On a Class of Unary Operators in Continuous-Valued Logic*

József Dombi[ab] and Tamás Jónás[c]

Abstract

The unary operators play an important role in continuous-valued logic and in artificial intelligence as well. Based on our previous results concerning these operators, we prove here that (a) the Pliant negation operator (also known as the Dombi form of negation); (b) the substantiating, weakening, modal and linguistic hedge operators; (c) the sharpness operator; and (d) the preference operator can all be written in a common form, which is called the kappa function. The kappa function is an operator class-dependent, universal operator. Here, a sufficient condition for the identity of two kappa functions is presented. Also, we provide the condition for which the conjunctive and disjunctive forms of the kappa function coincide. Next, we demonstrate that the inverse of a kappa function is a kappa function as well. Then, we show that for certain conditions, a set of kappa functions is closed under the composition and conjunctive (or disjunctive) operations. Also, we briefly describe two special cases of the kappa function: the product and the Dombi operator case; and we point out that its extended versions can be applied in various areas.

Keywords: unary operators, modifier operators, sharpness operator, kappa function, generalized preference operator

1 Introduction

In continuous-valued logic, the unary operators like the negation operator (see, e.g., the papers of Esteva *et al.* [19] and Cintula *et al.* [5]), the substantiating

*This work was supported by the European Union project RRF-2.3.1-21-2022-00004 within the framework of the Artificial Intelligence National Laboratory and project TKP2021-NVA-09, implemented with the support provided by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, financed under the TKP2021-NVA funding scheme.

[a]Department of Computer Algorithms and Artificial Intelligence, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary, E-mail: dombi@inf.u-szeged.hu, ORCID: 0000-0001-9459-912X

[b]HUN-REN SZTE Research Group on Artificial Intelligence

[c]Faculty of Economics, ELTE Eötvös Loránd University, Rákóczi út 7, H-1088 Budapest, Hungary, E-mail: jonas@gtk.elte.hu, ORCID: 0000-0001-8241-2321

DOI: 10.14232/actacyb.300495

and weakening operators (see, e.g., Łukasiewicz [26, 27, 28], Mattila [30, 29]), the necessity and possibility operators (see, e.g., Banerjee and Dubois [2], Cattaneo *et al.* [3], Vidal [33] and Jain *et al.* [23]), and the linguistic hedge operators (see, e.g., De Cock and Kerre [6], Huynh, Ho and Nakamori [22], Yan *et al.* [34], Rubin [31] and Esteva *et al.* [20]) are of high importance. This is due to the fact that these operators can be used to modify the logical value of a statement or transform the membership function of a fuzzy set to a new membership function. Also, if in a preference relation, one of the arguments has a fixed value, then the preference relation may be treated as a unary operator (see, e.g., Dombi and Baczyński [13]).

In this study, we will introduce a generator-function dependent operator, called the kappa function, and present its main properties. Here, we will present the necessary and sufficient conditions for the identity of two kappa functions that have the same parameter values, but are induced by generator functions that may differ. Also, we will demonstrate that the conjunctive and disjunctive forms of the kappa function coincide if $f_c(x)f_d(x) = 1$ holds for any $x \in [0, 1]$, where f_c and f_d are generator functions of the conjunctive and disjunctive forms, respectively. Then, we will show that the inverse of a kappa function is a kappa function as well, and demonstrate that composition of kappa functions results in a kappa function. We will prove that for certain conditions, a set of kappa functions is closed under conjunctive or disjunctive operations. Next, we will briefly describe two special cases of the kappa function: the product operator case and the Dombi operator case. Also, we will highlight that the transformed or extended versions of the kappa function induced by the generator function of Dombi operators can be applied in various areas of science. Here, we will demonstrate that (a) the Pliant negation operator (also known as the Dombi form of negation); (b) the substantiating, weakening, modal and linguistic hedge operators; (c) the sharpness operator; and (d) the preference operator are special cases of the kappa function. Therefore, the kappa function may be viewed as a general unary modifier operator.

This paper is structured as follows. In Section 2, the basic notions and notations, which will be used later on, are described. In Section 3, we will introduce the kappa function and present its main properties. In Section 4, we will provide a brief overview of our previous results on unary operators and demonstrate that these operators are special ceases of the kappa function. Lastly, our conclusions are summarized in Section 5.

2 Notions and notations

Here, we will briefly review the basic notions and notations, which we will use later on.

In continuous-valued logic, the concepts of strict triangular norm (strict t-norm) and strict triangular conorm (strict t-conorm) play an important role. The following definitions of strict t-norms and strict t-conorms is based on the application of Aczél's results on the associative functional equation [1] (also see [24]).

Definition. *A function* $c\colon [0,1]^2 \to [0,1]$ *is a strict t-norm if and only if* c *is continuous, and there exists a continuous and strictly decreasing function* $f_c\colon [0,1] \to [0,\infty]$ *with the properties* $f_c(1) = 0$ *and* $f_c(0) = \infty$ *such that for any* $x, y \in [0,1]$*,*

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right).$$

Definition. *A function* $d\colon [0,1]^2 \to [0,1]$ *is a strict t-conorm if and only if* d *is continuous, and there exists a continuous and strictly increasing function* $f_d\colon [0,1] \to [0,\infty]$ *with the properties* $f_d(0) = 0$ *and* $f_d(1) = \infty$ *such that for any* $x, y \in [0,1]$*,*

$$d(x,y) = f_d^{-1}\left(f_d(x) + f_d(y)\right).$$

The function f_c (f_d, respectively) is called an additive generator of the strict t-norm c (strict t-conorm d, respectively), which is uniquely determined up to a positive constant multiplier of f_c (f_d, respectively).
Note that the strict t-norm and strict t-conorm are special cases of the general t-norm and t-conorm classes, respectively. In our study:

(a) We do not use the pseudo-inverse and ordinal sum to construct a general t-norm and t-conorm;

(b) We do not use the commutativity axiom of the t-norm and t-conorm because it is always valid for the strict t-norm;

(c) We do not use the boundary condition of the t-norm and t-conorm, just the compatibility condition with binary logic.

Remark. In this article, we will refer to strict t-norms and t-conorms as conjuctive and disjunctive operators denoted by c and d, respectively. From now on, the mapping $f\colon [0,1] \to [0,\infty]$ will always be a continuous, strictly decreasing (increasing, respectively) generator function of a conjunctive (disjunctive, respectively) operator. If f is strictly decreasing, then we will interpret $f(0) = \infty$ and $f^{-1}(\infty) = 0$. Similarly, if f is strictly increasing, then we will interpret $f(1) = \infty$ and $f^{-1}(\infty) = 1$. Also, we will make use of the extended arithmetic operations $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Here, we will use the following definition of a strong negation (see, e.g., Definition 1.2 in [21], or Definition 11.3 in [24]).

Definition. *We say that* $\eta\colon [0,1] \to [0,1]$ *is a strong negation if and only if* η *satisfies the following requirements:*

(a) η *is continuous*	*(Continuity)*
(b) $\eta(0) = 1$, $\eta(1) = 0$	*(Boundary conditions)*
(c) $\eta(x) < \eta(y)$ *for* $x > y$	*(Monotonicity)*
(d) $\eta(\eta(x)) = x$ *for any* $x \in [0,1]$	*(Involution).*

Remark. It should be added that the requirements (a) and (b) in Definition 2 can be omitted (see Theorem 3.1 in the book of Klir and Yuan [25]).

We will use the concept of Pliant negation (also known as the Dombi form of negation), which is defined as follows (see [8, 11]).

Definition. *Let* $f\colon [0,1] \to [0,\infty]$ *be a generator function of a conjunctive or disjunctive operator and let* $\nu \in (0,1)$*. The mapping* $\eta_\nu\colon [0,1] \to [0,1]$ *is said to be a Pliant negation operator with the parameter* ν *if* η_ν *is given by*

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right). \tag{1}$$

Remark. The Pliant negation given in Definition 2 is a strong negation in the sense of Definition 2 (see [8]). Since for any $\nu \in (0,1)$, $\eta_\nu(\nu) = \nu$, the Pliant negation η_ν is characterized by its fixed point, which is its parameter value ν.

Definition. *We will say that the strong negation* $\eta_1\colon [0,1] \to [0,1]$ *is stricter than the strong negation* $\eta_2\colon [0,1] \to [0,1]$ *if and only if for any* $x \in (0,1)$*,* $\eta_1(x) < \eta_2(x)$*.*

Proposition. *Let* $\nu_1, \nu_2 \in (0,1)$ *and let the generator functions of the Pliant negation operators* η_{ν_1} *and* η_{ν_2} *be identical up to a positive multiplicative constant. Then,* η_{ν_1} *is stricter than* η_{ν_2} *if and only if* $\nu_1 < \nu_2$*.*

Proof. Let η_{ν_1} be induced by a generator function f and let η_{ν_2} be induced by the generator function $g = af$, where $a \in \mathbb{R}$, $a > 0$ is a constant. This means that $g(x) = af(x)$ for any $x \in [0,1]$ and $g^{-1}(x) = f^{-1}\left(\frac{1}{a}x\right)$ for any $x \in [0,\infty]$. Hence, using Eq. (1), we have

$$\eta_{\nu_1}(x) = f^{-1}\left(\frac{f^2(\nu_1)}{f(x)}\right) \tag{2}$$

and

$$\begin{aligned}\eta_{\nu_2}(x) =& g^{-1}\left(\frac{g^2(\nu_2)}{g(x)}\right) = f^{-1}\left(\frac{1}{a}\frac{a^2 f^2(\nu_2)}{af(x)}\right)\\ =& f^{-1}\left(\frac{f^2(\nu_2)}{f(x)}\right)\end{aligned} \tag{3}$$

for any $x \in [0,1]$. Noting the strict monotonicity of f, the rest of the proof is straightforward. □

It should be mentioned that a representation theorem for the strong negation given in Definition 2 was first presented by [32].

Here, we will utilize the Pliant system that is defined as follows (see [8, 10, 11]).

Definition. *Let* $c\colon [0,1]^2 \to [0,1]$ *be a conjunctive operator with a generator function* $f_c\colon [0,1] \to [0,\infty]$ *and let* $d\colon [0,1]^2 \to [0,1]$ *be a disjunctive operator with a generator function* $f_d\colon [0,1] \to [0,\infty]$*. The triplet* (c, d, η_ν) *is called a Pliant system if*

$$f_c(x) f_d(x) = 1 \tag{4}$$

holds for any $x \in [0,1]$ *and* η_ν *is a Pliant negation operator induced by* f_c *or* f_d*.*

Remark. If Eq. (4) holds for any $x \in [0,1]$, then the Pliant negations induced by f_c and f_d coincide (see Theorem 8 in [8]). That is, in a Pliant logical system, the conjunction, disjunction and negation operators are all determined by one generator function (which is uniquely determined up to a multiplicative constant). Also, if Eq. (4) holds, then c, d and η_ν form a De Morgan system (see Theorem 9 in [8]).

3 The kappa function

Here, we introduce the kappa function and present its main properties. Later, we will show that there are important unary operators in continuous-valued logic that can be viewed as special cases of the kappa function.

Definition. *Let f be a generator function of a conjunctive or disjunctive operator. We say that the mapping $\kappa^{(\lambda)}_{\nu,\nu_0,f} \colon [0,1] \to [0,1]$, which is given by*

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \tag{5}$$

where $\nu, \nu_0 \in (0,1)$ and $\lambda \in \mathbb{R}$, is a kappa function induced by f.

We shall call the function f in Eq. (5) a generator function of the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f}$.

Remark. In this paper, we will use the notation $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ for the kappa function, which has the parameters ν, ν_0, λ and is induced by a generator function f. However, if we consider the kappa function as being a unary operator in a logical system and all the operators in this system (conjunction, disjunction, negation) are induced by a common generator f, then f could be omitted from the notations as for example in the case of the negation operator.

Now, we will provide a sufficient condition for the equality of two kappa functions, which have the same parameter values, but are induced by generator functions that may differ.

Proposition. *Let f and g be generator functions of conjunctive or disjunctive operators. Let the kappa functions $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ and $\kappa^{(\lambda)}_{\nu,\nu_0,g}$ be induced by f and g, respectively, where $\nu, \nu_0 \in (0,1)$ and $\lambda \in \mathbb{R} \setminus \{0\}$. For any $x \in [0,1]$, if*

$$f(x) = \beta g^{\alpha}(x), \tag{6}$$

then

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \kappa^{(\lambda)}_{\nu,\nu_0,g}(x), \tag{7}$$

where $\alpha \neq 0$ and $\beta > 0$.

Proof. Let us assume that Eq. (6) holds for any $x \in [0,1]$. From this equation, we have

$$f^{-1}(z) = g^{-1}\left(\left(\frac{z}{\beta}\right)^{\frac{1}{\alpha}}\right)$$

for any $z \in [0, \infty]$. Therefore, using Definition 3, for any $x \in [0, 1]$, we can write

$$\begin{aligned}\kappa_{\nu,\nu_0,f}^{(\lambda)}(x) =& f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right) \\ =& g^{-1}\left(\left(\frac{\beta g^{\alpha}(\nu_0)\left(\frac{\beta g^{\alpha}(x)}{\beta g^{\alpha}(\nu)}\right)^{\lambda}}{\beta}\right)^{\frac{1}{\alpha}}\right) \\ =& g^{-1}\left(g(\nu_0)\left(\frac{g(x)}{g(\nu)}\right)^{\lambda}\right) = \kappa_{\nu,\nu_0,g}^{(\lambda)}(x),\end{aligned}$$

which means that the kappa functions generated by function f and g are identical for any $x \in [0, 1]$. □

It should be emphasized that the kappa function is generator function-dependent. That is, the kappa functions induced by generator functions of conjunctive and disjunctive logical operators may differ. The following proposition demonstrates that the kappa functions induced by generator functions of conjunctive and disjunctive operators of a Pliant logical system coincide.

Proposition. *Let $\nu, \nu_0 \in (0, 1)$ and $\lambda \in \mathbb{R}$. Let f_c be a generator function of a conjunctive operator c and let f_d be a generator function of a disjunctive operator d. Furthermore, let the kappa functions $\kappa_{\nu,\nu_0,f_c}^{(\lambda)}$ and $\kappa_{\nu,\nu_0,f_d}^{(\lambda)}$ be induced by f_c and f_d, respectively. If*

$$f_c(x) f_d(x) = 1$$

holds for any $x \in [0, 1]$ (i.e., c and d are conjunctive and disjunctive operators of a Pliant system, respectively), then

$$\kappa_{\nu,\nu_0,f_c}^{(\lambda)}(x) = \kappa_{\nu,\nu_0,f_d}^{(\lambda)}(x)$$

holds for any $x \in [0, 1]$.

Proof. Since $f_c(x) f_d(x) = 1$ can be written as $f_c(x) = (f_d(x))^{-1}$ for any $x \in [0, 1]$, this proposition immediately follows from Proposition 3. □

3.1 Properties of the kappa function

Here, we will summarize the main properties of the kappa function $\kappa_{\nu,\nu_0,f}^{(\lambda)}$, which is induced by the generator function f, where $\nu, \nu_0 \in (0, 1)$, $\lambda \in \mathbb{R}$.

3.1.1 Basic properties of the kappa function

(a) $\kappa_{\nu,\nu_0,f}^{(\lambda)}(x)$ is a continuous on $(0, 1)$.

(b) $\kappa_{\nu,\nu_0,f}^{(\lambda)}(\nu) = \nu_0$.

(c) If $\lambda < 0$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ is strictly decreasing.

(d) If $\lambda = 0$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \nu_0$ for any $x \in [0, 1]$.

(e) If $\lambda > 0$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ is strictly increasing.

(f) If $\lambda > 0$, then

$$\lim_{x\to 0} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = 0 \quad \text{and} \quad \lim_{x\to 1} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = 1.$$

(g) If $\lambda < 0$, then

$$\lim_{x\to 0} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = 1 \quad \text{and} \quad \lim_{x\to 1} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = 0.$$

(h) If $\lambda > 0$, then
a) $\nu < x$ if and only if $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) > \nu_0$
b) $x < \nu$ if and only if $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) < \nu_0$.

(i) If $\lambda < 0$, then
a) $x < \nu$ if and only if $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) > \nu_0$
b) $\nu < x$ if and only if $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) < \nu_0$.

(j) If $\nu = \nu_0$ and $\lambda = 1$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = x$.

(k)

$$\lim_{\lambda\to+\infty} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \begin{cases} 0, & \text{if } x < \nu \\ \nu_0 & \text{if } x = \nu \\ 1, & \text{if } x > \nu \end{cases}$$

(l)

$$\lim_{\lambda\to-\infty} \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \begin{cases} 1, & \text{if } x < \nu \\ \nu_0 & \text{if } x = \nu \\ 0, & \text{if } x > \nu \end{cases}$$

Figure 1 shows example plots of kappa functions with various parameter values for the generator function $f(x) = \frac{1-x}{x}$, where $x \in [0, 1]$.

3.1.2 Involutivity of the kappa function

Here, we will show that the inverse of a kappa function is a kappa function as well and give sufficient conditions for the involutivity of the kappa function.

Proposition. *The inverse of the kappa function* $\kappa^{(\lambda)}_{\nu,\nu_0,f} \colon [0, 1] \to [0, 1]$ *induced by a generator function* f *is the kappa function* $\kappa^{(1/\lambda)}_{\nu_0,\nu,f} \colon [0, 1] \to [0, 1]$*, where* $\nu, \nu_0 \in (0, 1)$ *and* $\lambda \neq 0$.

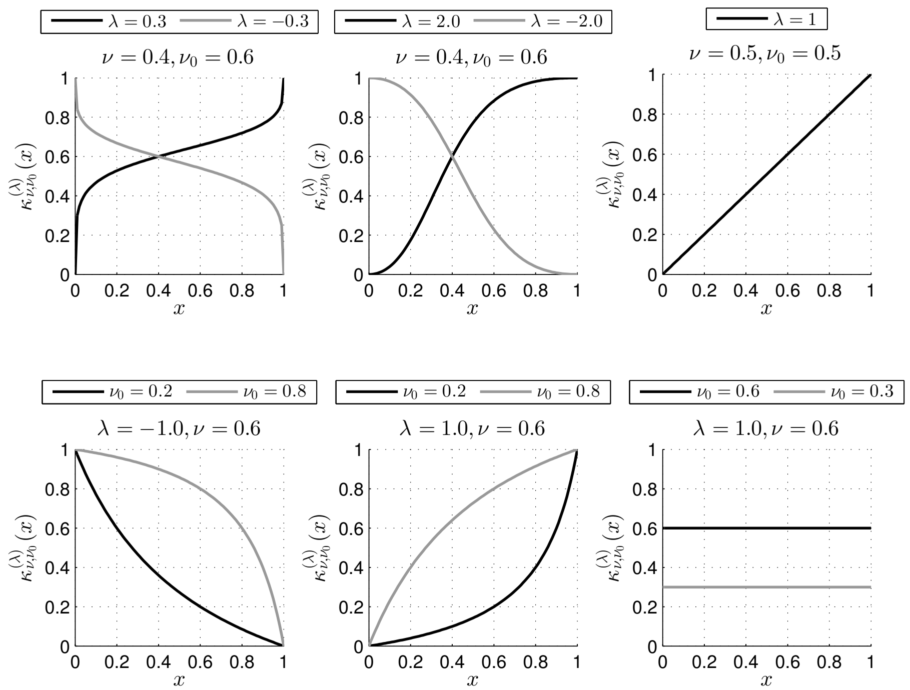

Figure 1: Example plots of $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ with various parameter values, $f(x) = \frac{1-x}{x}$.

Proof. If $\lambda \neq 0$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ is a bijective function, and so its inverse function exists. Let $h\colon [0,1] \to [0,1]$ be the inverse function of $\kappa^{(\lambda)}_{\nu,\nu_0,f}$. Noting the definition for a kappa function, function h needs to satisfy the equation

$$f^{-1}\left(f(\nu_0)\left(\frac{f((h(x)))}{f(\nu)}\right)^{\lambda}\right) = x$$

for any $x \in [0,1]$. From this equation, we get

$$h(x) = f^{-1}\left(f(\nu)\left(\frac{f(x)}{f(\nu_0)}\right)^{\frac{1}{\lambda}}\right)$$

for any $x \in [0,1]$; which means that

$$h(x) = \kappa^{(1/\lambda)}_{\nu_0,\nu,f}(x).$$ □

The following proposition provides sufficient conditions for the involutivity of the kappa function, i.e., for the property that

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}\left(\kappa^{(\lambda)}_{\nu,\nu_0,f}(x)\right) = x \tag{8}$$

holds for any $x \in [0,1]$.

Proposition. *Let* $\nu, \nu_0 \in (0,1)$ *and* $\lambda \neq 0$. *If*

(a) $\lambda = -1$
or

(b) $\lambda = 1$ *and* $\nu = \nu_0$ *(i.e.,* $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = x$, $x \in [0,1]$*)*

then the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f} \colon [0,1] \to [0,1]$ *induced by a generator function* f *is involutive.*

Proof. Noting Proposition 3.1.2, Eq. (8) is equivalent to

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \kappa^{(1/\lambda)}_{\nu_0,\nu,f}(x), \tag{9}$$

where $x \in [0,1]$.

(a) If $\lambda = -1$, then we have

$$\begin{aligned}\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) =& f^{-1}\left(f(\nu_0)\frac{f(\nu)}{f(x)}\right) \\ =& f^{-1}\left(f(\nu)\frac{f(\nu_0)}{f(x)}\right) = \kappa^{(1/\lambda)}_{\nu_0,\nu,f}(x)\end{aligned}$$

for any $x \in [0,1]$.

(b) If $\lambda = 1$ and $\nu = \nu_0$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = x$, and so Eq. (8) trivially holds for any $x \in [0,1]$. □

3.1.3 Composition of kappa functions

Here, will demonstrate that the composition of kappa functions, which are both induced by a generator function f, results in a kappa function induced by f as well. This means that for a given generator function, which is uniquely determined according to Proposition 3, the set of kappa functions is closed under the composition operation.

Proposition. *If* $\kappa^{(\lambda_1)}_{\nu_1,\nu_{0_1},f}$ *and* $\kappa^{(\lambda_1)}_{\nu_2,\nu_{0_2},f}$ *are two kappa functions, both induced by a generator function* f*, with the parameters* $\nu_{0_1}, \nu_{0_2}, \nu_1, \nu_2 \in (0,1)$*,* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1, \lambda_2 \neq 0$*, then*

$$\kappa^{(\lambda_1)}_{\nu_1,\nu_{0_1},f} \circ \kappa^{(\lambda_2)}_{\nu_2,\nu_{0_2},f} = \kappa^{(\lambda)}_{\nu,\nu_0,f},$$

where $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ *is a kappa function with the parameters*

$$\lambda = \lambda_1\lambda_2$$

$$\nu = f^{-1}\left(f^{\frac{1}{\lambda_2}}(\nu_1)f(\nu_2)\right) \tag{10}$$

$$\nu_0 = f^{-1}\left(f(\nu_{0_1})f^{\lambda_1}(\nu_{0_2})\right). \tag{11}$$

Proof. After direct calculation, we get

$$\begin{aligned} &\kappa^{(\lambda_1)}_{\nu_1,\nu_{0_1},f}\left(\kappa^{(\lambda_2)}_{\nu_2,\nu_{0_2},f}(x)\right) \\ &= f^{-1}\left(f(\nu_{0_1})f^{\lambda_1}(\nu_{0_2})\frac{f^{\lambda_1\lambda_2}(x)}{f^{\lambda_1}(\nu_1)f^{\lambda_1\lambda_2}(\nu_2)}\right) \end{aligned}$$

for any $x \in [0,1]$. The right hand side of this equation can be written in the form

$$\begin{aligned} &f^{-1}\left(f(\nu_{0_1})f^{\lambda_1}(\nu_{0_2})\frac{f^{\lambda_1\lambda_2}(x)}{f^{\lambda_1}(\nu_1)f^{\lambda_1\lambda_2}(\nu_2)}\right) \\ &\qquad = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \end{aligned}$$

where

$$\lambda = \lambda_1\lambda_2$$

$$f^{\lambda_1\lambda_2}(\nu) = f^{\lambda_1}(\nu_1)f^{\lambda_1\lambda_2}(\nu_2) \tag{12}$$

$$f(\nu_0) = f(\nu_{0_1})f^{\lambda_1}(\nu_{0_2}). \tag{13}$$

Next, from Eq. (12) and Eq. (13), we get Eq. (10) and Eq. (11). □

3.2 Logical operations over kappa functions

In the following, we will show that if in a set of kappa functions every element is induced by a generator function of a conjunctive (respectively disjunctive) operator and the parameters ν_0 and λ of the kappa functions have fixed values, then this set is closed under the corresponding conjunctive (respectively disjunctive) operation.

Proposition. *Let c be a conjunctive operator with a generator function f_c, let d be a disjunctive operator with a generator function f_d, let $\lambda > 0$ and let $\nu_0, \nu_1, \nu_2, \dots, \nu_n \in (0,1)$.*

(a) If the kappa functions $\kappa^{(\lambda)}_{\nu_1,\nu_0,f_c}, \kappa^{(\lambda)}_{\nu_2,\nu_0,f_c}, \dots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_c}$ are all induced by a generator function f_c, then

$$c\left(\kappa^{(\lambda)}_{\nu_1,\nu_0,f_c}(x), \kappa^{(\lambda)}_{\nu_2,\nu_0,f_c}(x), \dots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_c}(x)\right) = \kappa^{(\lambda)}_{\nu,\nu_0,f_c}(x)$$

for any $x \in [0,1]$, where $\nu = d_\lambda(\nu_1, \nu_2, \dots, \nu_n)$ and d_λ is the disjunctive operator induced by the generator function $\frac{1}{f_c^\lambda(x)}$.

(b) If the kappa functions $\kappa^{(\lambda)}_{\nu_1,\nu_0,f_d}, \kappa^{(\lambda)}_{\nu_2,\nu_0,f_d}, \dots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_d}$ are all induced by a generator function f_d, then

$$d\left(\kappa^{(\lambda)}_{\nu_1,\nu_0,f_d}(x), \kappa^{(\lambda)}_{\nu_2,\nu_0,f_d}(x), \dots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_d}(x)\right) = \kappa^{(\lambda)}_{\nu,\nu_0,f_d}(x)$$

for any $x \in [0,1]$, where $\nu = c_\lambda(\nu_1, \nu_2, \dots, \nu_n)$ and c_λ is the conjunctive operator induced by the generator function $\frac{1}{f_d^\lambda(x)}$.

Proof. Here, we will just prove (a), the proof of (b) being similar to that of (a).

Let the kappa functions $\kappa^{(\lambda)}_{\nu_1,\nu_0,f_c}, \kappa^{(\lambda)}_{\nu_2,\nu_0,f_c}, \ldots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_c}$ all be induced by a generator function f_c. Noting Definition 2 and the definition for a kappa function, we can write

$$c\left(\kappa^{(\lambda)}_{\nu_1,\nu_0,f_c}(x), \kappa^{(\lambda)}_{\nu_2,\nu_0,f_c}(x), \ldots, \kappa^{(\lambda)}_{\nu_n,\nu_0,f_c}(x)\right) = f_c^{-1}\left(\sum_{i=1}^{n} f_c\left(\kappa^{(\lambda)}_{\nu_i,\nu_0,f_c}(x)\right)\right) =$$

$$= f_c^{-1}\left(\sum_{i=1}^{n} f_c\left(f_c^{-1}\left(f_c(\nu_0)\left(\frac{f_c(x)}{f_c(\nu_i)}\right)^{\lambda}\right)\right)\right) =$$

$$= f_c^{-1}\left(f_c(\nu_0) f_c^{\lambda}(x) \sum_{i=1}^{n} \frac{1}{f_c^{\lambda}(\nu_i)}\right) = f_c^{-1}\left(f_c(\nu_0)\left(\frac{f_c(x)}{f_c(\nu)}\right)^{\lambda}\right) = \kappa^{(\lambda)}_{\nu,\nu_0,f_c}(x)$$

for any $x \in [0,1]$, where

$$\frac{1}{f_c^{\lambda}(\nu)} = \sum_{i=1}^{n} \frac{1}{f_c^{\lambda}(\nu_i)}. \tag{14}$$

Let the function g_λ be given by

$$g_\lambda(x) = \frac{1}{f_c^{\lambda}(x)}$$

for any $x \in [0,1]$. Then, since f_c is a generator function of a conjunctive operator and $\lambda > 0$, g_λ is a generator function of a disjunctive operator in $[0,1]$. Next, Eq. (14) can be written as

$$g_\lambda(\nu) = \sum_{i=1}^{n} g_\lambda(\nu_i),$$

from which we have

$$\nu = g_\lambda^{-1}\left(\sum_{i=1}^{n} g_\lambda(\nu_i)\right). \tag{15}$$

Based on Definition 2, the right hand side of Eq. (15) is a disjunctive operator d_λ with the generator function $g_\lambda(x) = \frac{1}{f_c^{\lambda}(x)}$, where $x \in [0,1]$.

□

3.3 The kappa functions induced by product and Dombi operators

As we saw before, the kappa function is a generator function-dependent unary operator. Here, we will describe two special cases of the kappa function. First, we sill show the case where a generator function is that of the product (product) conjunctive and disjunctive operators. Then, we will describe the case where the kappa function is induced by a generator function of the Dombi operators.

3.3.1 The kappa function in the product operator case

Let f_c and f_d be generator functions of the product conjunctive and disjunctive operators, respectively. That is, functions $f_c, f_d\colon (0,1) \to (0,\infty)$ can be given by

$$f_c(x) = -\ln(x) \quad \text{and} \quad f_d(x) = -\ln(1-x).$$

Let $\nu, \nu_0 \in (0,1)$. After direct calculation, we get that the kappa functions $\kappa^{(\lambda)}_{\nu,\nu_0,f_c}$ and $\kappa^{(\lambda)}_{\nu,\nu_0,f_d}$ induced by f_c and f_d, respectively, are

$$\kappa^{(\lambda)}_{\nu,\nu_0,f_c}(x) = \nu_0^{\left(\frac{\ln(x)}{\ln(\nu)}\right)^{\lambda}} \quad \text{and} \quad \kappa^{(\lambda)}_{\nu,\nu_0,f_d}(x) = 1 - (1-\nu_0)^{\left(\frac{\ln(1-x)}{\ln(1-\nu)}\right)^{\lambda}},$$

respectively, where $x \in (0,1)$.

3.3.2 The kappa function in the Dombi operator case

The generator function of the Dombi conjunction and disjunction operators is the function $f_\alpha\colon [0,1] \to [0,\infty]$ that is given by

$$f_\alpha(x) = \left(\frac{1-x}{x}\right)^{\alpha},$$

where $\alpha \neq 0$. If $\alpha > 0$, then f_α is a generator function of a conjunctive operator; and if $\alpha < 0$, then f_α is a generator function of a disjunctive operator (see, e.g. [7]). Now, let $\nu, \nu_0 \in (0,1)$, $\alpha \neq 0$. Then, the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f_\alpha}$ induced by f_α is

$$\kappa^{(\lambda)}_{\nu,\nu_0,f_\alpha}(x) = \frac{1}{1 + \left(\left(\frac{1-\nu_0}{\nu_0}\right)^{\alpha}\left(\frac{\left(\frac{1-x}{x}\right)^{\alpha}}{\left(\frac{1-\nu}{\nu}\right)^{\alpha}}\right)^{\lambda}\right)^{\frac{1}{\alpha}}} = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{\nu}{1-\nu}\frac{1-x}{x}\right)^{\lambda}},$$

where $x \in [0,1]$. Here, we can see that $\kappa^{(\lambda)}_{\nu,\nu_0}$ is independent of the parameter α. That is, the kappa functions induced by generator functions of conjunctive and disjunctive Dombi operators coincide.

Remark. Notice that for any $\alpha \neq 0$, one of the functions f_α and $f_{-\alpha}$ is a generator function of a conjunctive operator and the other one is a generator function of a disjunctive operator. Since $f_\alpha(x)f_{-\alpha}(x) = 1$ holds for any $x \in [0,1]$, based on Proposition 3, we also get that the kappa functions induced by f_α and $f_{-\alpha}$ coincide for any $x \in [0,1]$.

It should be noted that the Dombi form of kappa function extended to the interval (a,b), and its transformed versions can be utilized in various areas of science. A few examples are as follows. The kappa function can be used to approximate the cumulative distribution function of the standard normal probability distribution (see [14]). The so-called kappa regression, which is based on the kappa function, may be viewed as an alternative to logistic regression (see [15]). A special case of the kappa function can be used as a probability weighting function in prospect theory (see [16]).

4 The kappa function and the unary operators

Now, we will show that certain types of unary operators in continuous-valued logic can be viewed as special cases of the kappa function. The results in this section will all be stated for a generator function of a conjunctive operator. We should add that choosing a generator function of a disjunctive operator leads to equivalent conclusions.

4.1 Pliant negation

Let $\nu, \nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and let f be a generator function of a conjunctive operator. Then, noting Definition 2 and Definition 3, the Pliant negation η_ν and the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f}$, both induced by f, are given by

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \quad \text{and} \quad \kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right),$$

where $x \in [0,1]$. From these expressions, we readily get that if $\lambda = -1$ and $\nu_0 = \nu$, then $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ is identical with the Pliant negation operator η_ν. This means that the Pliant negation may be treated as a special case of the kappa function.

4.2 Modalities and linguistic hedges

Here, we will show that certain modal operators and linguistic hedges are special cases of the kappa function. In [17] (also see [12]), we interpreted the concept of a dual pair of modal operators in continuous-valued logic based on the criteria for an algebraic version of dual necessity and possibility operators on De Morgan lattices given in [3] (also, see [4]).

Let $\nu, \nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and let f be a generator function of a conjunctive operator. Previously we proved (see Theorem 5 in [18]) that the functions $\underline{\tau}_{\nu,\nu_0}\colon [0,1] \to [0,1]$ and $\overline{\tau}_{\nu,\nu_0}\colon [0,1] \to [0,1]$, which are given by

$$\underline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right) \quad \text{and} \quad \overline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu)\frac{f(x)}{f(\nu_0)}\right)$$

for any $x \in [0,1]$ are a dual pair of modal operators in the above sense. In [18], we also demonstrated that the function $\underline{\tau}_{\nu,\nu_0}$ (respectively $\overline{\tau}_{\nu,\nu_0}$) may also be viewed as a linguistic hedge. Noting these results and the definition for a kappa function, we get that if $\lambda = 1$, then

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = \underline{\tau}_{\nu,\nu_0}(x) \quad \text{and} \quad \kappa^{(\lambda)}_{\nu_0,\nu,f}(x) = \overline{\tau}_{\nu,\nu_0}(x)$$

for any $x \in [0,1]$. That is, the above-mentioned modal and linguistic hedge operators are special cases of the kappa function.

4.3 Modifier operators induced by connectives

Following the approach presented by Mattila [30], we interpret the substantiating modifier operator and the weakening modifier operator as follows.

Definition. *We say that $\tau\colon [0,1] \to [0,1]$ is a substantiating modifier operator if and only if*

$$\tau(x) < x$$

and τ is a weakening modifier operator if and only if

$$\tau(x) > x$$

for any $x \in (0,1)$, and $\tau(0) = 0$, $\tau(1) = 1$.

Let $\nu, \nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and let f be a generator function of a conjunctive operator c. In [18] (also see [9]), we showed that by repeating the arguments of the conjunctive operator c n-times, and then extending n to a real-valued number, we get the function $\tau_{\nu,\nu_0}\colon [0,1] \to [0,1]$, which is given by

$$\tau_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right). \tag{16}$$

In [18], we also demonstrated that if $\nu > \nu_0$ ($\nu < \nu_0$, respectively), then τ_{ν,ν_0} is a substantiating (weakening, respectively) modifier operator. Taking into account the definition for a kappa function, we immediately get that the modifier operator τ_{ν,ν_0} is none other than the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ with $\lambda = 1$.

Notice that the substantiating and weakening operators and the modal and linguistic hedge operators in Section 4.2 all have the form of the function τ_{ν,ν_0} given in Eq. (16). Hence, all these operators may be viewed as kappa functions with the parameter value $\lambda = 1$.

4.4 The sharpness operator derived from the multiplicative aggregative operator

In [9], Dombi introduced an operator called the sharpness operator by repeating the arguments of the aggregative operator $a_\nu\colon [0,1]^n \to [0,1]$, which is given by

$$a_\nu(x_1, x_2, \ldots, x_n) = f^{-1}\left(f(\nu)\prod_{i=1}^{n}\frac{f(x_i)}{f(\nu)}\right),$$

where f is a generator function of a conjunctive operator and $\nu \in (0,1)$ (see [11]). Namely, by repeating the arguments of a_ν n-times, and then extending n to a real-valued number, we get the sharpness operator $\chi_\nu^{(\lambda)}\colon [0,1] \to [0,1]$, which is given by

$$\chi_\nu^{(\lambda)}(x) = f^{-1}\left(f(\nu)\left(\frac{f(x)}{f(\nu)}\right)^\lambda\right), \tag{17}$$

where $\nu \in (0,1)$ and $\lambda > 0$.

Using basic considerations and direct calculations, it can be shown that the sharpness operator $\chi_\nu^{(\lambda)}$ has the following properties ($\lambda > 0$):

(a) $\chi_\nu^{(\lambda)}(0) = 0$

(b) $\chi_\nu^{(\lambda)}(1) = 1$

(c) $\chi_\nu^{(\lambda)}(\nu) = \nu$

(d) If $0 < \lambda < 1$, then $\chi_\nu^{(\lambda)}(x) \geq x$ for any $x \in (0,\nu]$, and $\chi_\nu^{(\lambda)}(x) \leq x$ for any $x \in [\nu, 1)$

(e) if $\lambda = 1$, then $\chi_\nu^{(\lambda)}(x) = x$ for any $x \in [0,1]$

(f) If $1 < \lambda$, then $\chi_\nu^{(\lambda)}(x) \leq x$ for any $x \in (0,\nu]$, and $\chi_\nu^{(\lambda)}(x) \geq x$ for any $x \in [\nu, 1)$

(g) If a generator function f of $\chi_\nu^{(\lambda)}$ is differentiable, $\nu \in (0,1)$ and $f(\nu) = 1$, then

$$\left.\frac{\mathrm{d}\chi_\nu^{(\lambda)}(x)}{\mathrm{d}x}\right|_{x=\nu} = \lambda.$$

Notice that property (g) means that by changing the value of parameter λ, we can modify the sharpness of the operator $\chi_\nu^{(\lambda)}$. Also, parameter ν determines the point where $\chi_\nu^{(\lambda)}$ intersects the diagonal line. Figure 2 shows example plots of the sharpness operator $\chi_\nu^{(\lambda)}$ for the generator function $f(x) = \frac{1-x}{x}$, where $x \in [0,1]$. This figure also shows the tangent line of $\chi_\nu^{(\lambda)}$ at ν, i.e., $t_\nu(x) = \lambda x + \nu(1-\lambda)$.

Noting the definition for a kappa function, we get that if $\lambda > 0$ and $\nu = \nu_0$, then $\kappa_{\nu,\nu_0,f}^{(\lambda)}$

$$\kappa_{\nu,\nu_0,f}^{(\lambda)}(x) = \chi_\nu^{(\lambda)}(x)$$

for any $x \in [0,1]$. That is, in this case the kappa function $\kappa_{\nu,\nu_0,f}^{(\lambda)}$ is identical to the sharpness operator $\chi_\nu^{(\lambda)}$.

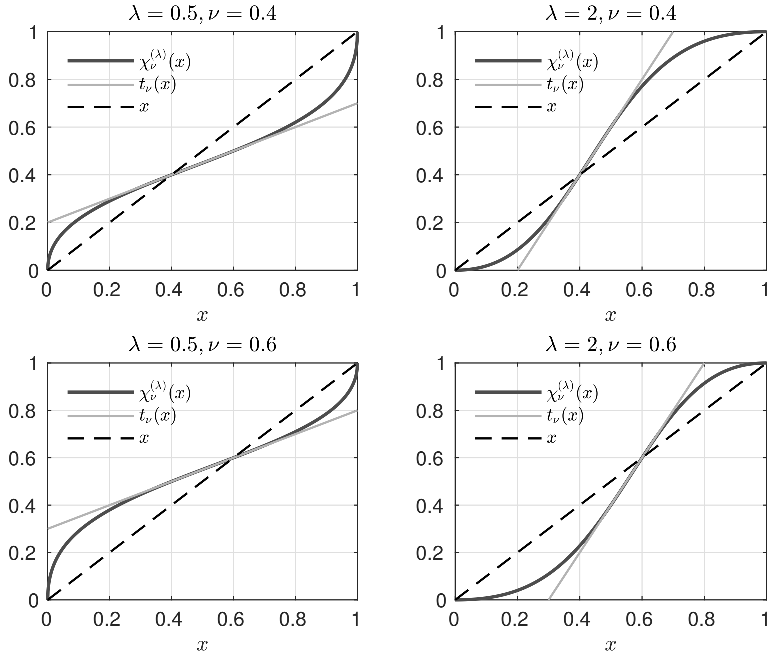

Figure 2: Example plots of the sharpness operator for $f(x) = \frac{1-x}{x}$.

4.5 The preference implication and the preference operator

Let $\nu \in (0,1)$ and let f be a generator function of a conjunctive operator. The preference implication operator $p_\nu : [0,1]^2 \to [0,1]$, which was introduced by Dombi and Baczyński in [13], is given by

$$p_\nu(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0),(1,1)\} \\ f^{-1}\left(f(\nu)\frac{f(y)}{f(x)}\right), & \text{otherwise,} \end{cases} \tag{18}$$

where $x, y \in [0,1]$. It can be shown that $p_\nu(x,y) \geq \nu$ if and only if $y \geq x$ (see Proposition 5 in [13]). Therefore, $p_\nu(x,y)$ may be viewed as a threshold-based implication for the continuous logical values x and y. This means that if the continuous logical value of the implication is greater that ν, then the implication is true. Also, $p_\nu(x,y)$ can be interpreted as the continuous logical value of the preference $x \leq y$.

Now, let $\nu, \nu_0 \in (0,1)$, let ν have a fixed value and let the operator $p_{\nu_0}(\nu;\cdot)\colon [0,1] \to [0,1]$ be given by

$$p_{\nu_0}(\nu;x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right). \tag{19}$$

Using Eq. (19), it can be shown that the following proposition is valid.

Proposition. *Let* $\nu \in (0,1)$ *have a fixed value. For any* $x \in [0,1]$,

$$p_{\nu_0}(\nu; x) \geq \nu_0 \quad \textit{if and only if} \quad x \geq \nu,$$

where $\nu_0 \in (0,1)$.

This means that $p_{\nu_0}(\nu; x)$ represents a threshold-based measure of the preference $x \geq \nu$, where $\nu \in (0,1)$ has a fixed value.

Notice that if $\lambda = 1$, then

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = p_{\nu_0}(\nu; x)$$

for any $x \in [0,1]$. That is, in this case, the kappa function $\kappa^{(\lambda)}_{\nu,\nu_0,f}$ coincides with the preference operator $p_{\nu_0}(\nu; \cdot)$ given in Eq. (19).

Remark. It is worth noting that following this line of thinking, we can extend the preference operator in Eq. (18) such that it also has a parameter λ:

$$p^{(\lambda)}_{\nu}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0),(1,1)\} \\ f^{-1}\left(f(\nu)\left(\frac{f(y)}{f(x)}\right)^{\lambda}\right), & \text{otherwise,} \end{cases}$$

where $x, y \in [0,1]$ and $\lambda > 0$. Here, the value of parameter λ determines the slope of the preference operator.

5 Conclusions and future research plans

In the previous sections, we demonstrated that (a) the Pliant negation operator η_ν given in Eq. (1); (b) the substantiating, weakening, modal and linguistic hedge operators τ_{ν,ν_0} given in Eq. (16); (c) the sharpness operator $\chi^{(\lambda)}_{\nu}$ given in Eq. (17); and (d) the preference operator $p_{\nu_0}(\nu; \cdot)$ given in Eq. (19) are special cases of the kappa function

$$\kappa^{(\lambda)}_{\nu,\nu_0,f}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right),$$

where $\nu, \nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and f is a generator function of a conjunctive or disjunctive operator. This means that the kappa function may be treated as a general unary modifier operator. Table 1 summarizes these results.

Remark. The unary operators listed in Table 1 are special cases of the kappa function. Therefore, by considering Proposition 3, we also have that the conjunctive and disjunctive forms of each of these unary operators coincide if $f_c(x)f_d(x) = 1$ holds for any $x \in [0,1]$, where f_c and f_d are generator functions of the conjunctive and disjunctive forms, respectively.

Table 1: Unary operators and the kappa function ($\nu, \nu_0 \in (0,1)$ and $\lambda \in \mathbb{R}$)

Parameters of $\kappa^{(\lambda)}_{\nu,\nu_0,f}$	Formula of $\kappa^{(\lambda)}_{\nu,\nu_0,f}$	The operator given by $\kappa^{(\lambda)}_{\nu,\nu_0,f}$
$\lambda = -1$, $\nu = \nu_0$	$f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right)$	negation (η_ν)
$\lambda = 1$	$f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right)$	substantiating, weakening, modal and linguistic hedge operators (τ_{ν,ν_0}); measure of the preference $x \geq \nu$ ($p_{\nu_0}(\nu;\cdot)$)
$\lambda > 0$, $\nu = \nu_0$	$f^{-1}\left(f(\nu)\left(\frac{f(x)}{f(\nu)}\right)^\lambda\right)$	sharpness operator ($\chi^{(\lambda)}_\nu$)

Example. In Section 3.3.2 we showed that the kappa function induced by the generator of Dombi operators (i.e. by $f(x) = \left(\frac{1-x}{x}\right)^\alpha$, $\alpha \neq 0$) is

$$\kappa^{(\lambda)}_{\nu,\nu_0}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{\nu}{1-\nu}\frac{1-x}{x}\right)^\lambda}, \quad x \in [0,1].$$

In this case, the unary operators in Table 1 have the following forms:

$$\eta_\nu(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}}, \qquad \tau_{\nu,\nu_0}(x) = p_{\nu_0}(\nu;x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\frac{\nu}{1-\nu}\frac{1-x}{x}}$$

$$\chi^{(\lambda)}_\nu(x) = \frac{1}{1 + \frac{1-\nu}{\nu}\left(\frac{\nu}{1-\nu}\frac{1-x}{x}\right)^\lambda}, \quad x \in [0,1].$$

5.1 Future research plans

In our future research, we intend to study how the kappa function can be utilized as an activation function in artificial neural networks. We also would like to develop a kappa function-based decision supporting method.

References

[1] Aczél, J. *Lectures on functional equations and their applications.* Academic Press, 1966. ISBN: 9780080955254.

[2] Banerjee, M. and Dubois, D. A simple logic for reasoning about incomplete knowledge. *International Journal of Approximate Reasoning*, 55(2):639–653, 2014. DOI: 10.1016/j.ijar.2013.11.003.

[3] Cattaneo, G., Ciucci, D., and Dubois, D. Algebraic models of deviant modal operators based on De Morgan and Kleene lattices. *Information Sciences*, 181(19):4075–4100, 2011. DOI: 10.1016/j.ins.2011.05.008.

[4] Chellas, B. F. *Modal logic: An introduction*. Cambridge University Press, 1980. DOI: 10.1017/CBO9780511621192.

[5] Cintula, P., Klement, E. P., Mesiar, R., and Navara, M. Fuzzy logics with an additional involutive negation. *Fuzzy Sets and Systems*, 161:390–411, 2010. DOI: 10.1016/j.fss.2009.09.003.

[6] De Cock, M. and Kerre, E. E. Fuzzy modifiers based on fuzzy relations. *Information Sciences*, 160(1-4):173–199, 2004. DOI: 10.1016/j.ins.2003.09.002.

[7] Dombi, J. Towards a general class of operators for fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 16(2):477–484, 2008. DOI: 10.1109/TFUZZ.2007.905910.

[8] Dombi, J. De Morgan systems with an infinitely many negations in the strict monotone operator case. *Information Sciences*, 181(8):1440–1453, 2011. DOI: 10.1016/j.ins.2010.11.038.

[9] Dombi, J. On a certain type of unary operators. In *IEEE International Conference on Fuzzy Systems*, pages 1–7, 2012. DOI: 10.1109/FUZZ-IEEE.2012.6251349.

[10] Dombi, J. Pliant operator system. In Fodor, J., Klempous, R., and Suárez Araujo, C. P., editors, *Recent Advances in Intelligent Engineering Systems*, pages 31–58. Springer Berlin Heidelberg, 2012. DOI: 10.1007/978-3-642-23229-9_2.

[11] Dombi, J. On a certain class of aggregative operators. *Information Sciences*, 245(1):313–328, 2013. DOI: 10.1016/j.ins.2013.04.010.

[12] Dombi, J. Modalities based on double negation. In Halaš, R., Gagolewski, M., and Mesiar, R., editors, *New Trends in Aggregation Theory*, pages 327–338. Springer, 2019. DOI: 10.1007/978-3-030-19494-9_30.

[13] Dombi, J. and Baczyński, M. General characterization of implication's distributivity properties: The preference implication. *IEEE Transactions on Fuzzy Systems*, 28(11):2982–2995, 2019. DOI: 10.1109\/TFUZZ.2019.2946517.

[14] Dombi, J. and Jónás, T. Approximations to the normal probability distribution function using operators of continuous-valued logic. *Acta Cybernetica*, 23(3):829–852, 2018. DOI: 10.14232/actacyb.23.3.2018.7.

[15] Dombi, J. and Jónás, T. Kappa regression: An alternative to logistic regression. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 28(02):237–267, 2020. DOI: 10.1142/S0218488520500105.

[16] Dombi, J. and Jónás, T. Towards a general class of parametric probability weighting functions. *Soft Computing*, 24:15967–15977, 2020. DOI: 10.1007/s00500-020-05335-3.

[17] Dombi, J. and Jónás, T. On a strong negation-based representation of modalities. *Fuzzy Sets and Systems*, 407:142–160, 2021. DOI: 10.1016/j.fss.2020.10.005.

[18] Dombi, J. and Jónás, T. A unified approach to four important classes of unary operators. *International Journal of Approximate Reasoning*, 133:80–94, 2021. DOI: 10.1016/j.ijar.2021.03.007.

[19] Esteva, F., Godo, L., Hájek, P., and Navara, M. Residuated fuzzy logics with an involutive negation. *Archive for Mathematival Logic*, 39(2):103–124, 2000. DOI: 10.1007/s001530050006.

[20] Esteva, F., Godo, L., and Noguera, C. A logical approach to fuzzy truth hedges. *Information Sciences*, 232:366–385, 2013. DOI: 10.1016/j.ins.2012.12.010.

[21] Fodor, J. and Roubens, M. *Fuzzy preference modelling and multicriteria decision support*, Volume 14. Springer Science & Business Media, 1994. DOI: 10.1007/978-94-017-1648-2.

[22] Huynh, V.-N., Ho, T. B., and Nakamori, Y. A parametric representation of linguistic hedges in Zadeh's fuzzy logic. *International Journal of Approximate Reasoning*, 30(3):203–223, 2002. DOI: 10.1016/S0888-613X(02)00075-0.

[23] Jain, M., Madeira, A., and Martins, M. A. A fuzzy modal logic for fuzzy transition systems. *Electronic Notes in Theoretical Computer Science*, 348:85–103, 2020. DOI: 10.1016/j.entcs.2020.02.006.

[24] Klement, E. P., Mesiar, R., and Pap, E. *Triangular Norms*. Trends in Logic. Springer Netherlands, 2013. DOI: 10.1007/978-94-015-9540-7.

[25] Klir, G. J. and Yuan, B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1995. ISBN: 9780131011717.

[26] Łukasiewicz, J. On the concept of possibility. *Ruch Filozoficzny*, 5:169–170, 1920.

[27] Łukasiewicz, J. On three-valued logic. *Ruch Filozoficzny*, 5:170–171, 1920.

[28] Łukasiewicz, J. Two-valued logic. *Przeglad Filozoficzny*, 23:189–205, 1921.

[29] Mattila, J. K. Modifier logics based on graded modalities. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 7(2):72–78, 2003. URL: https://www.academia.edu/download/115206070/Modifier_logics_based_on_graded_modalities_JACIII_2003_excerpt.pdf.

[30] Mattila, J. K. Modifiers based on some t-norms in fuzzy logic. *Soft Computing*, 8(10):663–667, 2004. DOI: 10.1007/s00500-003-0323-x.

[31] Rubin, S. H. Computing with words. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 29(4):518–524, 1999. DOI: 10.1109/3477.775267.

[32] Trillas, E. Sobre funciones de negación en la teoría de conjuntos difusos. *Stochastica*, 3(1):47–60, 1979. URL: https://eudml.org/doc/38807.

[33] Vidal, A. On transitive modal many-valued logics. *Fuzzy Sets and Systems*, 407:97–114, 2021. DOI: 10.1016/j.fss.2020.01.011.

[34] Yan, L., Pei, Z., and Ren, F. Constructing and managing multi-granular linguistic values based on linguistic terms and their fuzzy sets. *IEEE Access*, 7:152928–152943, 2019. DOI: 10.1109/ACCESS.2019.2948847.

Received 21th November 2022