

# Quantification of Time-Domain Truncation Errors for the Reinitialization of Fractional Integrators

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## Abstract

In recent years, fractional differential equations have received a significant increase in their use for modeling a wide range of engineering applications. In such cases, they are mostly employed to represent non-standard dynamics that involve long-term memory effects or to represent the dynamics of system models that are identified from measured frequency response data in which magnitude and phase variations are observed that could be captured either by low-order fractional models or high-order rational ones. Fractional models arise also when synthesizing CRONE (Commande Robuste d'Ordre Non Entier) and/or fractional PID controllers for rational or fractional systems. In all these applications, it is frequently required to transform the frequency domain representation into time domain. When doing so, it is necessary to carefully address the issue of the initialization of the pseudo state variables of the time domain system model. This issue is discussed in this article for the reinitialization of fractional integrators which arises among others when solving state estimation tasks for continuous-time systems with discrete-time measurements. To quantify the arising time-domain truncation errors due to integrator resets, a novel interval observer-based approach is presented and, finally, visualized for a simplified battery model.

**Keywords:** fractional differential equations (FDEs), observer design, uncertain cooperative dynamics, temporal truncation errors, state estimation

## 1 Introduction

Fractional differential equations (FDEs) are powerful modeling tools in many engineering applications in which non-standard dynamics, characterized by infinite horizon states, can be observed [21, 23, 37, 40]. Examples for such applications are modeling the charging and discharging dynamics of batteries [11], the identification of dynamic system models by means of impedance spectroscopy [2] if amplitude

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and phase variations do not correspond to integer multiples of  $\pm 20$  dB and  $\pm \frac{\pi}{2}$  per frequency decade, respectively, modeling of multi-robot systems [9], control design for flexible manipulators [4], and generally for the representation of dynamic systems with long-term memory effects. Moreover, advanced models of visco-elastic damping [13] can be described with the help of FDEs. In this domain, FDEs can be used to describe phenomena with non-integer time derivatives, which represent phenomena lying “between” Hooke’s law (with a proportionality of forces to the displacement) and Newton’s law (being proportional to the first time derivative of the displacement) [16]. Similar effects also exist in the domain of heat and mass transport, where non-standard dynamics may be related to phenomena “between” diffusion and wave propagation [36].

Previous work for an interval-based state estimation of such systems has accounted for a cooperativity preserving or cooperativity enforcing design of observers [3, 11]. These interval observers exploit specific monotonicity properties of positive dynamic systems and provide lower and upper bounding trajectories for all pseudo state variables<sup>1</sup> as soon as suitable initialization functions for the fractional dynamic system model are specified.

Moreover, FDEs arise naturally if the CRONE design methodology [15, 22] for efficient shaping of frequency response characteristics of linear control systems and/or if fractional PID controllers [20, 24] are employed. FDEs then arise independently of whether the plant to be controlled is represented by a rational or fractional system model, and if the resulting closed-loop system dynamics are subsequently represented in the time domain.

In contrast to the case of integer-order models, the time responses of fractional systems significantly depend on the initialization of the pseudo state. This is shown exemplarily in this paper with the help of the Grünwald-Letnikov definition of fractional derivatives [23, 32, 34, 39] to illustrate further that the Caputo initialization corresponds to the special case that an FDE model is initialized with an initial condition that also represents a perfectly constant, infinitely long history of the pseudo states in the past. Although this may hold (at least in good approximation) for the initialization of a dynamic system which is fully in rest, this is obviously not true when resetting integrators after a finitely long time interval.

Apart from the discussions above, interval-valued iteration procedures have been developed in [26–29] for a verified simulation of FDE models. These iteration procedures, based on Mittag-Leffler function parameterizations of the pseudo-state enclosures, are not *a priori* restricted to cooperative models but are applicable also to nonlinear systems with interval parameters. So far, this procedure assumes that — for the initialization — a fractional derivative definition according to Caputo is used. This verified simulation, however, allows for resetting the integration after a finite time span by applying (to our knowledge, for the first time in a verified simulation of FDEs) an error quantification originally published in the book [23]

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<sup>1</sup>The notion *pseudo state* is used throughout this manuscript to indicate the existence of the infinite memory problem of FDEs in contrast to the classical notion of *state variables* that only need to be specified at distinct points in time to unambiguously solve initial or boundary value problems for classical integer-order dynamic system models.

by Podlubny.

In this paper, we aim at improving this error quantification scheme by a novel interval observer-based approach that allows for estimating guaranteed interval bounds for time-domain truncation errors in scenarios in which fractional integrators need to be reset. Such cases occur when state estimation for continuous-time FDE models with discrete-time measurements is considered. So far, the state-of-the-art in the evaluation of observer-based pseudo-state estimation procedures for continuous-time FDE models supposes that measurements are also available in a continuous-time form or at least at each sampling period [3]. If measurements are available only at discrete time instants, continuous-time pseudo-state predictions need to be performed between the sampling instants of the measurements.

Then, the measured pseudo-state information (described by intervals to represent bounded measurement errors) can be intersected with the predicted state information to enhance the knowledge of the actual system dynamics. However, this intersection demands reinitializing the integration of the fractional model. As already mentioned above, a similar requirement is discussed in [26], where temporal sub-slices were considered to reduce the overestimation of interval-based simulation approaches. Moreover, such integrator resets also help to limit memory demands that would grow continuously with increasing integration times if solutions to FDEs were approximated with the help of series expansion techniques based on the Grünwald-Letnikov definition of fractional derivatives.

Due to the infinite horizon memory property of fractional systems, the reinitialization of time-domain simulations requires a rigorous consideration of the arising truncation errors. Although guaranteed outer bounds for these errors were derived by Podlubny in [23], they may be unnecessarily conservative due to an assumption of the time invariance of these bounds for all future points after the integrator reset. We aim at using Podlubny's initial bounds as a basis for a novel error refinement strategy between discrete reinitialization points in an observer-based setting.

After an introduction into the infinite memory problem of FDE models in Sec. 2, an approach that accounts for handling non-constant pseudo-state initializations from a bounded past time window in terms of uncertain initial conditions at a single point is derived. This approach is based on a conservative interval-valued correction of the FDE model. It forms the basis for implementing an observer-based quantification of truncation errors for simulations of FDEs in which a periodic reinitialization is employed in Sec. 3. This approach is then applied in Sec. 4 to an academic benchmark example as well as to the interval contractor-based state estimation of a continuous-time battery model [11] with discrete-time measurements, before the paper is concluded with an outlook on future work in Sec. 5.

## 2 Influence of the Initialization of FDE Models

To visualize the influence of the initialization of the pseudo state of FDE models on their future behavior, consider the representation of the solution of a commensurate

autonomous FDE

$$\mathbf{x}^{(\nu)}(t) = \mathbf{f}(\mathbf{x}(t)) \quad , \quad \mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad (1)$$

in terms of the infinite series

$$\mathbf{x}(t_{k+1}) = \nu \mathbf{I} \cdot \mathbf{x}(t_k) + \Delta T_k^\nu \cdot \mathbf{f}(\mathbf{x}(t_k)) - \sum_{i=2}^{\infty} c_i \cdot \mathbf{x}(t_{k+1-i}) \quad (2)$$

with the sufficiently short step size  $\Delta T_k = t_{k+1} - t_k$ . This series expansion results from the Grünwald-Letnikov definition<sup>2</sup> of a non-integer derivative of order  $0 < \nu \leq 1$  with the coefficients

$$c_i = (-1)^i \cdot \binom{\nu}{i} = (-1)^i \cdot \frac{\Gamma(\nu+1)}{\Gamma(i+1) \cdot \Gamma(\nu-i+1)}, \quad (3)$$

in which the term  $\binom{\nu}{i}$  is the Newton binomial coefficient with the gamma function

$$\Gamma(\nu) = \int_0^{\infty} \xi^{\nu-1} e^{-\xi} d\xi \quad (4)$$

as a generalization of the factorial to the case of non-integer arguments. To avoid excessive numerical errors when evaluating the coefficients  $c_i$ , they are typically computed in a recursive manner according to [39]

$$c_i = c_{i-1} \cdot \left(1 - \frac{1+\nu}{i}\right) \quad \text{with} \quad c_0 = 1 \quad \text{and} \quad i \in \mathbb{N}. \quad (5)$$

As it can be seen already in Eq. (2), future pseudo states  $\mathbf{x}(t_{k+1})$  do not only depend on the current state  $\mathbf{x}(t_k)$  as a kind of initialization (as it would be the case for integer-order system models), but they also depend on an infinite horizon of pseudo states from previous points of time  $t < t_k$ . Note, stability properties of this series expansion and properties of its convergence toward the true solution of a fractional system model have been analyzed in detail in [32]. In fact, the Grünwald-Letnikov discretization can be interpreted as a generalization of the well-known Euler discretization scheme for integer-order models so that the true state evolution can be approximated accurately for sufficiently small values of  $\Delta T_k$ . For methods that allow a rigorous quantification of time discretization errors, the reader is referred to [27–29], where an exponential state enclosure technique is generalized to fractional models by using an iteration scheme exploiting an interval extension of Mittag-Leffler functions [10, 19], or to [1, 18, 26] where series expansion approaches and Picard iteration schemes were generalized to the fractional case.

For linear FDEs of Caputo type, typically only initial conditions  $\mathbf{x}(t_k)$  are specified explicitly at a point of time  $t_0$  that is set to  $t_0 = 0$  without loss of generality in the remainder of this paper. As shown in the following example, this specification

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<sup>2</sup>This representation corresponds to the one discussed in [26], except for the correction of a small typo in the quoted previous work of the first author.



implicitly imposes that the pseudo state of the system showed an exactly constant behavior for an infinitely long time window in the past.

To perform this investigation, consider the FDE

$$x^{(0.5)}(t) = -x(t) + u(t) \quad (6)$$

with the pseudo state initialization

$$x(t) = x_0 \quad \text{for } t \leq 0 \quad (7)$$

and the constant external control input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ u_0 & \text{for } t \geq 0. \end{cases} \quad (8)$$

Due to the fact that fractional derivatives of constant values in the Caputo sense are zero, the linear change of variables

$$x(t) = y(t) + u_0 \quad \text{with} \quad x^{(0.5)}(t) = y^{(0.5)}(t) \quad (9)$$

yields the equivalent FDE

$$y^{(0.5)}(t) = -y(t) \quad (10)$$

with

$$y(t) = x_0 - u_0 \quad \text{for } t \leq 0 \quad (11)$$

for which the exact solution is given by

$$y(t) = (x_0 - u_0) \cdot E_{0.5,1}(-t^{0.5}) \quad \text{for } t \geq 0 \quad (12)$$

being equivalent to

$$x(t) = x_{\text{ML}}(t) = (x_0 - u_0) \cdot E_{0.5,1}(-t^{0.5}) + u_0 \quad \text{for } t \geq 0. \quad (13)$$

In (12) and (13),  $E_{\nu,1}(\cdot)$  is the Mittag-Leffler function with the fractional derivative order  $\nu = 0.5$  as parameter.

In Fig. 1, different approximations of the solution to the FDE model (6)–(8) are computed by using the Grünwald-Letnikov approximation with the constant discretization step size  $\Delta T_k = 0.01$  and approximations of the infinitely long constant initialization of the pseudo state in (7). These latter approximations are defined by

$$x(t) = x_0 \quad \text{for } t \in [-\Delta T_k \cdot 10^N; 0] \quad (14)$$

with  $N \in \{1, 2, \dots, 8\}$ , where the corresponding approximations to the true solution  $x(t)$  are denoted by  $x_N(t)$ . As described above, the exact solution  $x_{\text{ML}}(t)$  corresponds to a solution representation in terms of the Mittag-Leffler function according to (13), which has been evaluated in Fig. 1 by the MATLAB implementation by R. Garrappa in [8].

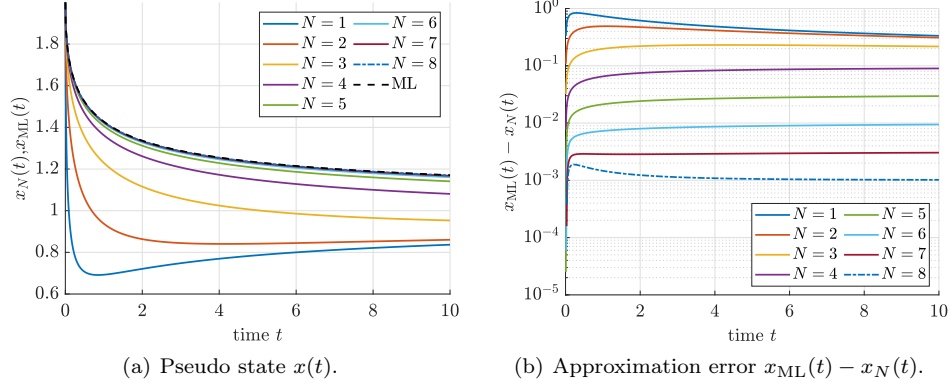


Figure 1: Influence of the memory length of the pseudo state initialization.

It can be seen that insufficiently long memory lengths in the state initialization lead to large deviations between the Grünwald-Letnikov approximation and the true solution. These deviations due to an insufficiently long initialization window are much larger than the influence of the finitely long discretization step size  $\Delta T_k$ .

In the following section, time domain truncation errors resulting from resetting a numerical integration of an FDE model after a finitely long time span are quantified in a rigorous manner under the assumption that the system behavior for  $t \leq 0$  is known in advance. For the sake of simplicity, we rely on the temporally constant initialization according to (7) in the following.

**Remark 1.** Due to the fact that the numerical solution of an FDE with non-constant initialization functions is influenced by the change of coordinates in (9), this linear shift of the coordinate system will form a potential basis for identifying the history of the pseudo states in future work.

### 3 Interval Observer Technique for the Identification of Improved Bounds of Time-Domain Truncation Errors

#### 3.1 Constant Bounds for Time-Domain Truncation Errors

So far, we have assumed that initial conditions for the pseudo state of the FDE system model (1) are specified at the instant  $t = 0$  with a temporally constant past for all times  $t < 0$  in accordance to the Caputo definition of fractional derivatives. To allow for a notation denoting the influence of the point of time at which the derivative operator  ${}_0\mathcal{D}_t^{(\nu)}$  is initialized, the notation of Eq. (1) is changed to

$${}_0\mathcal{D}_t^{(\nu)} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (15)$$

in the following, where the left subscript of  ${}_0\mathcal{D}_t^{(\nu)}$  specifies the initialization point of time (in the case above,  $t = 0$ ).

According to the work of Podlubny [23, Eq. (7.5)], guaranteed bounds for the influence of shifting this initialization point from the time instant  $t_k$  to another point  $t_k + T$ ,  $T > 0$ , can be computed component-wise according to

$$\left| {}_{t_k}\mathcal{D}_t^{(\nu)}\mathbf{x}(t) - {}_{t_k+T}\mathcal{D}_t^{(\nu)}\mathbf{x}(t) \right| \leq \frac{\mathcal{X}T^{-\nu}}{|\Gamma(1-\nu)|} =: \boldsymbol{\mu} \quad (16)$$

with

$$\mathcal{X} = [\mathcal{X}_1 \quad \dots \quad \mathcal{X}_n]^T \quad (17)$$

comprising the suprema

$$\mathcal{X}_i = \sup_{t \in [t_k; t_k+T]} |x_i(t)|, \quad i \in \{1, \dots, n\} \quad (18)$$

of the reachable pseudo states over the time interval  $t \in [t_k; t_k + T]$  for each element of the vector  $\mathbf{x}(t)$ .

As shown in [28], these error bounds can be employed to reset interval-based verified solution procedures for FDE models after a certain time span and to use the solution enclosures determined after the reset to reduce overestimation arising due to pessimism that is introduced by long integration time intervals. For that purpose, the right-hand side of the system model (15) is inflated by the error bound interval  $[-\boldsymbol{\mu}; \boldsymbol{\mu}]$  to obtain the uncertain system model

$$\tilde{\mathbf{f}}(\mathbf{x}(t)) \in \mathbf{f}(\mathbf{x}(t)) + [-\boldsymbol{\mu}; \boldsymbol{\mu}]. \quad (19)$$

Using this modification, the simulation is continued after the point  $t = t_k + T$  for the differential inclusion model defined by the expression  $\tilde{\mathbf{f}}(\mathbf{x}(t))$  and the pseudo state values  $\mathbf{x}(t_k)$  as initial condition, while the entire past for  $t < t_k$  is no longer required for a further system simulation.

Under the assumption of cooperativity of the state equations, see [3, 5, 25, 33] for further details, independent lower and upper bounding trajectories can be extracted from the modified system model (19) so that set-based integration routines such as the one based on interval extensions of the Mittag-Leffler function from [27–29] can be avoided when solving the corresponding initial value problem for the differential inclusion problem (19) after the inflation of the right-hand side  $\mathbf{f}(\mathbf{x}(t))$  of the original system.

**Remark 2.** To limit the pessimism introduced by the additive error bounds in (19), the following two aspects should be accounted for:

- Define the pseudo state  $\mathbf{x}(t)$  in such a way that  $\mathbf{x} = \mathbf{0}$  corresponds to the equilibrium of an asymptotically stable FDE. If  $\mathbf{x} = \mathbf{0}$  is not the corresponding steady state after a first-principle modeling, perform a shift of coordinates as inspired by Eq. (11) so that the absolute values for the bounds  $\mathcal{X}$  do not increase for sufficiently large values of  $t_k + T$ .

- Set the initial point  $t_k$  in (16) to  $t_k = 0$ . Together with the first aspect in this remark, this allows for a computation of values for the error bounds  $\mu$  that decrease after sufficiently long integration times and thus lead to less conservative system models than always recomputing the bounds  $\mu$  with respect to a previous reset point  $t_k > 0$ .

Even though these two aspects can be accounted for in many practical situations, the bounds  $\mu$  given in (16) remain conservative due to the fact that they are temporally constant. This property does not explicitly account for the observation that shifting the initialization point of the fractional derivative operator becomes less important for increasing integration times<sup>3</sup>. Therefore, an observer-based refinement of the bounds  $\mu$  — to our knowledge not yet considered in any other publication — is presented in the following subsection. A similar approach, however, can be found in [35], where the authors propose an observer to initialize fractional system models consistently.

### 3.2 Observer-Based Enhancement of the Bounds for Time-Domain Truncation Errors

For the observer-based enhancement of the time-domain truncation error bounds when resetting fractional integrators, we restrict ourselves to the case of cooperative system models in this paper.

As a generator for *virtual measurements* of a cooperative dynamic system model, we compute pseudo state enclosures

$$\mathbf{x}(t) \in [\mathbf{v}(t) ; \mathbf{w}(t)] \quad (20)$$

for the FDE model (15) with the temporally constant initialization

$$\mathbf{x}(t) \in [\mathbf{x}_0] , \quad \dot{\mathbf{x}}(t) = \mathbf{0} \quad \text{for } t < 0. \quad (21)$$

This setting corresponds to uncertain initial conditions in the sense of Caputo, while the influence of temporally varying initializations is taken into account as soon as the first integrator reset has been performed.

Cooperativity of the system model (15) is guaranteed as a sufficient condition if all off-diagonal elements of the Jacobian of the right-hand side of the system model with respect to the pseudo state vector  $\mathbf{x}(t)$  ( $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ ) satisfy the inequalities

$$\frac{\partial f_i(\mathbf{x})}{\partial x_j} \geq 0. \quad (22)$$

Then, all reachable pseudo states can be enclosed by the lower and upper bounding systems

$$\begin{aligned} {}_0\mathcal{D}_t^{(\nu)} \mathbf{v}(t) &= \mathbf{f}_v(\mathbf{v}(t)) , \quad \mathbf{v}(t \leq 0) = \underline{\mathbf{x}}_0 \quad \text{and} \\ {}_0\mathcal{D}_t^{(\nu)} \mathbf{w}(t) &= \mathbf{f}_w(\mathbf{w}(t)) , \quad \mathbf{w}(t \leq 0) = \bar{\mathbf{x}}_0, \end{aligned} \quad (23)$$

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<sup>3</sup>This observation is denoted as *short memory principle* in [23].

respectively, where the inequalities

$$v_j(t) \leq x_j(t) \leq w_j(t) \quad (24)$$

hold for all  $j \in \{1, \dots, n\}$ .

If the integration of a cooperative FDE model is reinitialized at a point  $t = T > 0$ , an observer-based approach

$${}_T\mathcal{D}_t^{(\nu)} \mathbf{z}(t) = \begin{bmatrix} \mathbf{f}_v(\tilde{\mathbf{v}}(t)) + \boldsymbol{\mu}_v(t) \\ \mathbf{f}_w(\tilde{\mathbf{w}}(t)) + \boldsymbol{\mu}_w(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \mathbf{H} \cdot \begin{bmatrix} \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \\ \mathbf{w}(t) - \tilde{\mathbf{w}}(t) \end{bmatrix} \quad (25)$$

with the augmented state vector

$$\mathbf{z}(t) = \begin{bmatrix} \tilde{\mathbf{v}}(t) \\ \tilde{\mathbf{w}}(t) \\ \boldsymbol{\mu}_v(t) \\ \boldsymbol{\mu}_w(t) \end{bmatrix} \in \mathbb{R}^{4n} \quad (26)$$

can be used to enhance the pseudo state enclosures and the truncation error bounds in comparison with the ones obtained by the integrator resetting approach according to the previous subsection that only employs temporally constant truncation error bounds.

The observer (25) is initialized with the pseudo state vector

$$\mathbf{z}(T) = \begin{bmatrix} \mathbf{v}(T) \\ \mathbf{w}(T) \\ -\boldsymbol{\mu}_T \\ \boldsymbol{\mu}_T \end{bmatrix}, \quad (27)$$

where the truncation error bounds  $[-\boldsymbol{\mu}_T; \boldsymbol{\mu}_T]$  are computed as described in the previous subsection. Due to the inclusion of the truncation error bounds  $\boldsymbol{\mu}_v(t)$  and  $\boldsymbol{\mu}_w(t)$  by means of so-called integrator disturbance models in (25), leading to constant values if  $\mathbf{H} = \mathbf{0}$ , the case of the error quantification according to the previous subsection is included as a special case in this formulation. If  $\mathbf{H} \neq \mathbf{0}$ , the bounds  $\boldsymbol{\mu}_v(t)$  and  $\boldsymbol{\mu}_w(t)$  are enhanced in such a way that the *virtual measurements* and the enhanced bounds for the pseudo state variables approach each other as close as possible. For that purpose, the augmented system model (25) must be a valid interval observer.

To make the augmented system model (25) with the estimated lower and upper bounding trajectories  $\tilde{\mathbf{v}}(t)$  and  $\tilde{\mathbf{w}}(t)$  a valid observer, the gain matrix  $\mathbf{H}$  needs to be chosen so that the error dynamics associated with the bounding trajectories remain asymptotically stable with

$$\|\mathbf{v}(t) - \tilde{\mathbf{v}}(t)\| \rightarrow 0 \quad \text{and} \quad \|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\| \rightarrow 0 \quad \text{for } t \rightarrow \infty \quad (28)$$

and that

$$[\mathbf{v}(t) ; \mathbf{w}(t)] \subseteq [\tilde{\mathbf{v}}(t) ; \tilde{\mathbf{w}}(t)] \quad (29)$$

is ensured for all  $t \geq T$ .

For corresponding stability criteria for linear fractional differential equations, see the eigenvalue domains summarized in [11, 35]. Due to the fact that these domains can be expressed effectively by linear matrix inequality constraints and that nonlinear models can be bounded by quasi-linear system models with polytopic uncertainty representations, stability requirements for the gain matrix  $\mathbf{H}$  cannot only be obtained for linear system models. They can also be obtained from the existing literature for nonlinear ones as it has been shown, for example in [3, 11], for the design of robust state estimation schemes for FDEs.

To ensure the enclosure property (29) and to verify the decoupled nature of the equations in (25) with respect to  $\tilde{\mathbf{v}}(t)$  and  $\tilde{\mathbf{w}}(t)$  as well as  $\boldsymbol{\mu}_v(t)$  and  $\boldsymbol{\mu}_w(t)$ , respectively, the Jacobian of the right-hand side of the augmented model (25) with respect to the pseudo state vector  $\mathbf{z}(t)$  needs to satisfy the sign property presented in the inequalities (22) with now  $4n$  as the dimension of the augmented model.

### 3.3 Periodic Reset of Fractional Integrators and Their Application to Predictor–Corrector State Estimation

The observer approach from the previous subsection is the basis for a predictor–corrector technique for state estimation if measurements

$$\mathbf{y}(t_{m,k}) \in [\mathbf{y}](t_{m,k}) = [\underline{\mathbf{y}}(t_{m,k}) ; \bar{\mathbf{y}}(t_{m,k})] \quad (30)$$

(including interval uncertainty to represent bounded measurement errors with unknown distributions) are available at the time instants  $t_{m,k}$  in the form

$$\mathbf{g}(\mathbf{x}(t_{m,k})) \in [\underline{\mathbf{y}}(t_{m,k}) ; \bar{\mathbf{y}}(t_{m,k})]. \quad (31)$$

Then, the same observer (25) as in the previous subsection is employed with *virtual measurements* obtained from a simulation of the original system dynamics. The *actual state measurements*  $[\mathbf{y}](t_{m,k})$  are then used to tighten the bounds included in the pseudo state initialization  $\mathbf{z}(T)$  at each point  $T = t_{m,k}$ .

This tightening is either obtained by a direct intersection of the measured intervals  $[\mathbf{y}](t_{m,k})$  with the already computed state bounds  $[\tilde{\mathbf{v}}(t_{m,k}) ; \tilde{\mathbf{w}}(t_{m,k})]$  in the case of a direct pseudo state measurement or by applying a suitable contraction scheme [12] (forward–backward contractor or Krawczyk-type contractor) to the relation (31), where the bounds for  $\mathbf{x}(t_{m,k})$  are initialized with the interval  $[\tilde{\mathbf{v}}(t_{m,k}) ; \tilde{\mathbf{w}}(t_{m,k})]$  as in the first case. After this tightening step, the procedure is continued as described in the previous subsection, where the modification of the pseudo state reinitialization is the only modification in comparison with the previous subsection.

This approach allows for directly handling the continuous-time dynamics of the system model between two subsequent discrete time instants at which measured

data are available. In such a way, the sampling times both for the numerical evaluation of the FDE model and the measurements can be decoupled.

**Remark 3.** Cases in which the measurement step size is not an integer multiple of the numerical integration step, or in which the measurement times themselves are uncertain, can be handled with the same procedure as in [31].

**Remark 4.** Future work will aim at removing the precondition of cooperativity of the original as well as the observed system dynamics in (15) and (25). To solve this task, the so-called TNL approach for the parameterization of interval observers as derived in [38] is a promising solution which — due to its direct applicability to descriptor models — can extend the approach presented in this paper also to cases in which only some of the pseudo state variables are described by the explicit FDE models studied in (15) and others are expressed implicitly by using algebraic constraints. This approach is named after three matrices  $\mathbf{T}$ ,  $\mathbf{N}$ , and  $\mathbf{L}$  which are included in the observer as design degrees of freedom instead of using purely the observer gain for defining the observer dynamics.

## 4 Illustrating Example: Observer Approach

In this section, the observer-based identification of bounds for time-domain truncation errors of FDE models is presented for both a nonlinear academic simulation scenario and for a close-to-life quasi-linear model for the charging and discharging dynamics of Lithium-ion batteries.

### 4.1 Nonlinear Academic Benchmark System

#### 4.1.1 Observer-Based Quantification of Time-Domain Truncation Errors

As a first example, consider the uncertain FDE model

$$x^{(0.5)}(t) = -x(t) - p \cdot x^3(t) \quad (32)$$

with the interval-based temporally constant initialization

$$x(t) \in [x_0] = [0.9 ; 1.0] \quad \text{for } t \leq 0 \quad (33)$$

and the uncertain, time-invariant parameter  $p \in [0.1 ; 0.2]$ . This system model is simulated over the time interval  $t \in [0 ; 10]$  with integrator resets at the time instants

$$T \in \{T', 2T', 3T', \dots\}, \quad \text{where } T' = 1. \quad (34)$$

Due to its scalar nature, this system model satisfies the property of cooperativity, so that (without integrator resetting) the true pseudo state enclosure  $x(t) \in [v(t) ; w(t)]$  according to (23) can be determined by means of the craps system models

$${}_0\mathcal{D}_t^{(0.5)}v(t) = -v(t) - \bar{p} \cdot v^3(t), \quad v(t \leq 0) = \underline{x}_0 \quad (35)$$

and

$${}_0\mathcal{D}_t^{(0.5)}w(t) = -w(t) - \underline{p} \cdot w^3(t), \quad w(t \leq 0) = \bar{x}_0. \quad (36)$$

These bounds, computed with the help of the numerical integration routine `fde12` [7], are visualized by the solid lines in Figs. 2 and 3.

To investigate the observer approach given in Eq. (25), the time- and state-independent gain matrix

$$\mathbf{H} = 20 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (37)$$

is chosen which ensures stability of the estimation error dynamics and cooperativity of the augmented pseudo state equations.

Fig. 2(a) presents a comparison between the integrator resetting in combination with piecewise constant bounds  $\mu_T$  for each time slice  $t \in [(i-1)T'; iT']$ ,  $i \in \{1, 2, \dots, 10\}$  (determined according to (16), where  $t_k = 0$  is chosen in each reinitialization point  $T$  defined in (34)), while Fig. 2(b) shows the observer-based enhancement of the pseudo state enclosures due to the temporal adaptation of the truncation error bounds according to Eq. (25) in Sec. 3.2.

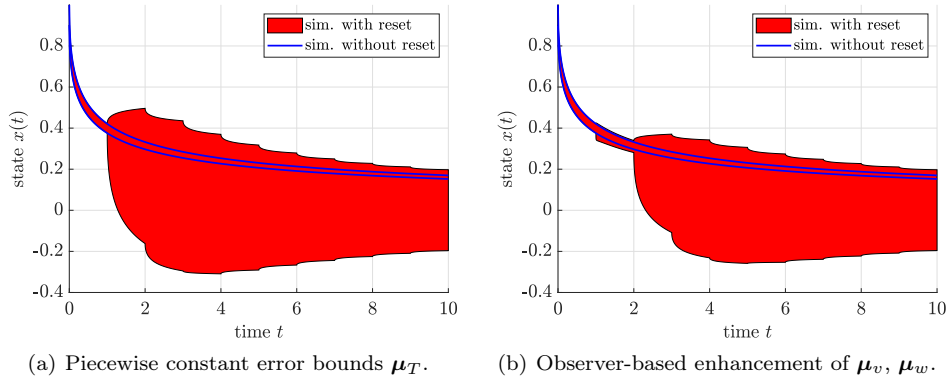


Figure 2: Simulation of the uncertain, nonlinear benchmark system (32).

#### 4.1.2 Predictor–Corrector State Estimation

For the implementation of the predictor–corrector state estimator according to Sec. 3.3, we assume that pseudo state measurements are available at the time instants  $t_{m,k} = T$  listed in (34).

The results in Fig. 3(a) distinguish the following two cases:



- The measured pseudo state information at the time instants  $t_{m,k}$  corresponds to the enclosures from (35) and (36) with

$$y(t_{m,k}) \in [v(t_{m,k}) ; w(t_{m,k})] . \quad (38)$$

This scenario is depicted in Figs. 3(a) and 3(b).

- The measured pseudo state information at the time instants  $t_{m,k}$  is obtained as

$$y(t_{m,k}) \in \hat{x}(t_{m,k}) + 0.001 \cdot [-1 ; 1] \quad (39)$$

with  $\hat{x}(t)$  as the simulation of a nominal parameter model

$$\hat{x}^{(0.5)}(t) = -\hat{x}(t) - 0.15 \cdot \hat{x}^3(t) \quad (40)$$

with  $\hat{x}(t) = 0.95$  for  $t \leq 0$ , see Figs. 3(c) and 3(d).

From a comparison of Figs. 3(a) and 3(b), it is obvious that the observer-based approach in combination with resetting the pseudo state to the measured data leads to significantly tighter enclosures of the solutions than the use of piecewise constant error bounds  $\boldsymbol{\mu}_T$ .

A further tightening of the simulated bounds becomes possible if the uncertainty in the measured data is reduced in Figs. 3(c) and 3(d) in accordance with the second case above. Then, the new solution approach is capable of determining pseudo state enclosures that are partially tighter (in this case for the upper bounding trajectory) than a pure simulation of the uncertain nonlinear model (32) that still serves as the *virtual measurement* generator between the points  $T$  at which the *actual discrete-time measurements* are available. In such a way, the proposed observer-based enhancement of the time-domain truncation error bounds as well as the predictor–corrector state estimation scheme form the basis for the development of set-based parameter identification schemes that are part of our ongoing research activities.

## 4.2 Simplified Fractional Battery Model

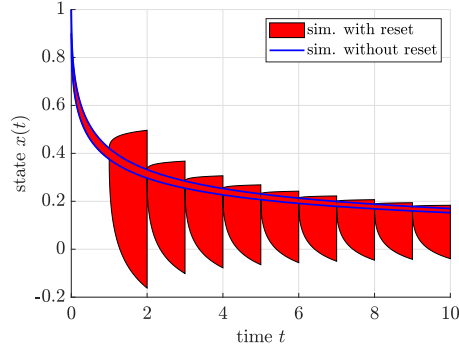
As a final application scenario, consider the fractional-order equivalent circuit model for the charging and discharging dynamics of Lithium-ion batteries depicted in Fig. 4.

Using the parameter values identified experimentally in [30], continuous-time state equations

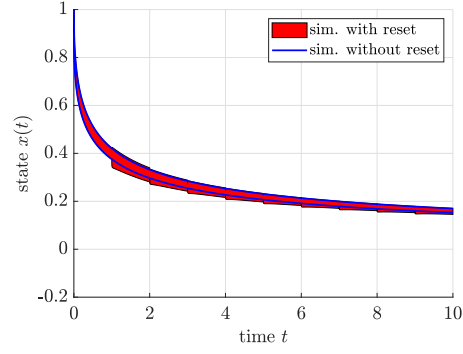
$${}_0\mathcal{D}_t^{(0.5)} \mathbf{x}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot i(t) \quad (41)$$

with the system and input matrices

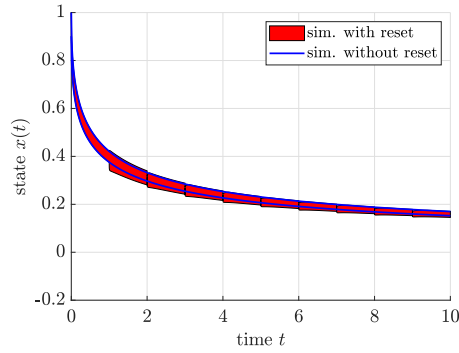
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\eta_1 \cdot \text{sign}(i(t))}{3600C_N} & 0 & 0 \\ 0 & 0 & -\frac{1}{RQ} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -\frac{\eta_0}{3600C_N} \\ \frac{1}{Q} \end{bmatrix} \quad (42)$$



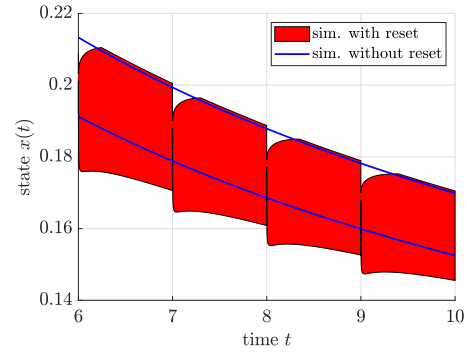
(a) Piecewise constant error bounds  $\mu_T$ , measured pseudo state information according to Eq. (38).



(b) Observer-based enhancement of  $\mu_v, \mu_w$  for the measured pseudo state information according to Eq. (38).



(c) Observer-based enhancement of  $\mu_v, \mu_w$  for the measured pseudo state information according to Eq. (39).



(d) Enlarged view of Fig. 3(c).

Figure 3: Predictor–corrector state estimation for the uncertain, nonlinear benchmark system (32).

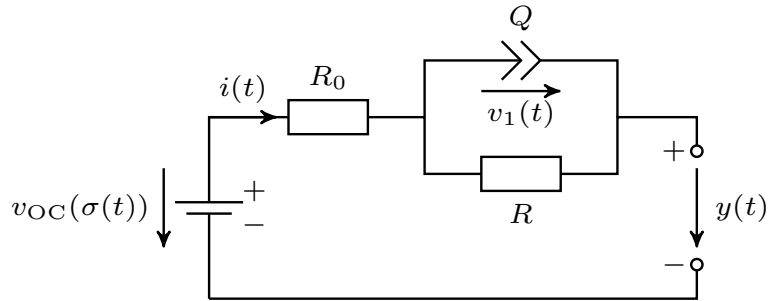


Figure 4: Equivalent circuit representation of a simplified fractional battery model.

as well as the pseudo state vector

$$\mathbf{x}(t) = \begin{bmatrix} \sigma(t) & {}_0\mathcal{D}_t^{(0.5)}\sigma(t) & v_1(t) \end{bmatrix}^T \in \mathbb{R}^3 \quad (43)$$

can be derived by applying Kirchhoff's voltage and current laws. In (43),  $\sigma(t)$  denotes the state of charge of the battery, its fractional derivative is included in the vector  $\mathbf{x}(t)$  to represent long-term memory phenomena, and  $v_1(t)$  is the voltage across a non-integer constant phase element  $Q$  serving as a generalization of capacitors that are typically employed to represent polarization effects and the transportation of charge carriers in Thevenin equivalent circuit models of batteries [6].

For state estimation purposes, the terminal voltage (given in a quasi-linear representation)

$$y(t) = g(\mathbf{x}(t)) = \begin{bmatrix} \sum_{k=0}^4 c_k \sigma^{k-1}(t) & 0 & -1 \end{bmatrix} \cdot \mathbf{x}(t) + \left(-R_0 + d_0 e^{d_1 \sigma(t)}\right) \cdot i(t) \quad (44)$$

is assumed to be available as a measured system output at specific discrete points in time.

To obtain further a cooperative system model, we consider the special case with  $\eta_1 = 0$  and a controlled discharging process of the battery with the terminal current

$$i(t) = -\mathbf{k}^T \cdot \mathbf{x}(t) \quad (45)$$

as the system input in which the controller gain vector  $\mathbf{k}^T$  is determined by pole assignment so that the eigenvalues of the closed-loop system are located at the points  $\lambda \in \{0; -0.0002; -\frac{1}{RQ}\}$ .

This leads to the linear autonomous system model

$${}_0\mathcal{D}_t^{(0.5)}\mathbf{x}(t) = \mathbf{A}_C \cdot \mathbf{x}(t) \quad (46)$$

with

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \quad (47)$$

in which the entries  $a_{22}$ ,  $a_{32}$ , and  $a_{33}$  are converted into interval parameters (displayed after outward rounding of the corresponding bounds) according to

$$\begin{aligned} a_{22} &\in [-0.000220; -0.000179] \\ a_{32} &\in [0.097557; 0.119237] \\ a_{33} &\in [-0.531557; -0.434910] \end{aligned} \quad (48)$$

to account for independent uncertainties of each of these quantities in the intervals of  $\pm 10\%$  around the respective nominal values obtained with the help of the parameters given in [28] and [30].

The initial conditions of the system are assumed to be uncertain according to the Caputo definition

$$\mathbf{x}(t) = [0.5 \quad 0.01 \quad 0.1]^T \cdot [0.9 ; 1.1] , \quad t \leq 0. \quad (49)$$

To set up the observer-based enhancement of the time-domain truncation error bounds according to Eq. (25) with the resetting time instants

$$T \in \{T', 2T', 3T', \dots\} , \quad \text{where } T' = 60 \text{ s}, \quad (50)$$

the gain matrix

$$\mathbf{H} = 5 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (51)$$

is chosen in the following subsections. According to the description in Sec. 3.2, this matrix is specified so that stability and cooperativity of the estimation error dynamics are ensured.

#### 4.2.1 Observer-Based Quantification of Time-Domain Truncation Errors

Fig. 5 summarizes a simulation of the fractional battery model in terms of a direct evaluation of the pseudo state equations (46)–(48) by directly exploiting the property of cooperativity. These results are shown by solid lines, indicating the lower and upper bounds of each state variable, respectively. As for the previous academic example, the numerical solver `fde12` [7] has been used for this purpose.

When resetting the fractional integrator at the time instants (50) and using constant bounds for the time-domain truncation errors, a rapid inflation of the pseudo state enclosures can be observed. This inflation is reduced in Fig. 6, where it has been assumed that the bounds, resulting from the cooperative system simulation, are available as initial conditions for each of the time slices. This resetting of the integrator, together with a reinitialization of the pseudo state for each point in time  $T$  is shown in the left column of Fig. 6 for piecewise constant bounds of the time-domain truncation errors.

Activating the observer-based quantification of the truncation according to Sec. 3.2 additionally, as illustrated in the right column of Fig. 6, leads to significantly tighter outer enclosures that satisfy the relation (29) with certainty.

#### 4.2.2 Predictor–Corrector State Estimation

In practical situations, the resetting of the fractional integrator is often combined with a pseudo state estimation approach as presented in Sec. 3.3. To visualize the applicability of this technique for the model of a controlled Lithium-ion battery, it

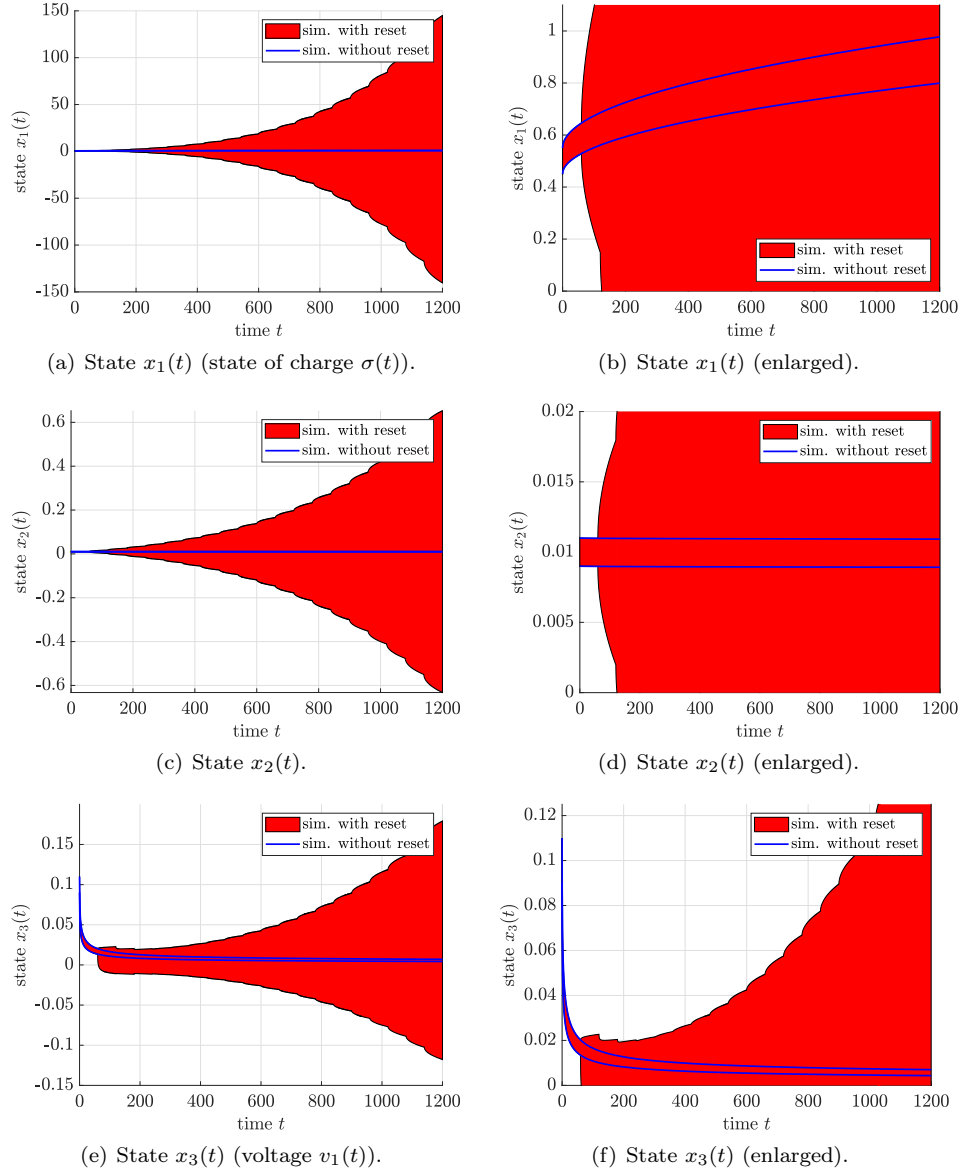


Figure 5: Simulation of the simplified battery model with piecewise constant error bounds  $\mu_T$ ; integrator reset at the points  $T$  defined in (50).

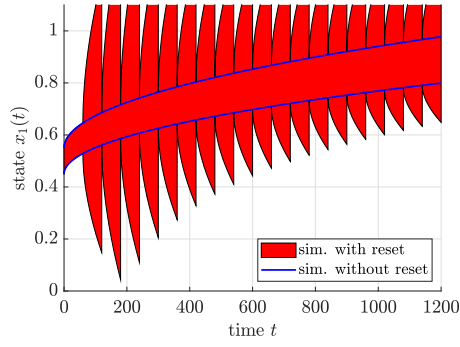
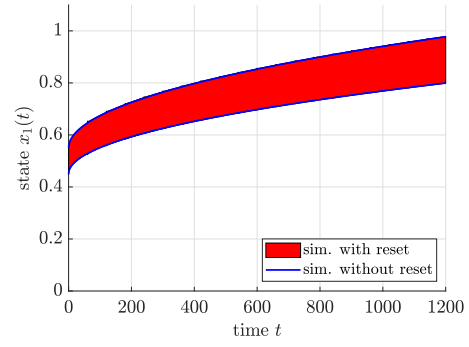
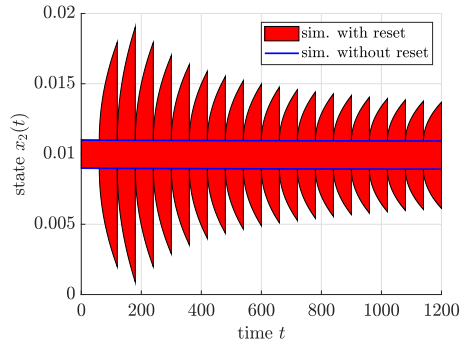
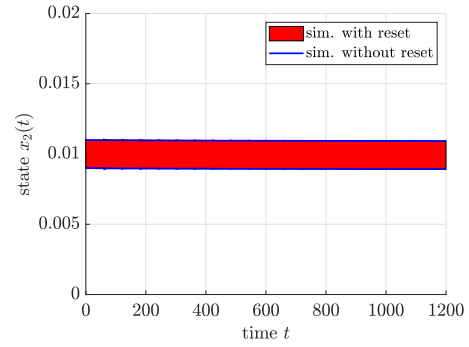
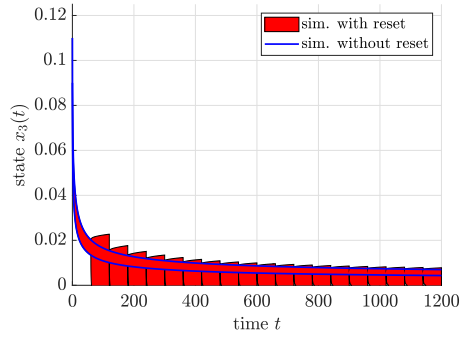
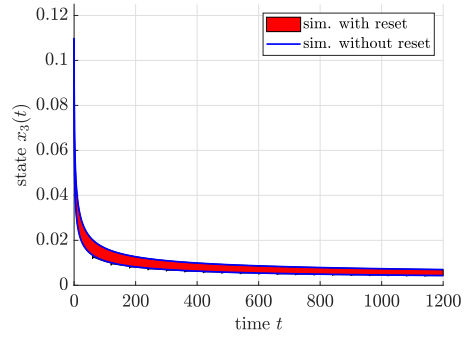
(a) State  $x_1(t)$  (state of charge  $\sigma(t)$ ).(b) State  $x_1(t)$  (state of charge  $\sigma(t)$ ).(c) State  $x_2(t)$ .(d) State  $x_2(t)$ .(e) State  $x_3(t)$  (voltage  $v_1(t)$ ).(f) State  $x_3(t)$  (voltage  $v_1(t)$ ).

Figure 6: Comparison of piecewise constant error bounds  $\mu_T$  (left column) with the observer-based enhancement  $\mu_v, \mu_w$  (right column) according to Sec. 3.2; resetting to the true state enclosures  $[\mathbf{v}(t); \mathbf{w}(t)]$  at each time instant  $T$  defined in (50).

is assumed that uncertain measurements of the terminal voltage of the battery are available at each time instant  $t_{m,k} = T$ . These uncertain measurements are chosen as the intervals

$$[y](t_{m,k}) = \text{mid} \left( [g] \left( [\mathbf{v}(t_{m,k}) ; \mathbf{w}(t_{m,k})] \right) \right) + [-10 ; 10] \text{ mV}, \quad (52)$$

where  $\text{mid}([x]) = \frac{1}{2} \cdot (\underline{x} + \bar{x})$  defines the midpoint of an interval  $[x]$ .

At each measurement instant  $T$ , the new pseudo state bounds are then initialized with the enclosure  $[\tilde{\mathbf{v}}(T) ; \tilde{\mathbf{w}}(T)]$  obtained at the end of the previous time slice. From these bounds, an axis-aligned interval box is extracted by a SIVIA-like state reconstruction for the measured system output (44) that eliminates subboxes that are incompatible with the measurement intervals (52). To continue the simulation further, a tight axis-aligned interval hull around the not eliminated boxes is formed, so that the pseudo state enclosures shown in Fig. 7 are obtained.

Future work will make use of these bounds for an identification of interval parameters included in both the FDE model and the algebraic output equation of a dynamic system.

## 5 Conclusions

In this paper, a novel observer-based approach for the quantification of time-domain truncation errors of FDE models has been presented. These errors arise inevitably when resetting fractional integrators. Integrator resets are necessary for the numerical evaluation of FDE models both to restrict the growth of memory demands when evaluating FDEs over long time spans and to take into account measured state information at distinct points in time between which the system dynamics are evolving continuously.

Future work will make use of the presented approach to solve the tasks of identifying past pseudo state information from an observed evolution of these quantities into the future and to identify uncertain system parameters on the basis of predictor-corrector state estimators. Moreover, the TNL interval observer design approach [38] for non-cooperative system models will be taken into consideration to avoid the currently existing necessity to transform non-cooperative models into cooperative ones by using the approaches presented in [14]. Although these approaches are useful for many practical applications, they always lead to conservative state enclosures due to the wrapping effect [17] that is inevitable when transforming the state equations and the domains of uncertain initial conditions with the help of (static) similarity transformations. This pessimism can be reduced by the TNL approach due to the introduction of further degrees of freedom for the observer parameterization. Moreover, this approach will also make the proposed methodology applicable to fractional descriptor systems.

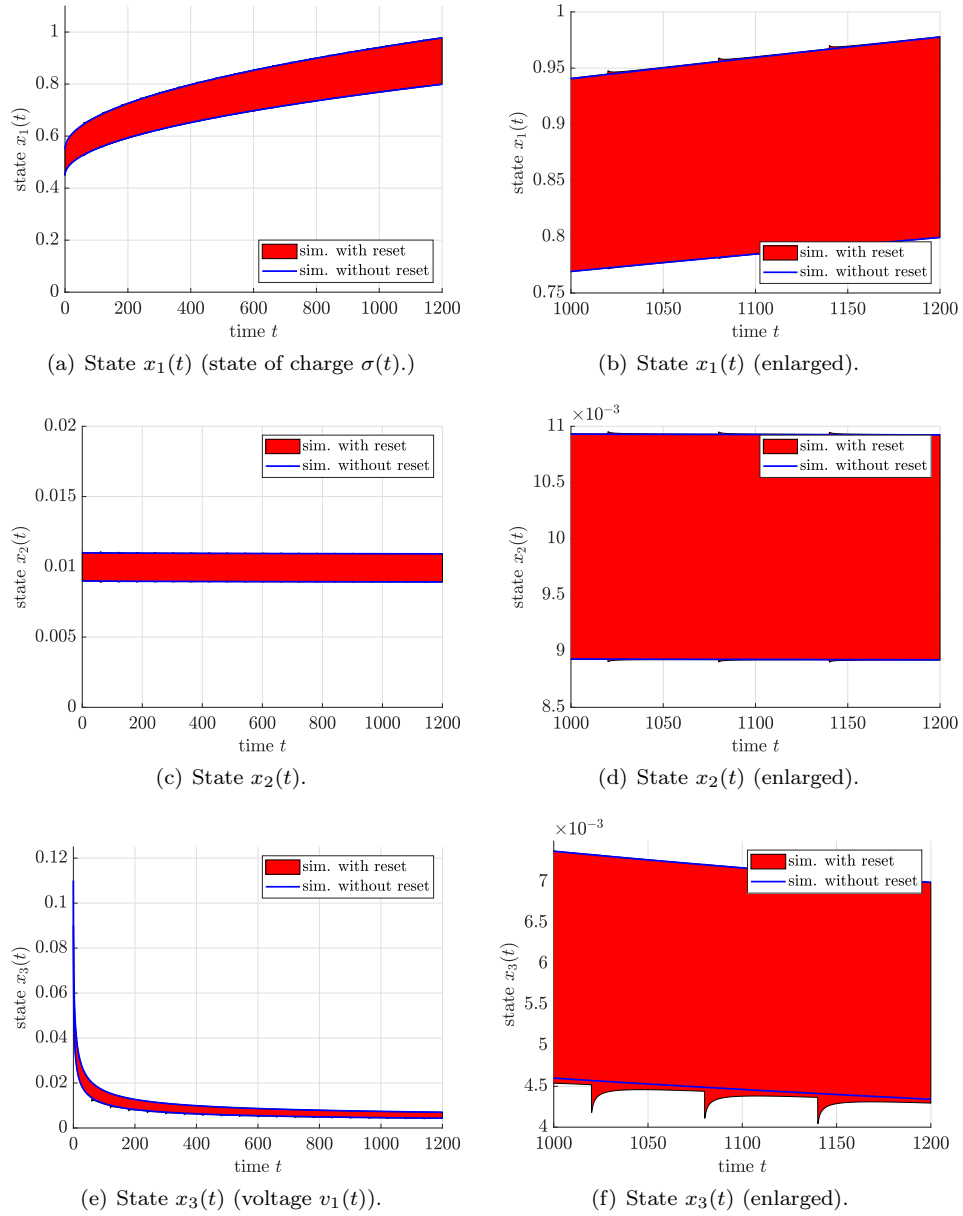


Figure 7: Simulation using the observer-based enhancement  $\mu_v$ ,  $\mu_w$  according to Sec. 3.2 and contractor-based resetting of the state variables at each measurement instant  $T$  according to Sec. 3.3.



## References

- [1] Amairi, M., Aoun, M., Najar, S., and Abdelkrim, M.N. A constant enclosure method for validating existence and uniqueness of the solution of an initial value problem for a fractional differential equation. *Applied Mathematics and Computation*, 217(5):2162–2168, 2010. DOI: [10.1016/j.amc.2010.07.015](https://doi.org/10.1016/j.amc.2010.07.015).
- [2] Andre, D., Meiler, M., Steiner, K., Wimmer, Ch., Soczka-Guth, T., and Sauer, D.U. Characterization of high-power lithium-ion batteries by electrochemical impedance spectroscopy. I. Experimental investigation. *Journal of Power Sources*, 196(12):5334–5341, 2011. DOI: [10.1016/j.jpowsour.2010.12.102](https://doi.org/10.1016/j.jpowsour.2010.12.102).
- [3] Bel Haj Frej, G., Malti, R., Aoun, M., and Raïssi, T. Fractional interval observers and initialization of fractional systems. *Communications in Nonlinear Science and Numerical Simulation*, 82:105030, 2020. DOI: [10.1016/j.cnsns.2019.105030](https://doi.org/10.1016/j.cnsns.2019.105030).
- [4] Delavari, H., Lanusse, P., and Sabatier, J. Fractional order controller design for a flexible link manipulator robot. *Asian Journal of Control*, 15(3):783–795, 2013. DOI: [10.1002/asjc.677](https://doi.org/10.1002/asjc.677).
- [5] Efimov, D., Raïssi, T., Chebotarev, S., and Zolghadri, A. Interval state observer for nonlinear time varying systems. *Automatica*, 49(1):200–205, 2013. DOI: [10.1016/j.automatica.2012.07.004](https://doi.org/10.1016/j.automatica.2012.07.004).
- [6] Erdinc, O., Vural, B., and Uzunoglu, M. A dynamic lithium-ion battery model considering the effects of temperature and capacity fading. In *Proc. of International Conference on Clean Electrical Power*, pages 383–386, Capri, Italy, 2009. DOI: [10.1109/ICCEP.2009.5212025](https://doi.org/10.1109/ICCEP.2009.5212025).
- [7] Garrappa, R. Predictor-corrector PECE method for fractional differential equations. MATLAB Central File Exchange. URL: <https://www.mathworks.com/matlabcentral/fileexchange/32918-predictor-corrector-pece-method-for-fractional-differential-equations> (accessed: Feb. 22, 2022).
- [8] Garrappa, R. Numerical evaluation of two and three parameter Mittag-Leffler functions. *SIAM Journal on Numerical Analysis*, 53(3):1350–1369, 2015. DOI: [10.1137/140971191](https://doi.org/10.1137/140971191).
- [9] Goodwine, B. Modeling a multi-robot system with fractional-order differential equations. In *Proc. of the IEEE International Conference on Robotics and Automation (ICRA)*, pages 1763–1768, Hong Kong, China, 2014. DOI: [10.1109/ICRA.2014.6907089](https://doi.org/10.1109/ICRA.2014.6907089).
- [10] Haubold, H.J., Mathai, A.M., and Saxena, R.K. Mittag-Leffler functions and their applications. *Journal of Applied Mathematics*, 2011:298628, 2011. DOI: [10.1155/2011/298628](https://doi.org/10.1155/2011/298628).

- [11] Hildebrandt, E., Kersten, J., Rauh, A., and Aschemann, H. Robust interval observer design for fractional-order models with applications to state estimation of batteries. In *Proc. of the 21st IFAC World Congress*, Berlin, Germany, 2020. DOI: [10.1016/j.ifacol.2020.12.2052](https://doi.org/10.1016/j.ifacol.2020.12.2052).
- [12] Jaulin, L., Kieffer, M., Didrit, O., and Walter, É. *Applied Interval Analysis*. Springer-Verlag, London, 2001. DOI: [10.1007/978-1-4471-0249-6](https://doi.org/10.1007/978-1-4471-0249-6).
- [13] Kempfle, S., Schäfer, I., and Beyer, H. Fractional differential equations and viscoelastic damping. In *2001 European Control Conference (ECC)*, pages 1738–1743, 2001. DOI: [10.23919/ECC.2001.7076172](https://doi.org/10.23919/ECC.2001.7076172).
- [14] Kersten, J., Rauh, A., and Aschemann, H. State-space transformations of uncertain systems with purely real and conjugate-complex eigenvalues into a cooperative form. In *Proc. of 23rd Intl. Conference on Methods and Models in Automation and Robotics*, pages 797–802, Miedzyzdroje, Poland, 2018. DOI: [10.1109/MMAR.2018.8486085](https://doi.org/10.1109/MMAR.2018.8486085).
- [15] Lanusse, P., Malti, R., and Melchior, P. CRONE control system design toolbox for the Control Engineering Community: Tutorial and case study. *Philosophical transactions. Series A, Mathematical, physical, and engineering sciences*, 371:20120149, 2013. DOI: [10.1098/rsta.2012.0149](https://doi.org/10.1098/rsta.2012.0149).
- [16] Lazarevic, M. P. Biologically inspired control and modeling of (Bio)Robotic systems and some applications of fractional calculus in mechanics. *Theoretical and Applied Mechanics*, 40:163–187, 2013. DOI: [10.2298/TAM1301163L](https://doi.org/10.2298/TAM1301163L).
- [17] Lohner, R. On the ubiquity of the wrapping effect in the computation of the error bounds. In Kulisch, U., Lohner, R., and Facius, A., editors, *Perspectives on Enclosure Methods*, pages 201–217, Wien, New York, 2001. Springer-Verlag. DOI: [10.1007/978-3-7091-6282-8\\_12](https://doi.org/10.1007/978-3-7091-6282-8_12).
- [18] Lyons, R., Vatsala, A.S., and Chiquet, R. Picard’s iterative method for Caputo fractional differential equations with numerical results. *Mathematics*, 5(4), 2017. DOI: [10.3390/math5040065](https://doi.org/10.3390/math5040065).
- [19] Miyajima, S. Computing enclosures for the matrix Mittag-Leffler function. *Journal of Scientific Computing*, 87:62, 2021. DOI: [10.1007/s10915-021-01447-6](https://doi.org/10.1007/s10915-021-01447-6).
- [20] Monje, C.A., Vinagre, B.M., Feliu, V., and Chen, Y. Tuning and auto-tuning of fractional order controllers for industry applications. *Control Engineering Practice*, 16(7):798–812, 2008. DOI: [10.1016/j.conengprac.2007.08.006](https://doi.org/10.1016/j.conengprac.2007.08.006).
- [21] Oustaloup, A. *La Dérivation Non Entière: Théorie, Synthèse et Applications*. Hermès, Paris, 1995. In French.
- [22] Oustaloup, A., Mathieu, B., and Lanusse, P. The CRONE control of resonant plants: Application to a flexible transmission. *European Journal of Control*, 1(2):113–121, 1995. DOI: [10.1016/S0947-3580\(95\)70014-0](https://doi.org/10.1016/S0947-3580(95)70014-0).

- [23] Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Mathematics in Science and Engineering. Academic Press, London, 1999.
- [24] Podlubny, I. Fractional-order systems and  $PI^\lambda D^\mu$ -controllers. *IEEE Transactions on Automatic Control*, 44(1):208–214, 1999. DOI: [10.1109/9.739144](https://doi.org/10.1109/9.739144).
- [25] Raïssi, T., Efimov, D., and Zolghadri, A. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57:260–265, 2012. DOI: [10.1109/TAC.2011.2164820](https://doi.org/10.1109/TAC.2011.2164820).
- [26] Rauh, A. and Jaulin, L. Novel techniques for a verified simulation of fractional-order differential equations. *Fractal and Fractional*, 5(1):17, 2021. DOI: [10.3390/fractalfract5010017](https://doi.org/10.3390/fractalfract5010017).
- [27] Rauh, A. and Kersten, J. Toward the development of iteration procedures for the interval-based simulation of fractional-order systems. *Acta Cybernetica*, 25(1):21–48, 2020. DOI: [10.14232/actacyb.285660](https://doi.org/10.14232/actacyb.285660).
- [28] Rauh, A. and Kersten, J. Verification and reachability analysis of fractional-order differential equations using interval analysis. In Dang, Thao and Ratschan, Stefan, editors, *Proceedings 6th International Workshop on Symbolic-Numeric methods for Reasoning about CPS and IoT*, Volume 331 of *Electronic Proceedings in Theoretical Computer Science*, pages 18–32. Open Publishing Association, 2021. DOI: [10.4204/EPTCS.331.2](https://doi.org/10.4204/EPTCS.331.2).
- [29] Rauh, A., Kersten, J., and Aschemann, H. Interval-based verification techniques for the analysis of uncertain fractional-order system models. In *Proc. of the 18th European Control Conference ECC2020*, St. Petersburg, Russia, 2020. DOI: [10.23919/ECC51009.2020.9143758](https://doi.org/10.23919/ECC51009.2020.9143758).
- [30] Reuter, J., Mank, E., Aschemann, H., and Rauh, A. Battery state observation and condition monitoring using online minimization. In *Proc. of 21st Intl. Conference on Methods and Models in Automation and Robotics*, pages 797–802, Miedzyzdroje, Poland, 2016. DOI: [10.1109/MMAR.2016.7575313](https://doi.org/10.1109/MMAR.2016.7575313).
- [31] Rohou, S. and Jaulin, L. Exact bounded-error continuous-time linear state estimator. *Systems & Control Letters*, 153:104951, 2021. DOI: [10.1016/j.sysconle.2021.104951](https://doi.org/10.1016/j.sysconle.2021.104951).
- [32] Scherer, R., Kalla, S.L., Tang, Y., and Huang, J. The Grünwald–Letnikov method for fractional differential equations. *Computers & Mathematics with Applications*, 62(3):902–917, 2011. DOI: [10.1016/j.camwa.2011.03.054](https://doi.org/10.1016/j.camwa.2011.03.054).
- [33] Smith, H.L. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Volume 41. Mathematical Surveys and Monographs, American Mathematical Soc., Providence, 1995. DOI: [10.1090/surv/041](https://doi.org/10.1090/surv/041).

- [34] Tavares, D., Almeida, R., and Torres, D.F.M. Caputo derivatives of fractional variable order: Numerical approximations. *Communications in Nonlinear Science and Numerical Simulation*, 35:69–87, 2016. DOI: [10.1016/j.cnsns.2015.10.027](https://doi.org/10.1016/j.cnsns.2015.10.027).
- [35] Trigeassou, J.-C. and Maamri, N. *Analysis, Modeling and Stability of Fractional Order Differential Systems 2*. John Wiley & Sons, Ltd, Hoboken, NJ, USA, 2019. DOI: [10.1002/9781119686859](https://doi.org/10.1002/9781119686859).
- [36] Troparevsky, M., Seminara, S., and Fabio, M. *A Review on Fractional Differential Equations and a Numerical Method to Solve Some Boundary Value Problems*. In *Nonlinear Systems — Theoretical Aspects and Recent Applications*. IntechOpen, 2020. DOI: [10.5772/intechopen.86273](https://doi.org/10.5772/intechopen.86273).
- [37] Wang, B., Liu, Z., Li, S., Moura, S., and Peng, H. State-of-charge estimation for lithium-ion batteries based on a nonlinear fractional model. *IEEE Transactions on Control Systems Technology*, 25(1):3–11, 2017. DOI: [10.1109/TCST.2016.2557221](https://doi.org/10.1109/TCST.2016.2557221).
- [38] Wang, Z., Lim, C.-C., and Shen, Y. Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116:41–46, 2018. DOI: [10.1016/j.sysconle.2018.04.003](https://doi.org/10.1016/j.sysconle.2018.04.003).
- [39] Yameni Noupoue, Y.Y., Tandoğdu, Y., and Awadalla, M. On numerical techniques for solving the fractional logistic differential equation. *Advances in Difference Equations*, 2019, 2019. DOI: [10.1186/s13662-019-2055-y](https://doi.org/10.1186/s13662-019-2055-y).
- [40] Zou, Ch., Zhang, L., Hu, X., Wang, Z., Wik, T., and Pecht, M. A review of fractional-order techniques applied to lithium-ion batteries, lead-acid batteries, and supercapacitors. *Journal of Power Sources*, 390:286–296, 2018. DOI: [10.1016/j.jpowsour.2018.04.033](https://doi.org/10.1016/j.jpowsour.2018.04.033).