_{m-1} \geq 2$. □

Claim 3. *Let L be a CLS defined for $M_{m,n}$ such that $m, n \geq 2$ (where each element of L is of the form (i, j)). If $L \cap C_n = \emptyset$, then L is a CLS for $M_{m,n-1}$. If $L \cap R_m = \emptyset$, then L is a CLS for $M_{m-1,n}$.*

Proof. Consider two vertices (i_1, j_1) and (i_2, j_2) . Any shortest path between these two vertices traverses only vertices on columns $\min\{j_1, j_2\}, \dots, \max\{j_1, j_2\}$ and rows $\min\{i_1, i_2\}, \dots, \max\{i_1, i_2\}$, that is, on columns and rows between the columns of these vertices and rows between the rows of these vertices. This implies the validity of the claim. □

Proof. We now prove Lemma 1. We start this proof with several simple cases, which will allow us to use induction for the remaining cases (on $n + m$).

Case 1. Consider the case $m = 2$. In this case we show $CMD(M_{m,n}) \geq n$. To prove this, we show that $\mu_j \geq 1$ for $1 \leq j \leq n$. By Claim 1, the only vertices that separate the two vertices $(1, j)$ and $(2, j)$ are on column j , that is, one of these two vertices. Therefore, any CLS either contains at least one of $(1, j)$ and $(2, j)$.

Case 2. Consider the case $n \leq m$ (where $m \geq 3$). In this case we show $CMD(M_{m,n}) \geq 2$. If $\mu_1 \geq 1$ and $\mu_n \geq 1$, we are done. Otherwise, at least one of the columns C_1 and C_n does not have a landmark. Assume without loss of generality (by rotating the mesh by 180 degrees or by reflecting it across a vertical line) that $L \cap C_1 = \emptyset$. By Claim 2, $N_{m-1} \geq 2$, so $|L| \geq 2$.

We are left with the case $n > m \geq 3$. We say that a *gap* (with respect to a CLS L) is a sequence of $m - 1$ columns that do not contain elements of L , that is, there is an index $1 \leq k \leq n - m + 2$ such that $\{C_k, C_{k+1}, \dots, C_{k+m-2}\} \cap L = \emptyset$. The cases $k = 1$ and $k = n - m + 2$ are not possible due to Claim 2, as the first $m - 1$ columns have at least one element of L , and symmetrically, the last $m - 1$ columns have at least one element of L . Thus, k satisfies $2 \leq k \leq n - m + 1$, and in particular, a gap is possible only if $n \geq m + 1$ (since all cases where $n \leq m$ were already considered, a gap is defined for all remaining cases).

We use the first two cases as the induction base, and prove the remaining cases via induction.

Case 3. There is a gap in L . Let k ($2 \leq k \leq n - m + 1$) be such that $\{C_k, C_{k+1}, \dots, C_{k+m-2}\} \cap L = \emptyset$, that is, all elements of L are on columns $C_1, C_2, \dots, C_{k-1}, C_{k+m-1}, \dots, C_n$. Let $L_1 = \{C_1, C_2, \dots, C_k\} \cap L$ and let $L_2 = \{C_{k+m-2}, C_{k+m-1}, \dots, C_n\} \cap L$ (so $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$ since $m \geq 3$). We claim that L_1 is a CLS for the sub-mesh of k columns and m rows consisting of the first $k \geq 2$ columns. Since L is a CLS, for every pair of distinct vertices v_1 and v_2 on one of the first k columns, there is a vertex of L separating them. By Claim 1, such a vertex is not on columns $k + m - 1, \dots, n$. As columns $k, k + 1, \dots, k + m - 2$ have no elements of L , there is an element of L separating v_1 and v_2 on one of the columns $1, 2, \dots, k - 1$, that is, it is an element of L_1 . Thus, L_1 is a CLS for a mesh of k columns and m rows. Analogously, it is possible to prove that L_2 is a CLS for a mesh of $n - m + 3 - k \geq 2$ columns and m rows. Using induction, and as the numbers of columns of L_1 and L_2 are k and $n - m - k + 3$, respectively (with m rows), we find $|L| = |L_1| + |L_2| \geq \lceil \frac{k-1}{m-1} \rceil + 1 + \lceil \frac{(n-m+3-k)-1}{m-1} \rceil + 1 \geq 2 + \lceil \frac{n-m+1}{m-1} \rceil = \lceil \frac{n}{m-1} \rceil + 1 \geq \lceil \frac{n-1}{m-1} \rceil + 1$, which holds by properties of rounding up.

We are left with the case where there is no gap in L . For an integer $c \geq 0$, let $h_c = m + (m - 1)c$ and $h'_c = m - 1 + (m - 1)c = h_c - 1$. Note that by their definitions it holds that $m \leq h_c$ and $m - 1 \leq h'_c$ for any value of c . We claim that in this case, if $h_c \leq n$, we have $N_{h_c} \geq c + 2$, and additionally, if $h'_c \leq n$ and $N_1 = 0$ hold, $N_{h'_c} \geq c + 2$. This is proved by induction on c . For $c = 0$, if $N_1 = 0$, by Claim 2, $N_m \geq N_{m-1} \geq 2$. Otherwise, $N_1 \geq 1$, and since there are no gaps, at least one of the columns C_2, C_3, \dots, C_m has an element of L , so $N_m \geq 2$. The inductive step is proved in a similar manner. If $N_{h_{c-1}} \geq c + 1$ for some integer $c \geq 1$, as there is no gap, there is at least one element of L on columns $h_{c-1} + 1, h_{c-1} + 2, \dots, h_{c-1} + m - 1$. Since $h_{c-1} = m + (m - 1)(c - 1) = 1 + (m - 1)c$ we get that $h_{c-1} + m - 1 = h_c$, and as a result $N_{h_c} \geq N_{h_{c-1}} + 1 \geq c + 2$. If $N_{h'_{c-1}} \geq c + 1$ for some $c \geq 1$, as there is no gap, there is at least one element of L on columns $h'_{c-1} + 1, h'_{c-1} + 2, \dots, h'_{c-1} + m - 1$. Since $h'_{c-1} = h_{c-1} - 1 = (m - 1)c$ we get that $h'_{c-1} + m - 1 = h_{c-1} - 1 = h'_c$, and as a result $N_{h'_c} \geq N_{h'_{c-1}} + 1 \geq c + 2$.

Case 4. There is an element of L on the first column or on the last column (or both). Without loss of generality (by possibly rotating the mesh by 180 degrees) we assume $L \cap C_n \neq \emptyset$, i.e., $\mu_n > 0$. Let $c = \lceil \frac{n-1}{m-1} \rceil - 2$, where $c \geq 0$ as $n > m$. Consider columns $1, 2, \dots, m + (m - 1)c$, where $(m - 1)c + m < (m - 1) \cdot (\frac{n-1}{m-1} - 1) + m = n$. Thus, as $h_c \leq n - 1$, we have $|L| \geq N_{h_c} + \mu_n \geq c + 3 = \lceil \frac{n-1}{m-1} \rceil + 1$.

We are left with the case where $\mu_1 = 0$ and $\mu_n = 0$.

Case 5. The number of columns is sufficiently large, that is, $n \geq 2m$. Let $c = \lfloor \frac{n}{m-1} \rfloor - 2$, where $c \geq 0$ as $n > m$. Consider columns $1, 2, \dots, m - 1 + (m - 1)c$,

where $m - 1 + (m - 1)c = (m - 1)(c + 1) \leq (m - 1) \cdot (\frac{n}{m-1} - 1) = n - m + 1$. By Claim 2, the last $m - 1$ columns have at least two elements of L . Thus, $|L| \geq N_{h'_c} + 2 \geq c + 4 = \lfloor \frac{n}{m-1} \rfloor + 2 \geq \lceil \frac{n}{m-1} \rceil + 1 \geq \lceil \frac{n-1}{m-1} \rceil + 1$.

We are left with the case where n satisfies $3 \leq n \leq 2m - 1$.

Case 6. The number of columns satisfies $m + 2 \leq n \leq 2m - 1$. Since C_n has no elements of L , using Claim 3, L is a CLS for the sub-mesh consisting of first $n - 1 \geq m + 1$ columns and m rows, and by induction, $|L| \geq 3$.

Case 7. The number of columns satisfies $n = m + 1$, and at least one row out of R_1 and R_m has no element of L (i.e., at least one of the following holds: $R_1 \cap L = \emptyset$, $R_m \cap L = \emptyset$). In this case, using Claim 3, L is a CLS for a mesh of $m - 1$ rows and n columns. Since $n = (m - 1) + 2$, using induction we have $|L| \geq 3$.

We are left with the case where $n = m + 1$, $|L \cap R_1| \geq 1$ and $|L \cap R_m| \geq 1$. Notice that the case where $|L| \geq 3$ satisfies the demand since $|L| \geq \lceil \frac{m}{m-1} \rceil + 1 = 3$.

The last case to consider is $n = m + 1$, $|L| \leq 2$, so $|L \cap R_1| = 1$ and $|L \cap R_m| = 1$, and no other row has an element of L . We show that this scenario is not possible.

Case 8. Let $L = \{(1, y_1), (m, y_2)\}$. Since C_1 and C_n have no elements of L , we have $2 \leq y_1, y_2 \leq n - 1$. By Claim 2, L has at least one element on columns $C_1, C_2, \dots, C_{\lceil \frac{m}{2} \rceil}$ and symmetrically at least one element on columns $C_{\lfloor \frac{m}{2} \rfloor + 2}, \dots, C_{m+1}$. As $\lceil \frac{m}{2} \rceil < \lfloor \frac{m}{2} \rfloor + 2$, we find that one of y_1, y_2 is in $\{1, 2, \dots, \lceil \frac{m}{2} \rceil\}$, and the other is in $\{\lfloor \frac{m}{2} \rfloor + 2, \dots, m + 1\}$. Without loss of generality, by possibly reflecting the mesh across a vertical line, assume that $y_1 \leq \lceil \frac{m}{2} \rceil$ and $y_2 \geq \lfloor \frac{m}{2} \rfloor + 2$. Moreover, by possibly rotating the mesh by 180 degrees, we assume that y_1 is at least as far from the first column as y_2 is to the last column, that is, $y_1 - 1 \geq n - y_2 = m + 1 - y_2$. Consider the vertices $v_1 = (y_1 - 1, 1)$ and $v_2 = (y_1, 1)$. Note that v_1 is well defined since $y_1 \geq 2$. If L is a CLS, these two vertices are separated by $(1, y_1)$ or by (m, y_2) . We have $d((1, y_1), v_1) = d((1, y_1), v_2) = y_1 - 1$, so $(1, y_1)$ does not separate v_1 and v_2 . We also have $d((m, y_2), v_1) = \max\{m - y_1 + 1, y_2 - 1\}$ and $d((m, y_2), v_2) = \max\{m - y_1, y_2 - 1\}$. Since $y_2 - 1 \geq m - y_1 + 1 > m - y_1$, we get $d((m, y_2), v_1) = d((m, y_2), v_2) = y_2 - 1$, so v_1 and v_2 are not separated by a vertex of L , a contradiction. \square

3 An upper bound

Given $M_{m,n}$ with $n \geq m \geq 2$, we define a set $L \subseteq V$ and show that its cardinality is according to Theorem 1 (i.e., $\lceil \frac{n-1}{m-1} \rceil + 1$), and that it is a landmark set (from now

on we only consider LS not CLS like in the previous section). Let $f = \lfloor \frac{n-1}{m-1} \rfloor + 1$. For $1 \leq k \leq f$, if k is odd, let $z_k = (1, 1 + (k - 1)(m - 1))$, and if k is even, let $z_k = (m, 1 + (k - 1)(m - 1))$. Note that $1 + (f - 1)(m - 1) \leq 1 + \frac{n-1}{m-1} \cdot (m - 1) = n$, so z_1, z_2, \dots, z_f are well-defined. That is, there is an element of L every $m - 1$ columns, starting with the first column. The rows of these elements alternate between 1 and m . If $1 + (f - 1)(m - 1) = n$, that is, $n - 1$ is divisible by $m - 1$, vertex z_f is on column n , moreover, in this case, $\lceil \frac{n-1}{m-1} \rceil = \lfloor \frac{n-1}{m-1} \rfloor$, and we have defined L completely. If $1 + (f - 1)(m - 1) < n$, that is, $n - 1$ is not divisible by $m - 1$, we have $\lceil \frac{n-1}{m-1} \rceil = \lfloor \frac{n-1}{m-1} \rfloor + 1$, and we define $z_{f+1} = (1, n)$ if f is even (and $f + 1$ is odd), and otherwise we let $z_{f+1} = (m, n)$.

Let $f' = \lceil \frac{n-1}{m-1} \rceil + 1$. The elements of L are $z_1, \dots, z_{f'}$ (no matter whether $f' > f$ or $f' = f$). Figure 1 shows an example of the set L for a mesh with $m = 5$ and $n = 12$.

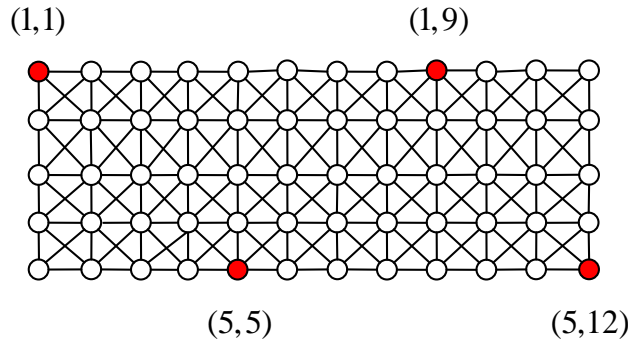


Figure 1: An example of the set $L = \{(1, 1), (5, 5), (1, 9), (5, 12)\}$ described in the above explanation, for a mesh with 5 rows and 12 columns.

Claim 4. Consider the case $n \geq 2m - 1$, and consider two vertices $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. If neither z_1 nor $z_{f'}$ separates v_1 and v_2 , then v_1 and v_2 are on the same column (that is, $y_1 = y_2$).

Proof. Assume by contradiction that $y_1 \neq y_2$ and assume without loss of generality that $y_1 < y_2$. If $y_2 \geq m + 1$, we have $d(v_2, z_1) = \max\{x_2 - 1, y_2 - 1\}$. Since $x_2 - 1 \leq m - 1$ while $y_2 - 1 \geq m$, we have $d(v_1, z_1) = d(v_2, z_1) = y_2 - 1$ where the first equality holds as z_1 does not separate v_1 and v_2 . Next, by $y_2 - 1 = d(v_1, z_1)$ and $d(v_1, z_1) = \max\{x_1 - 1, y_1 - 1\}$, and since $x_1 - 1 \leq m - 1$, we get $d(v_1, z_1) = y_1 - 1$. Thus $y_1 = y_2$ in this case, a contradiction.

If $y_2 \leq m$, we get $y_1 \leq m - 1$, and the proof is symmetric for the distances from z_2 instead of z_1 (by rotating the mesh and possibly reflecting it across a horizontal

line); the distance of v_1 to z_2 is at least m , and since v_2 has the same distance, the maximum value of the distance is achieved by the second term also for v_2 and they are on one column. \square

Claim 5. Consider two vertices $u_1 = (1, 1)$ and $u_2 = (m, q)$, such that $2 \leq q \leq n$ and $q \leq m$. Consider also two distinct vertices $v_1 = (x_1, y)$ and $v_2 = (x_2, y)$ (on one column), where $1 \leq x_1 < x_2 \leq m$ and $1 \leq y \leq q$. Then, at least one of u_1 and u_2 separates v_1 and v_2 .

Proof. If $x_1 \geq y$, we have $x_2 \geq y + 1$. Thus, for $i = 1, 2$, $d(v_i, u_1) = \max\{x_i - 1, y - 1\} = x_i - 1$, and since $x_1 - 1 \neq x_2 - 1$, we find that u_1 separates v_1 and v_2 . Otherwise, $x_1 < y$ holds, and we show that u_2 separates v_1 and v_2 . So $d(v_1, u_2) = \max\{m - x_1, q - y\}$, and since $q \leq m$, $-y < -x_1$, we find $d(v_1, u_2) = m - x_1$. On the other hand, $d(v_2, u_2) = \max\{m - x_2, q - y\}$, where $m - x_2 < m - x_1$ (as $x_1 < x_2$) and $q - y < m - x_1$, so $d(v_2, u_2) = \max\{m - x_2, q - y\} < d(v_1, u_2)$, and therefore u_2 separates v_1 and v_2 . \square

Corollary 1. Consider two vertices $u_1 = (1, 1)$ and $u_2 = (m, q)$, such that $2 \leq q \leq n$ and $q \geq m$. Consider two distinct vertices $v_1 = (x, y_1)$ and $v_2 = (x, y_2)$ (on one row), where $1 \leq y_1 < y_2 \leq q$ and $1 \leq x \leq m$. Then, at least one of u_1 and u_2 separates v_1 and v_2 .

The corollary holds by rotating the mesh by 90 degrees.

Claim 6. Consider two distinct vertices $v_1 = (x_1, y)$ and $v_2 = (x_2, y)$ (on one column). Let $z_s = (x_s, y_s)$ and $z_{s+1} = (x_{s+1}, y_{s+1})$ be such that $y_s \leq y \leq y_{s+1}$. Then, at least one of z_s and z_{s+1} separates v_1 and v_2 .

Proof. Without loss of generality it is sufficient to consider the sub-graph of the mesh consisting of columns $y_s, y_s + 1, \dots, y_{s+1}$ and all rows. The property holds by Claim 5, as no shortest path between two vertices in this sub-mesh traverses any vertex outside it, and by possibly reflecting the mesh across a vertical line. \square

Corollary 2. In the case $n \geq 2m - 1$, the set L is a landmark set.

Proof. By Claim 4, for every pair of vertices on different columns, L contains a vertex separating them. By Claim 6, every pair of vertices on one column is separated as well, by choosing an appropriate value of s , which is possible for any y since L has an element on C_1 and an element on C_n . \square

We are left with the case where n satisfies $m + 1 \leq n \leq 2m - 2$ (in particular, $n \geq 3$). In this case, $L = \{z_1 = (1, 1), z_2 = (m, m), z_3 = (1, n)\}$.

Claim 7. Consider two distinct vertices $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, in the case where $n \in \{m + 1, m + 2, \dots, 2m - 2\}$. At least one of the vertices z_1 and z_3 separates v_1 and v_2 .

Proof. Assume that z_1 does not separate v_1 and v_2 . We have $d(v_i, z_1) = \max\{x_i - 1, y_i - 1\}$.

If $x_1 = x_2$, we have $y_1 \neq y_2$. By $\max\{x_1 - 1, y_1 - 1\} = \max\{x_2 - 1, y_2 - 1\}$ we find that it cannot be the case that $d(v_i, z_1) = y_i - 1$ for $i = 1, 2$. Thus, we can assume without loss of generality that $d(v_1, z_1) = x_1 - 1 > y_1 - 1$. We claim that $x_2 \geq y_2$ holds. Indeed, $d(v_2, z_1) = d(v_1, z_1) = x_1 - 1 = x_2 - 1$, so $x_2 - 1 \geq y_2 - 1$. Thus, as $x_1 = x_2 \leq m$, we have $y_1 < m$ and $y_2 \leq m$, and by Corollary 1, $z_2 = (m, m)$ separates v_1 and v_2 .

If $y_1 = y_2$, by Claim 6, one of z_1, z_2 , and z_3 separates v_1 and v_2 .

We are left with the case $x_1 \neq x_2$ and $y_1 \neq y_2$. By the definition of distances, we cannot have that $d(v_i, z_1) = x_i - 1$ holds for $i = 1, 2$ or that $d(v_i, z_1) = y_i - 1$ holds for $i = 1, 2$, so we have either $d(v_1, z_1) = y_1 - 1$ and $d(v_2, z_1) = x_2 - 1$ (and $x_1 < y_1, x_2 > y_2$) or we have $d(v_1, z_1) = x_1 - 1$ and $d(v_2, z_1) = y_2 - 1$ (and $x_1 > y_1, x_2 < y_2$). Without loss of generality (by possibly swapping the roles of v_1 and v_2) we assume that $d(v_1, z_1) = y_1 - 1$ and $d(v_2, z_1) = x_2 - 1$ holds, so $y_1 = x_2$, and $x_1 < y_1, x_2 > y_2$.

Consider the distances to z_3 . We find $d(v_1, z_3) = \max\{x_1 - 1, n - y_1\}$ and $d(v_2, z_3) = \max\{x_2 - 1, n - y_2\}$. Assume that $d(v_1, z_3) = x_1 - 1$ holds. If $d(v_2, z_3) = x_2 - 1$, we are done by $x_1 \neq x_2$. Otherwise $d(v_2, z_3) = n - y_2$, and we have $x_1 - 1 > n - y_1$ and $x_2 - 1 < n - y_2$. We get $n - y_2 > x_2 - 1 = y_1 - 1 > x_1 - 1$. Assume that $d(v_1, z_3) = n - y_1$ holds. If $d(v_2, z_3) = n - y_2$, we are done by $y_1 \neq y_2$. Otherwise $d(v_2, z_3) = x_2 - 1$, and we have $x_1 - 1 < n - y_1$ and $x_2 - 1 > n - y_2$. We get $x_2 - 1 > n - y_2 > n - x_2 = n - y_1$, so z_3 separates v_1 and v_2 in all remaining cases. \square

4 Conclusion

In section 2 we have proved a lower bound of $\lceil \frac{n-1}{m-1} \rceil + 1$ on the column metric dimension of $M_{m,n}$, for all values of $m, n \geq 2$, implying the lower bound for the metric dimension for $n > m \geq 2$. In section 3 we have proved an upper bound of $\lceil \frac{n-1}{m-1} \rceil + 1$ on the metric dimension of $M_{m,n}$, for all values of $n > m \geq 2$, by defining a suitable landmark set. As mentioned in Section 1, the case of $m = n$ is a special case where $MD(M_{m,n}) = 3$ [15], which was known prior to our work, and we do not analyze this case. Our main result (Theorem 1) is proved by combining the lower bound and upper bound. Using the last remark we get a full characterization of the metric dimension of (two-dimensional) meshes.

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