Some measure problems concerning the retrospective sequential functions

By L. Klukovits

In this paper we shall study the mappings of the set of all infinite sequences the terms of which are from a given set $\mathfrak{X}$, so-called retrospective sequential functions. These are some mappings of $\mathfrak{X}^N$ into itself. We shall define a measure on the set $\mathfrak{X}^N$ in natural way so that the measure of the range of a retrospective sequential function may be considered as the measure of the maintenance of information by the automaton which realises it.

We shall prove, that a retrospective sequential funtion is measure-preserving if and only if it is an onto mapping (Theorem 2). After this we shall show that, although $\mathfrak{X}^N$ has non-measurable subsets (Theorem 3), the ranges of finite state retrospective sequential functions (namely which can be realised by a finite automaton) are all measurable (Theorem 4), and the corresponding measures can equal any rational numbers between zero and one (Theorem 5).

Finally, besides some remarks we shall illustrate by giving examples that among the algebraic and metric as well as measure-theoretic properties of the retrospective sequential functions we cannot expect close connection.

§ 1. Some fundamental concepts and notations

Let $\mathfrak{X}$ be a non-empty finite set. We shall denote by $\{\mathfrak{X}\}$ the set of all finite sequences (shortly: words), whose terms are from $\mathfrak{X}$. The elements of $\{\mathfrak{X}\}$ will be denoted by $p, q, \ldots$ and the elements of $\mathfrak{X}$ by $x, y, \ldots$. We remark that the empty sequence is an element of $\{\mathfrak{X}\}$. The *length* of the word $p = x_1 x_2 \ldots x_n$ is the natural number n $\big(l(p) = n\big)$, the length of the empty sequence is zero.

$\{\mathfrak{X}\}_{(k)}$ and $\{\mathfrak{X}\}_k$ will denote the set of all words composed by elements of $\{\mathfrak{X}\}$, the length of which is at most k and exactly k, respectively.

If $\mathfrak{M}, \mathfrak{N} \subseteqq \{\mathfrak{X}\}$, then

$$\mathfrak{M}\mathfrak{N} = \langle pq \,|\, p \in \mathfrak{M} \wedge q \in \mathfrak{N} \rangle.$$

The set of all ω-type sequences (shortly: sequences), whose terms are from $\mathfrak{X}$, will be denoted by $\mathfrak{X}^N$, and the elements of $\mathfrak{X}^N$ will be denoted by Greek letters. $\xi = \xi(1)\,\xi(2)\ldots$, where $\xi(i)$ $\big(\xi(i) \in \mathfrak{X}\big)$ is the i-th element of ξ will be used for the detailed description of the sequences. We shall use the same description also for the words.

If $p\in\{\mathfrak{X}\}$, $\alpha\in\mathfrak{X}^N$ then $p\alpha$ will denote that sequence ξ which satisfies the following conditions:

$$\xi(i)=p(i) \quad \text{if} \quad i\leqq l(p)$$

and

$$\xi(j)=\alpha(j-l(p)) \quad \text{if} \quad j>l(p).$$

If $\mathfrak{M}\subseteqq\{\mathfrak{X}\}$, $\mathfrak{N}\subseteqq\mathfrak{X}^N$ then

$$\mathfrak{M}\mathfrak{N}=\langle p\alpha|p\in\mathfrak{M}\wedge\alpha\in\mathfrak{N}\rangle;$$

if $\mathfrak{M}=\langle p\rangle$ then we shall write instead of $\mathfrak{M}\mathfrak{N}$ the symbol $p\mathfrak{N}$. ($\langle\ldots\rangle$ denotes set.)

A mapping of $\mathfrak{X}^N$ into itself will be called a *sequential function* (shortly: sf). In this paper we shall denote the sequential functions by Latin capitals.

If $\mathfrak{N}\subseteqq\mathfrak{X}^N$, then $F\mathfrak{N}$ denotes the set of all sequences, which may be written in the form $F\alpha(\alpha\in\mathfrak{N})$.

For any natural number n let us define the sf D_n in the following way

$$(D_n\alpha)(i)=\alpha(i+n)$$

for all $\alpha\in\mathfrak{X}^N$ and any natural number i.

A sf F will be called *retrospective* (shortly: rsf) (see [3]), if for any $p\in\{\mathfrak{X}\}$ there is a $q\in\{\mathfrak{X}\}$ such that $l(p)=l(q)$ and $F(p\mathfrak{X}^N)\subseteqq q\mathfrak{X}^N$ hold. It is easy to see that q is uniquely defined by p.

For all rsf F there exists a mapping $\bar{F}$ of $\{\mathfrak{X}\}$ into itself. $\bar{F}$ is defined in the following way: for any $p\in\{\mathfrak{X}\}$ let $\bar{F}p=q$ if $F(p\mathfrak{X}^N)\subseteqq q\mathfrak{X}^N$. This mapping $\bar{F}$ is called an automaton mapping (see [4]). It is easy to verify that for any $p, r, s\in\{\mathfrak{X}\}$, $r\neq s$, $(\bar{F}(pr))(i)=(\bar{F}(ps))(i)$ if $i\leqq l(p)$ holds.

Let F be any rsf. For an arbitrary word $p(\in\{\mathfrak{X}\})$ we shall define the sf F_p in the following way:

$$F_p\alpha=D_{l(p)}F(p\alpha)$$

for any $\alpha\in\mathfrak{X}^N$.

We know that for any $p\in\{\mathfrak{X}\}$, F_p is a rsf (see [3]). A rsf which can be written in the form F_p, is called a *state* of F. Of course it is possible that $F_p=F_q$ in spite of $q\neq p$. If all the different states of an rsf constitute a finite set, then it will be called a *finite state retrospective sequential function* (shortly: fsrsf). We shall call the the family

$$\Sigma_{F,k}=\langle F_p|\,l(p)=k\rangle$$

of the states of F the k-th *level* of F. It is easy to see that the set of all states of F coincides with the set of all rsf-s G, for which $G\in\Sigma_{F,i}$ holds for some i ($i=0, 1, 2,\ldots$). This set is denoted by Σ_F.

We know that $(F_p)_q=F_{pq}$ holds for any words p, q (see [3]). It follows that $\Sigma_{F_p}\subseteqq\Sigma_F$ is true for all $p\in\{\mathfrak{X}\}$.

The reader can prove easily that any state of any rsf is also retrospective. It is also easy to verify that the following assertions are equivalent

(i) $\bar{F}$ is one-to-one
(ii) $\bar{F}$ is onto.

Furthermore if F is onto, then F is one-to-one. A simple counter-example ($F:\alpha\to x\alpha;\ x\in\mathfrak{X}$) shows that the converse of this proposition if false. In the sequel if it does not make any misunderstanding, we shall write F instead of $\bar{F}$.

If the rsf F is not one-to-one, then there exists a word $p\in\{\mathfrak{X}\}$ such that $p\notin F\{\mathfrak{X}\}$. The length of the shortest word p which satisfies the preceding condition is denoted by $d(F)$.

The cardinal number of the set $\mathfrak{A}$ is denoted in this paper by $|\mathfrak{A}|$. We shall call the word p an *initial segment* of the sequence α, if

$$p(i)=\alpha(i)$$

for all $i\leqq l(p)$.

§ 2. Definition of a measure on the set $\mathfrak{X}^N$

Consider those subsets of $\mathfrak{X}^N$, which may be written in the form

$$p\mathfrak{X}^N$$

where p is arbitrary word. Enlarging the set of these subsets with the empty set, we get a semiring of sets, which we shall denote by $S(\mathfrak{X})$. The unit element of $S(\mathfrak{X})$ is $\mathfrak{X}^N$. Let us define a set function μ on the elements of $S(\mathfrak{X})$ in the following way:

(2.1) $$\mu(p\mathfrak{X}^N)=\frac{1}{n^{l(p)}}$$

where $n=|\mathfrak{X}|$. It is easy to see that this function is a measure (see [1]). We assert that this measure is σ-additive.

It is known that we can continue this measure so that its domain will be the minimal ring over the semiring $S(\mathfrak{X})$ (this ring will be denoted by $\mathfrak{R}(S(\mathfrak{X}))$), and the continuation of μ will be also σ-additive (see [1]).

In the next section we shall often refer to the definition of the Lebesgue measure, and a proposition on measurability. They are the following. Let a σ-additive measure m be given on some semiring of sets S_m with unit E. We shall define on the system $\mathfrak{S}$ of all subsets of the set E two functions $\mu^*(A)$ and $\mu_*(A)$ in the following way

Definition 1. The number

$$\mu^*(A)=\inf\sum_n m(B_n)$$

$$A\subseteqq\bigcup_n B_n$$

where the greatest lower bound is taken over all coverings of the set A by finite or countable systems of sets $B_n\in S_m$, is called the *outer measure* of the set $A\subseteqq E$.

Definition 2. The number

$$\mu_*(A)=m(E)-\mu^*(E\setminus A)$$

is called the *inner measure* of the set $A\subseteqq E$.

It is easy to see that $\mu_*(A)\leqq\mu^*(A)$ holds for every $A\subseteqq E$.

Definition 3. The set $A\subseteqq E$ is measurable (Lebesgue), if

$$\mu_*(A)=\mu^*(A).$$

If A is measurable, then we shall denote the common value $\mu_*(A)=\mu^*(A)$ by $\mu(A)$ and call it the (Lebesgue) measure of the set A. The next definition is equivalent to definition 3.

Definition 3'. The set $A \subseteq E$ is called measurable, if

$$\mu^*(A)+\mu^*(E\setminus A) = m(E).$$

Proposition 1. For the measurability of the set $A(\subseteq E)$ the following condition is necessary and sufficient: for any $\varepsilon>0$ there exists a $B\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(A\triangle B)<\varepsilon.$$

The proof of this theorem can be found for example in [1].

§ 3. The main results

First we prove a theorem which presents an analogy with some problems of the theory of real functions.

Theorem 1. If two retrospective sequential functions differ only on a set of measure zero then these functions are equal.

Proof. We shall prove the following assertion, equivalent to Theorem 1: if the rsf-s F and G are not equal, then there exists a set $\mathfrak{A}$, such that $\mu(\mathfrak{A})>0$ and

$$F\mathfrak{A}\cap G\mathfrak{A} = \emptyset.$$

Let $\alpha=\alpha(1)\alpha(2)\ldots$ be any sequence from $\mathfrak{X}^N$, for which there is a natural number i such that

$$(F\alpha)(i)\neq(G\alpha)(i).$$

After this consider the set $p\mathfrak{X}^N$, where $p=\alpha(1)\ldots\alpha(i)$. According to the definition of measure

$$\mu(p\mathfrak{X}^N) = \frac{1}{n^i}>0$$

$(n=|\mathfrak{X}|)$ and it is clear that

$$F(p\mathfrak{X}^N)\cap G(p\mathfrak{X}^N) = \emptyset.$$

Q. E. D.

Lemma 1. If the range of a rsf F is measurable, then the image of any measurable set under F is also measurable.

Proof. The reader can verify that, if the range of F is measurable, then $F\mathfrak{C}$ — where $\mathfrak{C}\in\mathfrak{R}(S(\mathfrak{X}))$ — is also measurable. Let now $\mathfrak{A}(\subseteq\mathfrak{X}^N)$ be any measurable set. According to the proposition 1, for any $\varepsilon>0$ there exists a set $\mathfrak{B}\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(\mathfrak{A}\triangle\mathfrak{B})<\varepsilon. \tag{3.1}$$

Let $\varepsilon > 0$ be an arbitrary number and $\mathfrak{B} \in \mathfrak{R}(S(\mathfrak{X}))$ a set satisfying (3. 1). It is simple to verify that

$$F\mathfrak{A} \triangle F\mathfrak{B} \subseteqq F(\mathfrak{A} \triangle \mathfrak{B})$$

is valid. Take the outer measure of both sides:

$$\mu^*(F\mathfrak{A} \triangle F\mathfrak{B}) \leqq \mu^*(F(\mathfrak{A} \triangle \mathfrak{B})) \leqq \mu^*(\mathfrak{A} \triangle \mathfrak{B}) < \varepsilon.$$

The reader can easily verify that

(3. 2) $$|\mu^*(F\mathfrak{A}) - \mu^*(F\mathfrak{B})| \leqq \mu^*(F\mathfrak{A} \triangle F\mathfrak{B}) < \varepsilon,$$

and since

$$(\mathfrak{X}^N \setminus F\mathfrak{A}) \triangle (\mathfrak{X}^N \setminus F\mathfrak{B}) = F\mathfrak{A} \triangle F\mathfrak{B},$$

we obtain

(3. 3) $$|\mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) - \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B})| < \varepsilon.$$

From (3. 2) and (3. 3) we get

(3. 4) $$|\mu^*(F\mathfrak{A}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) - (\mu^*(F\mathfrak{B}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B})| < 2\varepsilon.$$

Since $F\mathfrak{B}$ is measurable,

$$\mu^*(F\mathfrak{B}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B}) = \mu(\mathfrak{X}^N).$$

Since ε was arbitrary, it follows from (3. 4) that

$$\mu^*(F\mathfrak{A}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) = \mu(\mathfrak{X}^N)$$

and this means that the image of the set $\mathfrak{A}$ under F is measurable.

In the following theorem we give a necessary and sufficient condition in order that a retrospective sequential function be measure-preserving.

Theorem 2. A rsf is measure-preserving if and only if it is an onto mapping.

Proof.

Sufficiency: First let $\mathfrak{A} \in S(\mathfrak{X})$, i.e.

$$\mathfrak{A} = p\mathfrak{X}^N$$

where $p \in \{\mathfrak{X}\}$. In this case

(3. 5) $$F\mathfrak{A} = q\mathfrak{B}$$

where $q = Fp$ and $\mathfrak{B} \subseteqq \mathfrak{X}^N$. Since F is onto (and a fortiori one-to-one), its inverse F^{-1} exists, which is a rsf (see [3]) and

$$F^{-1}(q\mathfrak{X}^N) \subseteqq p\mathfrak{X}^N.$$

By (3. 5), it follows from this relation that

$$q\mathfrak{X}^N \subseteqq q\mathfrak{B},$$

i. e.

$$\mathfrak{X}^N \subseteqq \mathfrak{B}.$$

We have obtained, that $\mathfrak{B} \subseteqq \mathfrak{X}^N$ and $\mathfrak{X}^N \subseteqq \mathfrak{B}$, from which

$$\mathfrak{B} = \mathfrak{X}^N$$

follows and so for any $\mathfrak{A} \in S(\mathfrak{X})$ the sufficiency of the condition is proved. Now let $\mathfrak{A}$ be an arbitrary measurable set. According to the precedings we obtain

$$\mu^*(F\mathfrak{A}) = \mu^*(\mathfrak{A}) = \mu(\mathfrak{A}).$$

The set $F\mathfrak{A}$ is measurable by the lemma 1, and so

$$\mu^*(F\mathfrak{A}) = \mu(F\mathfrak{A}).$$

The sufficiency of the condition is proved.

Necessity: First we suppose that F is not one-to-one. In this case there are such sequences $\alpha, \beta \in \mathfrak{X}^N$ $(\alpha \neq \beta)$ for which

$$F\alpha = F\beta.$$

Let

$$\alpha = p\bar{\alpha} = px\xi,$$

$$\beta = p\bar{\beta} = py\zeta$$

where $p \in \{\mathfrak{X}\}$ (it is possible that $l(p)=0$), $x, y \in \mathfrak{X}$ but $x \neq y$ and $\bar{\alpha}, \bar{\beta}, \xi, \zeta \in \mathfrak{X}^N$. According to the assumption we obtain

$$F(px\xi) = qz\eta \tag{3.6}$$

and

$$F(py\zeta) = qz\eta \tag{3.7}$$

where $q \in \{\mathfrak{X}\}$ $(l(q)=l(p))$, $z \in \mathfrak{X}$ and $\eta \in \mathfrak{X}^N$. Consider the sets

$$px\mathfrak{X}^N \quad \text{and} \quad py\mathfrak{X}^N$$

which are disjoint elements of $S(\mathfrak{X})$. According to (3. 6) and (3. 7) we obtain

$$F(px\mathfrak{X}^N) = qz\mathfrak{A}$$

and

$$F(py\mathfrak{X}^N) = qz\mathfrak{B}$$

where $\mathfrak{A}, \mathfrak{B} \subseteqq \mathfrak{X}^N$ and

$$\mu(qz\mathfrak{A}) = \mu(qz\mathfrak{B}) = \mu(px\mathfrak{X}^N) = n^{-(l(p)+1)}$$

$(n=|\mathfrak{X}|)$. We remark that $\mu(\mathfrak{A}) = \mu(\mathfrak{B}) = 1$ and so $\mu(\mathfrak{A} \triangle \mathfrak{B}) = 0$. In this case we can suppose by the Theorem 1 that

$$\mathfrak{A} = \mathfrak{B}.$$

After these we consider the set $F(px\mathfrak{X}^N \cup py\mathfrak{X}^N)$, which is equal to $qz\mathfrak{A}$. We have a contradiction, since

$$\mu(px\mathfrak{X}^N \cup py\mathfrak{X}^N) = 2n^{-(l(p)+1)}$$

and

$$\mu(qz\mathfrak{A}) = n^{-(l(p)+1)}.$$

This makes the first part of the proof complete because $px\mathfrak{X}^N \cup py\mathfrak{X}^N$ is measurable.

Now let the rsf F be one-to-one but not onto, and $\alpha \in \mathfrak{X}^N \setminus F\mathfrak{X}^N$. We assert that there exists a natural number k such that

$$x_1 x_2 \dots x_k \mathfrak{X}^N \not\subseteq F\mathfrak{X}^N$$

where $x_i = \alpha(i)$ $(i = 1, 2, \dots, k)$. In the contrary case there exists a sequence of elements $\alpha_j \in F\mathfrak{X}^N$ $(j = 1, 2, \dots)$ such that $\alpha_j(i) = \alpha(i)$ whenever $1 \leqq i \leqq j$. Consider the sequence $\beta_j \in \mathfrak{X}^N$ $(j = 1, 2, \dots)$ defined by the equations

$$F\beta_j = \alpha_j.$$

Form the initial-segments of length j for every β_j. From these segments we can construct a sequence β, for which

$$F\beta = \alpha$$

holds. This is a contradiction. If the natural number k above exists, then we have relation

$$x_1 x_2 \dots x_k \mathfrak{X}^N \subseteq \mathfrak{X}^N \setminus F\mathfrak{X}^N.$$

We know that

$$\mu(\mathfrak{X}^N \setminus F\mathfrak{X}^N) = 0$$

because the rsf F is measure preserving. We see that our preceding relation contradicts to the measure preserving of the rsf F. The proof is complete.

We may ask the following question: Are all the subsets of $\mathfrak{X}^N$ measurable? The answer is negative. The construction which leads to a non-measurable subset of $\mathfrak{X}^N$ is analogous to that of Zermelo concerning the interval [0, 1] (see [2]).

Theorem 3. The set $\mathfrak{X}^N$ has non-measurable subsets.

Proof: Let us define a relation on $\mathfrak{X}^N$: $\alpha \sim \beta$ $(\alpha, \beta \in \mathfrak{X}^N)$ if and only if there exists an onto fsrsf F, for which

$$F\alpha = \beta.$$

This relation is an equivalence relation, and let the equivalence classes be $\mathfrak{A}_\iota$ $(\iota \in I)$. We choose one and only one element from every set $\mathfrak{A}_\iota$, and we denote the set of these elements by $\mathfrak{B}$. We shall show that $\mathfrak{B}$ is not measurable.

Consider the sequence of all onto fsrsf-s

$$F_1, F_2, \dots$$

and let

$$\mathfrak{B}_j = F_j \mathfrak{B} \qquad j = 1, 2, \dots.$$

The sets $\mathfrak{B}_j$ $(j = 1, 2, \dots)$ satisfy the condition

$$\mathfrak{B}_j \cap \mathfrak{B}_k = \emptyset \qquad j \neq k.$$

Namely, if $\alpha \in \mathfrak{B}_j \cap \mathfrak{B}_k$, then $F_j^{-1}\alpha$, $F_k^{-1}\alpha \in \mathfrak{B}$ and

$$F_j^{-1} F_k F_k^{-1} \alpha = G F_k^{-1} \alpha = F_j^{-1} \alpha$$

so $F_j^{-1}\alpha \sim F_k^{-1}\alpha$. This is a contradiction.

If the set $\mathfrak{B}$ is measurable, then the sets $\mathfrak{B}_j$ $(j=1, 2, \ldots)$ are measurable and $\mu(\mathfrak{B})=\mu(\mathfrak{B}_j)$ by Theorem 2. Furthermore,

$$\mathfrak{X}^N = \bigcup_{j=1}^{\infty} \mathfrak{B}_j.$$

In fact, for any $\alpha \in \mathfrak{X}^N$ one and only one index $\iota\,(\in I)$ exists such that $\alpha \in \mathfrak{A}_\iota$. Let $\beta = \mathfrak{A}_\iota \cap \mathfrak{B}$. On the basis of the definition of the equivalence classes $\mathfrak{A}_\iota$ there is an onto fsrsf H, such that

$$\alpha = H\beta.$$

Since H is among $F_1, F_2, \ldots$, there exists a natural number m, such that $\alpha \in \mathfrak{B}_m$ holds.

The series $\sum_{j=1}^{\infty} \mu(\mathfrak{B}_j)$ is convergent. Furthermore, $\mu(\mathfrak{B}_1)=\mu(\mathfrak{B}_2)=\ldots$ whence it follows that $\mu(\mathfrak{B}_j)=0$ for $j=1, 2, \ldots$. Because

$$\sum_{j=1}^{\infty} \mu(\mathfrak{B}_j) = \mu(\mathfrak{X}^N) = 1,$$

we have a contradiction, thus the proof is complete.

In the following we look for an answer to the question: for which classes of rsf will be the range measurable. We shall obtain that the range of any fsrsf is measurable.

Lemma 2. The range of any fsrsf without one-to-one state is measurable.

Proof. For any non-negative integer t let us define the subsets $\Lambda_{1,t}$, $\Lambda_{2,t}$ of $\Sigma_{F,t}$ in the following way: let $F_{p_1} \in \Lambda_{1,t}$ $\big(l(p_1)=t\big)$ if and only if there exist words $p_2, \ldots, p_k \in \{\mathfrak{X}\}_t$ such that

$$Fp_1 = Fp_2 = \ldots = Fp_k$$

and

$$\bigcup_{i=1}^{k} F_{p_i} \mathfrak{X}^N = \mathfrak{X}^N$$

and let $\Lambda_{2,t} = \Sigma_{F,t} \setminus \Lambda_{1,t}$. We see, if $F_{p_1} \in \Lambda_{1,t}$ and $l(q)=u$, then $F_{p_1 q} \in \Lambda_{1,t+u}$.

In fact, in this case

$$\bigcup_{i=1}^{k} F_{p_i} \mathfrak{X}^N = \mathfrak{X}^N \supset (F_{p_1} q)\mathfrak{X}^N,$$

therefore, for all $r \in \{\mathfrak{X}\}_u$ for which

$$F_{p_i} r = F_{p_i} q \quad (i=1, 2, \ldots, k),$$

we obtain

$$\bigcup_r F_{p_i r} \mathfrak{X}^N = \mathfrak{X}^N,$$

and for any such word

$$F(p_i r) = (Fp_i)(F_{p_i} r) = (Fp_1)(F_{p_1} r) = F(p_1 q).$$

Let us denote by $\mathfrak{B}_t$ the set of all sequences $\alpha \in F\mathfrak{X}^N$, for which there exists a word $p \in \{\mathfrak{X}\}_t$ such that $F_p \in \Lambda_{2,t}$ and $Fp = \alpha(1)\ldots\alpha(t)$. We shall show that if $t < v$, then

$\mathfrak{B}_v \subseteq \mathfrak{B}_t$. In fact, let $\alpha \in \mathfrak{B}_v$. Now there is a $q \in \{\mathfrak{X}\}_v$, such that $F_q \in \Lambda_{2,v}$ and $Fq = \alpha(1)\ldots\alpha(v)$. In this case for the word $r = q(1)\ldots q(t)$ we obtain $F_r \in \Lambda_{2,t}$ (otherwise it follows, that $F_q \in \Lambda_{1,v}$), and, owing to the retrospectivity of F

$$Fr = \alpha(1)\ldots\alpha(t),$$

and so $\alpha \in \mathfrak{B}_t$ holds.

Let $\Gamma = \langle G_1, \ldots, G_m \rangle$ be any set of states of F. If $\bigcup_{i=1}^{m} G_i \mathfrak{X}^N \subset \mathfrak{X}^N$, then there exists a natural number $f = f(\Gamma)$ such that

$$\bigcup_{i=1}^{m} G_i \{\mathfrak{X}\}_f \subset \{\mathfrak{X}\}_f. \tag{3.8}$$

In fact, if for all natural numbers j

$$\bigcup_{i=1}^{m} G_i \{\mathfrak{X}\}_j = \{\mathfrak{X}\}_j,$$

then for any $\alpha \in \mathfrak{X}^N$ there is a G_i such that the sets $G_i\{\mathfrak{X}\}_j$ $(j = 1, 2, \ldots)$ contain an infinite number of initial segments of α, and — since G_i is retrospective — these sets contain all initial segments of α, because Γ is a finite set. Let now

$$p_1, p_2, \ldots, p_j, \ldots \tag{3.9}$$

be words such that

$$G_i p_j = \alpha(1)\ldots\alpha(j).$$

Since $\mathfrak{X}$ is a finite set, (3. 9) has a sub-sequence

$$p_{11}, p_{12}, \ldots, p_{1j}, \ldots \tag{3.10}$$

such that all words in (3. 10) begin with the same letter x_1. An easy induction shows that, for any natural number k, (3. 9) has a sub-sequence

$$p_{k1}, p_{k2}, \ldots, p_{kj}, \ldots$$

such that all words in this sequence begin with the same word $x_1 \ldots x_k \in \{\mathfrak{X}\}_k$. In this way we define a sequence

$$\xi = x_1 \ldots x_k \ldots \in \mathfrak{X}^N$$

which for any k satisfies the condition

$$G_i(x_1 \ldots x_k) = \alpha(1)\ldots\alpha(k).$$

So we have $G_i \xi = \alpha$, namely $\alpha \in G_i \mathfrak{X}^N$ and $\alpha = \bigcup_{i=1}^{m} G_i \mathfrak{X}^N$. We obtain

$$\mathfrak{X}^N \subseteq \bigcup_{i=1}^{m} G_i \mathfrak{X}^N.$$

namely $\bigcup_{i=1}^{m} G_i \mathfrak{X}^N = \mathfrak{X}^N$ and thus, we have a contradiction.

If there exists a natural number $f=f(\Gamma)$, for which (3. 8) holds, then there exists at least such a number and let us denote this number by f. If $\bigcup_{i=1}^{m} G_i\mathfrak{X}^N=\mathfrak{X}^N$ holds, then let $f(\Gamma)=0$.

Let now

$$k = \max_{\Gamma\subseteqq\Sigma_F} f(\Gamma).$$

(If Γ has only one element, $\Gamma=\langle G\rangle$, then according to the definition $f(\Gamma)=d(G)$. So for any state G of F, $k\geqq d(G)$ holds.)

We shall denote by B_t the set of all initial segments of the sequences from $\mathfrak{B}_t$, whose lengths are equal to t. Furthermore B_t contains all the words q $\big(l(q)=t\big)$ for which there is a word p $\big(l(p)=t\big)$ such that

$$F_p\in A_{2,t} \quad \text{and} \quad Fp=q.$$

We shall show that, for any natural number i,

$$|B_{ik}| \leqq (n^k-1)^i$$

holds, where $n=|\mathfrak{X}|$. It is also true, that $B_k\subseteqq F\{\mathfrak{X}\}_k$ and $d(F)\leqq k$ hold, and because of the retrospectivity of F, we have

$$|F\{\mathfrak{X}\}_k| \leqq n^k-1,$$

consequently

$$|B_k|< n^k-1.$$

Let $|B_{(i-1)k}|\leqq(n^k-1)^{i-1}$, and let $p\in B_{(i-1)k}$. It suffices to show that there are at least n^k-1 words $q\in\{\mathfrak{X}\}_k$ such that $pq\in B_{ik}$. The following more strong assertion is also true: the number of those words $q\in\{\mathfrak{X}\}_k$ which satisfy this condition $pq\in F\{\mathfrak{X}\}_{ik}$, is at least n^k-1. In fact, consider all the words $r_1,\dots,r_m\in\{\mathfrak{X}\}_{(i-1)k}$ for which $Fr_1=\dots=Fr_m=p$. In this case

$$F_{r_1},\dots,F_{r_m}\in A_{2,(i-1)k},$$

therefore

$$\bigcup_{j=1}^{m} F(r_j\{\mathfrak{X}\}_k)=\bigcup_{j=1}^{m} pF_{r_j}\{\mathfrak{X}\}_k = p\bigcup_{j=1}^{m} F_{r_j}\{\mathfrak{X}\}_k\subset p\{\mathfrak{X}\}_k$$

according to the definition of k. On the basis of the preceding we obtain for any natural number t

$$\mu^*(\mathfrak{B}_{tk})\leqq\mu^*\Big(\bigcup_{q\in B_{tk}} q\mathfrak{X}^N\Big)\leqq\sum_{q\in B_{tk}}\mu^*(q\mathfrak{X}^N)=\sum_{q\in B_{tk}}\mu(q\mathfrak{X}^N)\leqq\left(\frac{n^k-1}{n^k}\right)^t.$$

We have seen that if $t<v$, then $\mathfrak{B}_v\subseteqq\mathfrak{B}_t$ has been satisfied. Thus for any $\varepsilon>0$ there exists a t_ε such that for any $t>t_\varepsilon$ the relation

$$\mu^*(\mathfrak{B}_t)<\varepsilon$$

is satisfied.

We observe that for any natural number t

$$F\mathfrak{X}^N=\Big(\bigcup_{F_p\in A_{1,t}}(Fp)\mathfrak{X}^N\Big)\cup\mathfrak{B}_t=\mathfrak{A}_t\cup\mathfrak{B}_t$$

and $\mathfrak{A}_t\cap\mathfrak{B}_t=\emptyset$.

Since $F\mathfrak{X}^N \triangle \mathfrak{A}_t = \mathfrak{B}_t$ and $\mathfrak{A}_t \in \mathfrak{R}(S(\mathfrak{X}))$ hold (for $\mathfrak{A}_t$ is a finite union of elements from $S(\mathfrak{X})$ and see [1]),

$$\mu^*(F\mathfrak{X}^N \triangle \mathfrak{A}_t) = \mu^*(\mathfrak{B}_t) < \varepsilon.$$

By the proposition 1 from this relation the measurability of the range of F follows.

Q. E. D.

Theorem 4. The range of any fsrsf is measurable.

Proof. Let be F an arbitrary fsrsf. We define fo rany natural number t three sets: $\Lambda_{1,t}$ is the set of all fsrsf $F_p(p \in \{\mathfrak{X}\}_t)$ which are onto mappings; $\Lambda_{2,t}$ is the set of all fsrsf $F_q(q \in \{\mathfrak{X}\}_t)$ which are not onto mappings, but there exists a word $r \in \{\mathfrak{X}\}$ $(l(r) > 0)$, such that F_{qr} is onto and finally $\Lambda_{3,t} = \Sigma_{F,t} \setminus \setminus (\Lambda_{1,t} \cup \Lambda_{2,t})$.

It is easy to see: if $F_p \in \Lambda_{1,t}$ and $r \in \{\mathfrak{X}\}$ is any word, then $F_{pr} \in \Lambda_{1,t+l(r)}$. Let $k(H)$ $(H \in \Sigma_F)$ the smallest natural number for which

$$\Sigma_H = \bigcup_{j=1}^{k(H)} \bigcup_{i=1}^{3} \Lambda_{i,j}$$

holds, and let $k = \max_{H \in \Sigma_F} k(H)$. For the sake of simplicity put

$$\bigcup_{G \in \Lambda_{1,t}} G\mathfrak{X}^N = \mathfrak{A}_t$$

$$\bigcup_{H \in \Lambda_{2,t}} H\mathfrak{X}^N = \mathfrak{B}_t$$

$$\bigcup_{J \in \Lambda_{3,t}} J\mathfrak{X}^N = \mathfrak{C}_t$$

for any natural number t, and denote by B_t the set of all initial segments of the sequences from $\mathfrak{B}_t$ whose length is exactly t.

The following inequality is valid:

$$|B_{2k}| \leqq |B_k|(n^k - 1).$$

In fact, for any $G \in \Lambda_{2,k}$ there exists a word $r \in \{\mathfrak{X}\}_{(k)}$ such that $l(r) > 0$ and $G_r \in \Lambda_{1,t+l(r)}$ hold.

A simple induction shows, that for any natural number t

$$|B_{tk}| \leqq |B_k|(n^k - 1)^{t-1}$$

is valid. Thus

$$\mu^*(\mathfrak{B}_{tk}) \leqq \mu^*\Big(\bigcup_{q \in B_{tk}} q\mathfrak{X}^N\Big) \leqq \sum_{q \in B_{tk}} \mu(q\mathfrak{X}^N) \leqq \frac{|B_k|}{n^k}\left(\frac{n^k - 1}{n^k}\right)^{t-1}$$

is true.

Now let $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ arbitrary numbers and $\varepsilon_1 + \varepsilon_2 < \varepsilon$.

It is obvious that there exists such a natural number t_{ε_1} for which

$$\mu^*(\mathfrak{B}_{tk}) \leqq \mu^*\Big(\bigcup_{q \in B_{tk}} q\mathfrak{X}^N\Big) = \mu^*(\mathfrak{B}) < \varepsilon_1$$

holds if $t > t_{\varepsilon_1}$ (in the following we assume it).

Since $\mathfrak{C}_{tk}$ is measurable by Lemma 2, there is a set $\mathfrak{C}\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(\mathfrak{C}_{tk}\triangle\mathfrak{C})<\varepsilon_2$$

is valid (see proposition 1).

Now we obtain that

$$\mu^*(F\mathfrak{X}^N\triangle(\mathfrak{A}_{tk}\cup\mathfrak{B}\cup\mathfrak{C}))\leqq\mu^*(\mathfrak{B})+\mu^*(\mathfrak{C}_{tk}\triangle\mathfrak{C})<\varepsilon_1+\varepsilon_2<\varepsilon$$

holds. Since ε was arbitrary, in virtue of Proposition 1, this relation gives the measurability of the range of F.

Q. E. D.

The following theorems give answer to what the set of the values of the function μ can be like.

Theorem 5. For any rational number r, $0\leqq r\leqq 1$, there exists such a fsrsf F for which we have

$$\mu(F\mathfrak{X}^N)=r.$$

Proof. If $r=0$ or $r=1$, then the statement is trivial, thus we can suppose that $0<r<1$. The proof will be constructive and we make the construction in the special case $\mathfrak{X}=\langle 0, 1\rangle$. The reader can show that this condition does not restrict the generality.

Let $\underbrace{11\ldots1}_{i\text{ times}}=1^i$ $(i=0, 1,\ldots)$ and define two sequential functions

$$G_0\alpha=00\ldots \tag{3.11}$$

$$G_1\alpha=\alpha$$

for any sequence $\alpha\in\mathfrak{X}^N$.

Let

$$0, a_1\,a_2\ldots a_k\,b_1\,b_2\ldots$$

be the dyadic form of the rational number r, where there is a natural number m for which

$$b_j=b_{j-\left[\frac{j}{m}\right]m} \tag{3.12}$$

for any natural number j $\left(\left[\dfrac{j}{m}\right]\right.$ denotes here the integer part of $\left.\dfrac{j}{m}\right)$.

After that, let

$$F_{1^i0}=\begin{cases}G_{a_{i+1}} & \text{if } 0\leqq i<k,\\ G_{b_{i-k+1}} & \text{if } k\leqq i.\end{cases} \tag{3.13}$$

Owing to the condition (3.12) we have only finitely many different states and these are

$$F_{1^0}, F_{1^1},\ldots, F_{1^{k+m}}.$$

A simple calculation shows that for the fsrsf F for which

$$\Sigma_F=\langle G_0, G_1, F_{1^0},\ldots, F_{1^{k+m}}\rangle$$

is valid, where G_0, G_1 and F_{1^i} are defined by (3. 11) and (3. 13), we have

$$\mu(F\mathfrak{X}^N) = \sum_{j=1}^{k} \frac{a_j}{2^j} + \sum_{i=1}^{\infty} \frac{b_i}{2^{k+i}} = r.$$

Theorem 6. For any number p, $0<p<1$ there exists a rsf F for which

$$\mu(F\mathfrak{X}^N) = p.$$

Proof. We may suppose that p is an irrational number, and let its dyadic form be

$$0; a_1 a_2 \dots$$

Let G_0 and G_1 be defined, as in Theorem 5, by (3. 11). Instead of (3. 13) we consider

$$F_{1^i 0} = G_{a_{i+1}} \text{ for all } i = 0, 1, \dots.$$

We obtain by a simple calculation that if

$$\Sigma_F = \langle G_0, G_1, F_{1^0}, \dots \rangle$$

then

$$\mu(F\mathfrak{X}^N) = \sum_{i=1}^{a} \frac{a_i}{2^i} = p.$$

Q. E. D

§ 4. Two negative results

We can ask the following question. If F and H are two rsf-s, then what is the correspondence between $\mu(F\mathfrak{X}^N)\mu(H\mathfrak{X}^N)$ and $\mu(FH\mathfrak{X}^N)$? (We suppose that these measures exist).

The next example shows that in the general case we can assert nothing. Let $\mathfrak{X} = \langle x_1, x_2 \rangle$, and the rsf G_0, G_1 be defined by (3. 11). The functions F and H will be defined by the formulas

$$\Sigma_F = \langle F_{x_1}, F_{x_2} \rangle$$

where $F_{x_1} = G_1$ and $F_{x_2} = G_0$, furthermore

$$\Sigma_H = \langle H_{x_1}, H_{x_2} \rangle,$$

where $H_{x_1} = G_0$ and $H_{x_2} = G_1$.

It is easy to show that on the one hand

$$\mu(F\mathfrak{X}^N) = \mu(H\mathfrak{X}^N) = \frac{1}{2},$$

on the other hand, if for any $x \in \mathfrak{X}$

$$Fx = \begin{cases} x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2 \end{cases}$$

and

$$Hx = \begin{cases} x_2 & \text{if } x = x_1 \\ x_1 & \text{if } x = x_2 \end{cases}$$

hold, then

$$\mu(FH\mathfrak{X}^N) = \mu(x_2 x_1 x_1 \dots) + \mu(x_1 \mathfrak{X}^N) = \frac{1}{2}$$

and

$$\mu(HF\mathfrak{X}^N) = \mu(x_2x_1x_1\ldots)+\mu(x_1x_1\ldots) = 0.$$

Before the description of the second result it is necessary to give two definitions.

Definition 4 (see [5]). The distance of two rsf-s F and G is defined by

$$\varrho(F, G) = \frac{1}{m}$$

where m is the smallest natural number for which there exists a sequence $\alpha \in \mathfrak{X}^N$ such that

$$(F\alpha)(i) = (G\alpha)(i) \quad \text{for} \quad i < m$$

and

$$(F\alpha)(m) \neq (G\alpha)(m)$$

hold.

Definition 5. We say, that a sequence of rsf-s $F^{(k)}$ tends to an rsf F ($F^{(k)} \to F$), if

$$\lim_{k\to\infty} \varrho(F^{(k)}, F) = 0$$

holds.

Now the second problem is the following. If $F^{(k)} \to F$ then does

$$\lim_{k\to\infty} \mu(F^{(k)}\mathfrak{X}^N) = \mu(F\mathfrak{X}^N)$$

hold?

The answer is negative. Let x be a fixed element of the set $\mathfrak{X}$, and for any natural number k let us define the rsf $F^{(k)}$ in the following way:

$$F^{(k)}\alpha = \alpha(1)\ldots\alpha(k)xx\ldots$$

for any $\alpha \in \mathfrak{X}^N$. It is easy to see that this sequence tends to the identity of $\mathfrak{X}^N$, thus $\mu(F\mathfrak{X}^N) = 1$ but for any natural number k

$$\mu(F^{(k)}\mathfrak{X}^N) = 0.$$

BOLYAI INSTITUTE OF THE
JÓZSEF ATTILA UNIVERSITY,
ARADI VÉRTANÚK TERE 1,
SZEGED, HUNGARY.

References

[1] A. N. Kolmogorov and S. V. Fomin, *Measure, Lebesgue Integrals and Hilbert Space*, Academic Press, New York and London 1961.
[2] B. Sz. Nagy, *Introduction to Real Functions and Orthogonal Expansions*, Academic Press, Budapest, and Oxford University Press, New York, 1964.
[3] G. N. Raney: Sequential functions, *J. Assoc. Comp. Mach.*, 5:2 (1958), 177—180.
[4] В. М. Глущков, Абстрактная теория автоматов, *Успехи мат. наук*, 16:5 (101) (1961), 3—62.
[5] Б. Чакань—Ф. Гечег, О группе автоматных подстановок, *Кибернетика*, 1:5 (1965), 10—13.

(Received May 27, 1969)