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Domain Semirings United

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Abstract

Domain operations on semirings have been axiomatised in two different ways: by a map from an additively idempotent semiring into a boolean subalgebra of the semiring bounded by the additive and multiplicative unit of the semiring, or by an endofunction on a semiring that induces a distributive lattice bounded by the two units as its image. This note presents classes of semirings where these approaches coincide.

Keywords: semirings, quantales, domain operations

1 Introduction

Domain semirings and Kleene algebras with domain [1, 2] yield particularly simple program verification formalisms in the style of dynamic logics, algebras of predicate transformers or boolean algebras with operators (which are all related).

There are two kinds of axiomatisation. Both are inspired by properties of the domain operation on binary relations, but target other computationally interesting models such as program traces or paths on digraphs as well.

The initial two-sorted axiomatisation [1] models the domain operation as a map $\mathsf{d} : S \to B$ from an additively idempotent semiring $(S, +, \cdot, 0, 1)$ into a boolean subalgebra B of S bounded by 0 and 1. This seems natural as domain elements form powerset algebras in the target models mentioned. Yet the domain algebra B cannot be chosen freely: B must be the maximal boolean subalgebra of S bounded by 0 and 1 and equal to the set S_{d} of fixpoints of d in S.

The alternative, one-sorted axiomatisation [2] therefore models d as an endofunction on a semiring S that induces a suitable domain algebra on S_{d}—yet gener-

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ally only a bounded distributive lattice. An antidomain (or domain complementation) operation is needed to obtain boolean domain algebras.

In the model of binary relations over a set X, $+$ is set union and $\cdot$ relational composition; 0 is the empty relation and 1 the identity relation. The domain of relation $R \subseteq X \times X$ is $\mathsf{d}(R) = \{(x,x) \mid \exists y.\ (x,y) \in R\}$ while its antidomain is $\mathsf{a}(R) = \{(x,x) \mid \forall y.\ (x,y) \notin R\}$. In the path model over a directed graph $\sigma, \tau : E \to V$, the carrier set consists of all finite paths $(v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n)$ in the graph in which vertices $v_i \in V$ and edges $e_i \in E$ alternate and are compatible with the source map σ and target map τ. The operations $+$ and 0 are again $\cup$ and $\emptyset$, respectively; 1 is V with elements $v \in V$ seen as paths of length 1. Extending σ and τ to paths as expected, composition $\pi_1;\pi_2$ of paths π_1 and π_2 is defined if $\tau(\pi_1) = \sigma(\pi_2)$, and it then glues on this vertex. Path composition is lifted to sets of paths as $P;Q = \{\pi_1;\pi_2 \mid \pi_1 \in P, \pi_2 \in Q, \tau(\pi_1) = \sigma(\pi_2)\}$. Finally $\mathsf{d}(P) = \{\sigma(\pi) \mid \pi \in P\}$ and $\mathsf{a}(P) = \{v \mid \forall \pi.\ \sigma(\pi) = v \Rightarrow \pi \notin P\}$. Other models can be found in the literature.

This note revisits the two axiomatisations mentioned above to tie some loose ends together. We describe a natural algebraic setting in which they coincide, and which has so far been overlooked. It consists of additively idempotent semirings in which the sets of all elements below 1 form boolean algebras, as is the case, for instance, in boolean monoids and boolean quantales. We further take the opportunity to discuss domain axioms for arbitrary quantales.

The restriction to such boolean settings has little impact on applications: most models of interest are powerset algebras and hence (complete atomic) boolean algebras anyway. Yet the coincidence itself does make a difference: one-sorted domain semirings are easier to formalise in interactive proof assistants and apply in program verification and correctness.

2 Domain Axioms for Semirings

First we recall the two axiomatisations of domain semirings and their relevant properties. To distinguish them, we call the first class, introduced in [1], *test dioids with domain* and the second one, introduced in [2], *domain semirings*.

We assume familiarity with posets, lattices and semirings. A *dioid*, in particular, is an idempotent semiring $(S, +, \cdot, 0, 1)$, that is, $x + x = x$ holds for all $x \in S$. Its additive monoid $(S, +, 0)$ is then a semilattice ordered by $x \le y \Leftrightarrow x + y = y$ and with least element 0; multiplication preserves $\le$ in both arguments. (We generally omit the $\cdot$ for multiplication.)

We write $S_1 = \{x \in S \mid x \le 1\}$ for the set of *subidentities* in S and call S *bounded* if it has a maximal element, $\top$.

We call a dioid S *full* if S_1 is a boolean algebra, bounded by 0 and 1, with $+$ as sup, $\cdot$ as inf and an operation $(\_)'$ of complementation that is defined only on S_1.

Definition 1 ([1]). *A* test dioid (S, B) *is a dioid S that contains a boolean subalgebra B of S_1—the* test algebra *of S—with least element 0, greatest element 1, in which $+$ coincides with sup and that is closed under multiplication.*

Once again we write $(\_)'$ for complementation on B.

Lemma 1 ([1]). *In every test dioid, multiplication of tests is their meet.*

Lemma 2 ([1]). *Let (S, B) be a test dioid. Then, for all $x \in S$ and $p \in B$,*

1. $x \leq px \Leftrightarrow p'x = 0$,
2. $x \leq px \Leftrightarrow x \leq p\top$ *if S is bounded.*

Definition 2 ([1]). *A* test dioid with predomain *is a test dioid (S, B) with a* predomain operation $\mathsf{d} : S \to B$ *such that, for all $x \in S$ and $p \in B$,*

$$x \leq \mathsf{d}(x)x \qquad \textit{and} \qquad \mathsf{d}(px) \leq p.$$

It is a test dioid with domain *if it also satisfies, for $x, y \in S$, the* locality *axiom*

$$\mathsf{d}(x\mathsf{d}(y)) \leq \mathsf{d}(xy).$$

Weak locality $\mathsf{d}(xy) \leq \mathsf{d}(x\mathsf{d}(y))$ already holds in every test dioid with predomain. Thus $\mathsf{d}(x\mathsf{d}(y)) = \mathsf{d}(xy)$ in every test dioid with domain.

It is easy to check that binary relations and sets of paths satisfy the axioms of test dioids with domain, and that $B = S_1$ in both models.

Lemma 3 ([1]). *In every test dioid (S, B), the following statements are equivalent:*

1. *(S, B, d) is a test dioid with predomain,*
2. *the map $\mathsf{d} : S \to B$ on (S, B) satisfies, for all $x \in S$ and $p \in B$, the* least left absorption *property*
$$\mathsf{d}(x) \leq p \Leftrightarrow x \leq px, \qquad \text{(lla)}$$
3. *in case S is bounded, $\mathsf{d} : S \to B$ on (S, B) is, for all $x \in S$ and $p \in B$, the left adjoint in the adjunction*
$$\mathsf{d}(x) \leq p \Leftrightarrow x \leq p\top. \qquad \text{(d-adj)}$$

Interestingly, test algebras of test dioids with domain cannot be chosen ad libitum: they are formed by those subidentities that are complemented relative to the multiplicative unit [1]. This has the following consequences.

Proposition 1. *The test algebra B of a test dioid with domain (S, B, d) is the largest boolean subalgebra of S_1.*

We write $S_{\mathsf{d}} = \{x \mid \mathsf{d}(x) = x\}$ and $\mathsf{d}(S)$ for the image of S under d.

Lemma 4 ([2]). *Let (S, B, d) be a test dioid with domain. Then $B = S_{\mathsf{d}} = \mathsf{d}(S)$.*

Next we turn to the second type of axiomatisation.

Definition 3 ([2])**.** *A* domain semiring *is a semiring S with a map $\mathsf{d} : S \to S$ such that, for all $x, y \in S$ and with $\leq$ defined as for dioids,*

$$x \leq \mathsf{d}(x)x, \tag{d1}$$
$$\mathsf{d}(x\mathsf{d}(y)) = \mathsf{d}(xy), \tag{d2}$$
$$\mathsf{d}(x) \leq 1, \tag{d3}$$
$$\mathsf{d}(0) = 0, \tag{d4}$$
$$\mathsf{d}(x + y) = \mathsf{d}(x) + \mathsf{d}(y). \tag{d5}$$

Every domain semiring is a dioid: $\mathsf{d}(1) = \mathsf{d}(1)1 = 1 + \mathsf{d}(1)1 = 1 + \mathsf{d}(1) = 1$, where the second identity follows from (d1) and the last one from (d3), therefore $1+1 = 1+\mathsf{d}(1) = 1$ and finally $x + x = x(1+1) = x$. It follows that $\leq$ is a partial order and that axiom (d1) can be strengthened to $\mathsf{d}(x)x = x$.

Once again it is straightforward to check that binary relations and sets of paths form domain semirings.

In a domain semiring S, d induces the domain algebra: $\mathsf{d} \circ \mathsf{d} = \mathsf{d}$ and therefore $S_\mathsf{d} = \mathsf{d}(S)$. Moreover, $(S_\mathsf{d}, +, \cdot, 0, 1)$ forms a subsemiring of S, which is a bounded distributive lattice with $+$ as binary sup, $\cdot$ as binary inf, least element 0 and greatest element 1 [2], but not necessarily a boolean algebra.

Example 1 ([2])**.** The distributive lattice $0 < a < 1$ is a dioid with meet as multiplication, and a domain semiring with $\mathsf{d} = id$ and therefore $S_\mathsf{d} = S$. ☐

Proposition 2 ([2])**.** *The domain algebra of a domain semiring S contains the largest boolean subalgebra of S bounded by* 0 *and* 1*.*

Axiom (d5) implies that d is order preserving: $x \leq y \Rightarrow \mathsf{d}(x) \leq \mathsf{d}(y)$. In addition, $\mathsf{d}(px) = p\mathsf{d}(x)$ for all $p \in S_\mathsf{d}$, $\mathsf{d}(1) = 1$, and $\mathsf{d}(\top) = 1$ if S is bounded. More importantly, (lla) can now be derived for all $p \in S_\mathsf{d}$ (it need not hold for $p \in S_1$) [2]; it becomes an adjunction when S is bounded.

Lemma 5. *In any bounded domain semiring S,* (d-adj) *holds for all $p \in S_\mathsf{d}$.*

Proof. $\mathsf{d}(x) \leq p$ implies $x = \mathsf{d}(x)x \leq px \leq p\top$ and $\mathsf{d}(x) \leq \mathsf{d}(p\top) = p\mathsf{d}(\top) = p1 = p$ follows from $x \leq p\top$. ☐

As mentioned in the introduction, an antidomain operation is needed to make the bounded distributive lattice S_d boolean.

Definition 4 ([2])**.** *An* antidomain semiring *is a semiring S with a an operation $\mathsf{a} : S \to S$ such that, for all $x, y \in S$,*

$$\mathsf{a}(x)x = 0, \qquad \mathsf{a}(x) + \mathsf{a}(\mathsf{a}(x)) = 1, \qquad \mathsf{a}(xy) \leq \mathsf{a}(x\mathsf{a}(\mathsf{a}(y))).$$

Antidomain models boolean complementation in the domain algebra; the domain operation can be defined as $\mathsf{d} = \mathsf{a} \circ \mathsf{a}$ in any antidomain semiring S. The second and third antidomain axioms then simplify to $\mathsf{a}(x)+\mathsf{d}(x) = 1$ and $\mathsf{a}(xy) \leq \mathsf{a}(x\mathsf{d}(y))$. The domain algebra S_d of S is the maximal boolean subalgebra of S_1, as in Proposition 1. This leads to the following result.

Lemma 6 ([2]). *Let* (S, a) *be an antidomain semiring. Then* $(S, S_{\mathsf{d}}, \mathsf{d})$ *is a test dioid with domain.*

If the domain algebra S_{d} of a domain semiring S happens to be a boolean algebra, it must be the maximal boolean subalgebra of S_1 by Proposition 2, so that S is again a test dioid with $B = S_{\mathsf{d}}$. Antidomain is then definable.

Lemma 7. *Every domain semiring with boolean domain algebra is an antidomain semiring.*

Proof. With $\mathsf{a} = (\_)' \circ \mathsf{d}$, the first antidomain axiom follows immediately from Lemma 2(1); the remaining two axioms hold trivially. □

Example 2. In the dioid $0 < a < 1$ from Example 1, $\mathsf{d} : 0 \mapsto 0, a \mapsto 1, 1 \mapsto 1$ defines another domain semiring with $S_{\mathsf{d}} = \{0, 1\} = B$. So $S_{\mathsf{d}} \subset S_1$ is the maximal boolean subalgebra in S_1. In addition, $\mathsf{a} : 0 \mapsto 1, a \mapsto 0, 1 \mapsto 0$ defines the corresponding antidomain semiring. Finally, this dioid is a test dioid by Lemma 6 and in fact a test dioid with domain in which $B = S_{\mathsf{d}} \subset S_1$. □

As powerset algebras, relation and path domain semirings have of course boolean domain algebras with complement $x' = 1 \cap \overline{x}$, where $\overline{x}$ denotes complementation on the entire powerset algebra. Both are therefore antidomain semirings, with the operations shown in the introduction.

We finish this section with an aside on fullness:[1] While every test dioid with domain and every antidomain semiring is full whenever $S_{\mathsf{d}} = S_1$ by Proposition 1 and Lemma 6, in domain semirings, $S_{\mathsf{d}} = S_1$ need not imply that S_{d} is boolean (Example 1) and vice versa (Example 2). A domain semiring S is therefore full precisely when S_{d} is boolean and equal to S_1.

3 Coincidence Result

The results of Section 2 suggest that the two types of domain semiring coincide when the underlying dioid is full. We now spell out this coincidence.

Proposition 3. *Let* (S, B, d) *be a test dioid with domain. Then* (S, d) *is a domain semiring with* $S_{\mathsf{d}} = B$ *and an antidomain semiring with* $\mathsf{a} = (\_)' \circ \mathsf{d}$.

Proof. The domain semiring axioms are derivable in test dioids with domain [1]; the antidomain axioms follow by Lemma 7. Moreover, B is the maximal boolean subalgebra of S_1 by Proposition 1, and thus equal to S_{d} by Proposition 2 (alternatively Lemma 4). □

We know from Lemma 6 that every antidomain semiring is a test dioid with domain. Hence, by Proposition 3, antidomain semirings and test dioids with domain are interdefinable (see also [2]). For the other converse of Proposition 3 we consider full domain semirings S where $S_{\mathsf{d}} = S_1$ is a boolean algebra by Proposition 2. These are test dioids, hence (lla) can be used to define domain.

[1] We are grateful to a reviewer for reminding us of this fact.

Corollary 1. *Let S be a full dioid with map $\mathsf{d} : S \to S$. Then* (lla) *holds for all $x \in S$ and $p \in S_1$ if and only if the predomain axioms*

$$x \leq \mathsf{d}(x)x \qquad \textit{and} \qquad \mathsf{d}(px) \leq p$$

from Definition 2 hold for all $x \in S$ and $p \in S_1$.

Proof. As S is a test dioid with $B = S_1$, Lemma 3(1) applies. □

Lemma 8. *Let S be a full dioid with map $\mathsf{d} : S \to S$ that satisfies* (lla) *for all $x \in S$ and $p \in S_1$. Then (S, S_1, d) is a test dioid with predomain and $S_{\mathsf{d}} = S_1$.*

Proof. S is a test dioid with predomain by Corollary 1. $S_{\mathsf{d}} \subseteq S_1$ because $\mathsf{d}(x) \leq 1$ in any test dioid with predomain [1]. $S_1 \subseteq S_{\mathsf{d}}$ because $p \leq 1$ implies $p = \mathsf{d}(p)p \leq \mathsf{d}(p)$ and $\mathsf{d}(p) \leq p$ because $pp = p$, using (lla). □

Proposition 4. *Let (S, d) be a full domain semiring. Then $(S, S_{\mathsf{d}}, \mathsf{d})$ is a test dioid with domain.*

Proof. If (S, d) is a full domain semiring, then (lla) is derivable and locality holds. Then $(S, S_{\mathsf{d}}, \mathsf{d})$ is a test dioid with predomain by Lemma 8 and therefore a test dioid with domain because of locality. □

Our coincidence result, through which the two types of domain semirings are united, then follows easily from Propositions 3 and 4.

Theorem 1. *A full test dioid is a test dioid with domain if and only if it is a domain semiring.*

On full dioids, domain can therefore be axiomatised either equationally by the domain semiring axioms or those of test dioids with domain, or alternatively by (lla) and locality. The domain algebras of relation and paths domain semirings, in particular, are full.

In any dioid, hence in particular any domain semiring, fullness can be enforced, for instance, by requiring that every $p \in S_1$ be *complemented* within S_1, that is, there exists an element $q \in S_1$ such that $p + q = 1$ and $qp = 0$. It then follows that S_1 is a boolean algebra [2].

Alternatively, in any test dioid with domain or any antidomain semiring, $S_{\mathsf{d}} = S_1$ whenever $x \leq 1 \Rightarrow \mathsf{d}(x) = x$, for all $x \in S$. Yet Example 2 shows that this implication does not suffice to make S_{d} boolean in arbitrary domain semirings.

Finally, locality need not hold in full test dioids that satisfy (lla).

Example 3. Consider the full test dioid with $S = \{0, 1, a, \top\}$ in which a and 1 are incomparable with respect to $\leq$, $aa = 0$, multiplication is defined by $a\top = \top a = a$ and $\top\top = \top$, and d maps 0 to 0 and every other element to 1. Then (lla) holds, but $\mathsf{d}(a\mathsf{d}(a)) = \mathsf{d}(a1) = \mathsf{d}(a) = 1 > 0 = \mathsf{d}(0) = \mathsf{d}(aa)$. □

4 Examples

The restriction to full test dioids is natural for concrete powerset algebras, like the relation and path algebras mentioned. It is captured abstractly, for instance, by boolean monoids and quantales.

A *boolean monoid* [1] is a structure $(S, +, \sqcap, \cdot, ^{-}, 0, 1, \top)$ such that $(S, +, \cdot, 0, 1)$ is a semiring and $(S, +, \sqcap, ^{-}, 0, \top)$ a boolean algebra. As all sups, infs and multiplications of subidentities stay below 1, every boolean monoid is a full bounded dioid; boolean complementation on S_1 is given by $p' = 1 \sqcap \overline{p}$ for all $p \in S_1$.

Domain can now be axiomatised as an endofunction, either equationally using the domain semiring or test dioid with domain axioms, or by the adjunction (d-adj) and locality, as in Section 3. Once again, the antidomain operation a is complementation on S_1. Theorem 1 has the following instance.

Corollary 2. *A boolean monoid is a test dioid with domain if and only if it is a domain semiring.*

Quantales capture the presence of arbitrary sups and infs in powerset algebras more faithfully. Formally, a *quantale* $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ and a monoid $(Q, \cdot, 1)$ such that composition preserves all sups in its first and second argument. We write $\bigvee$ for the sup and $\bigwedge$ for the inf operator. We also write $0 = \bigwedge Q$ for the least and $\top = \bigvee Q$ for the greatest element of Q, and $\vee$ and $\wedge$ for binary sups and infs.

A quantale is *boolean* if its complete lattice is a boolean algebra. Every boolean quantale is obviously a boolean monoid, and every finite boolean monoid a boolean quantale. If Q is a boolean quantale, then Q_1 forms even a complete boolean algebra. In boolean quantales, predomain, domain and antidomain operations can therefore be axiomatised like in boolean monoids, and we obtain another instance of Theorem 1, analogous to Corollary 2, simply by replacing "boolean monoid" with "boolean quantale".

As for domain semirings, Q_{d} need neither be full nor boolean in an arbitrary domain quantale: the dioids in Examples 1 and 2 are defined over finite semilattices and hence complete lattices. They are therefore quantales. In this case, the identity $\mathsf{d}(x \wedge 1) = x \wedge 1$ forces $Q_{\mathsf{d}} = Q_1$, because this inequality implies $\mathsf{d}(x) = x$ for all $x \leq 1$, and in fact a domain semiring with a meet operation suffices for the proof.[2] In antidomain quantales, this identity thus implies fullness. Whether or how the fullness could be forced equationally in arbitrary domain semirings or antidomain semirings is left open.

5 Domain Quantales

Some loose ends remain to be tied together in this note as well:

- Does the interaction of domain with arbitrary sups and infs in quantales require additional axioms?

[2] Again we owe this observation to a reviewer.

- Why has domain not been axiomatised explicitly using the adjunction (d-adj), at least for boolean quantales?
- And why has domain in boolean monoids or quantales not been axiomatised explicitly by $\mathsf{d}(x) = 1 \wedge x\top$, as in relation algebra?

This section answers these questions.

First, we consider the domain semiring axioms in arbitrary quantales and argue that additional sup and inf axioms are unnecessary.

Definition 5. *A* domain quantale *is a quantale that is also a domain semiring.*

As every quantale is a bounded dioid, the adjunction (d-adj) holds for every $p \in Q_d$. In addition, domain interacts with sups and infs as follows.

Lemma 9. *In every domain quantale,*

1. $\mathsf{d}(\bigvee X) = \bigvee \mathsf{d}(X)$,
2. $\mathsf{d}(\bigwedge X) \leq \bigwedge \mathsf{d}(X)$,
3. $\mathsf{d}(x)(\bigwedge Y) = \bigwedge \mathsf{d}(x)Y$ *for all* $Y \neq \emptyset$.

Proof.

1. d is a left adjoint by Lemma 5 and therefore sup-preserving. Sups over X are taken in Q; those over $\mathsf{d}(X)$ in Q_{d}.
2. $(\forall x \in X.\ \bigwedge X \leq x) \Rightarrow (\forall x \in X.\ \mathsf{d}(\bigwedge X) \leq \mathsf{d}(x)) \Leftrightarrow \mathsf{d}(\bigwedge X) \leq \bigwedge \mathsf{d}(X)$.
3. Every $y \in Y \neq \emptyset$ satisfies
$$\mathsf{d}\left(\bigwedge \mathsf{d}(x)Y\right) \leq \mathsf{d}(\mathsf{d}(x)y) = \mathsf{d}(x)\mathsf{d}(y) \leq \mathsf{d}(x)$$
and therefore
$$\bigwedge \mathsf{d}(x)Y = \mathsf{d}\left(\bigwedge \mathsf{d}(x)Y\right)\left(\bigwedge \mathsf{d}(x)Y\right) \leq \mathsf{d}(x)\left(\bigwedge \mathsf{d}(x)Y\right) \leq \mathsf{d}(x)\left(\bigwedge Y\right).$$
The converse inequality holds because $x(\bigwedge Y) \leq \bigwedge xY$ in any quantale. □

If $Y = \emptyset$ in part (3) of the lemma, then $\mathsf{d}(x)(\bigwedge Y) = \mathsf{d}(x)\top$ need not be equal to
$$\top = \bigwedge \emptyset = \bigwedge \mathsf{d}(x)Y.$$
In the quantale of binary relations over the set $\{a, b\}$, for instance, $R = \{(a, a)\}$, satisfies $\mathsf{d}(R) = R$ and
$$\mathsf{d}(R)\top = \{(a, a)\} \cdot \{(a, a), (a, b), (b, a), (b, b)\} = \{(a, a), (a, b)\} \subset \top.$$

Moreover, part (1) of the lemma implies that the domain algebra Q_{d} is a *complete* distributive lattice: $\mathsf{d}(\bigvee \mathsf{d}(X)) = \bigvee \mathsf{d}(X)$ holds for all $X \subseteq Q$, so that any sup of domain elements is again a domain element. Yet the sups and infs in Q_d need not coincide with those in Q.

Second, the adjunction $\mathsf{d}(x) \leq p \Leftrightarrow x \leq p\top$ holds for all $p \in Q_1$ in a boolean quantale Q. General properties of adjunctions then imply that, for all $x \in Q$,

$$\mathsf{d}(x) = \bigwedge \{p \in Q_1 \mid x \leq p\top\}.$$

Lemma 8, in turn, guarantees that this identity defines predomain explicitly on boolean quantales. Yet Example 3 rules out that it defines domain: the full test dioid from this example is, in fact, a boolean quantale; it satisfies (lla) and thus (d-adj), but violates the locality axiom of domain quantales.

Finally, we give two reasons why the relation-algebraic identity

$$\mathsf{d}(x) = 1 \wedge x\top$$

cannot replace the domain axioms in boolean monoids and quantales.

It is too weak: In the boolean quantale $\{\bot, 1, a, \top\}$ with 1 and a incomparable and multiplication defined by $\top\top = \top$ and $aa = a\top = \top a = a$, it holds that $d(a) = \bot$ (when defined by $\mathsf{d}(x) = 1 \wedge x\top$), yet $\mathsf{d}(a)a = \bot a = \bot < a$. Therefore $\mathsf{d}(x)x = x$ is not derivable from $\mathsf{d}(x) = 1 \wedge x\top$ even in boolean quantales.

It is too restrictive: although $\mathsf{d}(x) = 1 \wedge x\top$ obviously holds in the quantale of binary relations, it fails, for instance, in the quantale formed by the sets of (finite) paths over a digraph $\sigma, \tau : E \to V$ mentioned in the introduction. Recall that the domain elements of a set P of paths are a subset of V given by the sources of the these paths. It is then obvious that $V \cap P\top = \emptyset$ unless P contains a path of length one and $\mathsf{d}(P) = \emptyset \Leftrightarrow P = \emptyset$, so that $\mathsf{d}(P) = V \cap P\top$ fails for any P in which all paths have length greater than 1.

This type of argument applies to all powerset quantales in which the composition of underlying objects (here: paths) is generally length-increasing and the quantalic unit and domain elements are formed by fixed-length objects.

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