On the behaviour of some cyclically symmetric networks

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Zusammenfassung. In diesem Artikel beschäftigen wir uns mit dem folgenden speziellen Typ von Netzwerken: die Punkte des Graphen werden durch $P_1, P_2, \ldots, P_n$ bezeichnet; es existiert ein Zahl k $(1 \leqq k < n)$ so daß von jedem Punkt P_i die Kanten zu den Punkten

$$P_{i-1}, P_{i-2}, \ldots, P_{i-k}$$

und nur zu diesen führen (wobei die Subtraktion modulo n gemeint wird). Wir setzen dasjenige kontinuierliche Modell fort, das im Abschnitt 3 der Arbeit [2] eingeführt wurde. Der Zustand $\mathfrak{A}$ eines derartigen Graphen heißt zyklisch, wenn es eine positive Zahl p gibt, so daß nach einem Zeit-Intervall der Länge p der aus $\mathfrak{A}$ entstehende Zustand mit $\mathfrak{A}$ übereinstimmt. Wir unterscheiden im § 1 reguläre und nicht-reguläre Zustände. In den §§ 2—3 wird das Funktionieren eines Graphen mit einem regulären Anfangszustand diskutiert; wir stellen fest, daß jeder reguläre Zustand zyklisch ist. Im § 4 beschäftigen wir uns mit dem Funktionieren eines Netzwerkes mit einem nicht-regulären Anfangszustand; unser Hauptergebnis besagt, daß kein nicht-regulärer Zustand zyklisch sein kann.

§ 1. Introduction

In this paper we deal with the function of a special graph-theoretical class of networks. (We speak of a *network* if numerical values or numerical functions are assigned to the vertices of a graph.) We shall point out that the behaviour of networks in question can be described more explicitly in comparation to the general model elaborated in Sect. 3 of [2]. It is throughout supposed that the reader is familiar with Sections 1—3 of the former article [2].

Now we delimit the graph-theoretical structure of the networks to be investigated. Let $G(n; m_1, m_2, \ldots, m_k)$ (where $1 \leqq m_1 < m_2 < \cdots < m_k < n$) denote the graph consisting of n vertices labelled as $P_1, P_2, \ldots, P_n$, so that the directed edge $\overrightarrow{P_i P_j}$ exists if and only if there is an integer h $(1 \leqq h \leqq k)$ for which the congruence

$$i - j \equiv m_h \pmod{n}$$

holds.[1] *We shall regard the graphs* $G(n; 1, 2, \ldots, k)$ (where $1 \leqq k < n$) in the whole

[1] For the isomorphism problem of these graphs see [1] and the most recent papers [3], [4].

paper. We note that the subscripts of the vertices of such a graph (and consequently, also the subscripts of the functions α_i assigned to them) are mostly understood modulo n.[2]

Let a state

$$\mathfrak{A} = \langle \alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t) \rangle$$

(at the instant[3] t) of a graph G (containing n vertices) be considered. Let us denote by $\mathfrak{A}[+p]$ the state of G at the instant $t+p$ where p is an arbitrary non-negative real number. (More precisely: let us apply the continuous model defined in Sect. 3 of [2] for G, starting with $\mathfrak{A}$ at t; let $\mathfrak{A}[+p]$ be the vector

$$\langle \alpha_1(t+p), \alpha_2(t+p), \ldots, \alpha_n(t+p) \rangle.)$$

We say that $\mathfrak{A}$ is a *cyclic* state (and p is its *period*) if there exists a positive p such that $\mathfrak{A} = \mathfrak{A}[+p]$. In the contrary case, $\mathfrak{A}$ is an *acyclic* state.

We use for $\alpha_i(0)$ the shorter notation β_i, too.

Let us consider a network $G(n;\ 1, 2, \ldots, k)$. Assume that there exists at least one vertex P_j with $\alpha_j(t) = 1$. (If this holds for P_j, then each of $\alpha_{j-1}(t), \alpha_{j-2}(t), \alpha_{j-3}(t), \ldots$ $\ldots, \alpha_{j-k}(t)$ is 0.) We say that the vertices

$$P_{i+1}, P_{i+2}, \ldots, P_{j-2}, P_{j-1}, P_j \tag{1}$$

form an *arc* (at the instant t) if

$$1 = \alpha_i(t) > \alpha_{i+1}(t) > \alpha_{i+2}(t) > \cdots > \alpha_{j-k-1}(t) \geqq$$
$$\geqq \alpha_{j-k}(t) = \alpha_{j-k+1}(t) = \alpha_{j-k+2}(t) = \cdots = \alpha_{j-1}(t) = 0$$

(and, of course, $\alpha_j(t) = 1$) hold. Evidently, the number of vertices of an arc is necessarily at least $k+1$. (We emphasize that P_i does *not* belong to the arc (1).) A state of a graph $G(n;\ 1, 2, \ldots, k)$ is called *regular* (at t) if each vertex is contained in an arc (obviously, it may be contained in only one). In a regular state, we denote by $\varphi(P_i, t)$ the first vertex P_j in the sequence

$$P_{i+1}, P_{i+2}, P_{i+3}, \ldots$$

which satisfies $\alpha_j(t) = 1$; in other words, $\varphi(P_i, t)$ is that vertex P_j in the arc containing P_{i+1} which fulfils $\alpha_j(t) = 1$. (P_i and P_{i+1} are in the same arc unless $\alpha_i(t) = 1$.)

In what follows, we shall obtain that *a state of a network* $G(n;\ 1, 2, \ldots, k)$ *is cyclic if and only if it is regular* (Propositions 2, 8).

§ 2. Discussion of the behaviour of a network starting with a regular state

Let us consider a regular state of a network $G(n;\ 1, 2, \ldots, k)$ at the instant 0. Our next aim is to give a detailed discussion of the function α_i associated to a vertex P_i (chosen arbitrarily) of G during the time interval $[0, \tau]$. Our treatment is based

[2] For example, we write simply "the vertex P_{i+l}" instead of "the vertex P_j whose subscript is determined by $j \equiv i+l \pmod n$, $1 \leqq j \leqq n$".

[3] In what follows, t will be almost everywhere 0.

upon Sect. 3 of [2]. We shall formulate several consequences of the present discussion in § 3; one of these consequences is anticipated just now:

Proposition 1. If

$$\mathfrak{A} = \langle \alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0)\rangle$$

is a regular state, then we have

$$\alpha_i(\tau) = \alpha_{i+k+1}(0)$$

for each i (i can be $1, 2, \ldots, n$).

We are going to perform the discussion. We distinguish three cases according to the possibilities $0<\beta_i<1$, $\beta_i=0$, $\beta_i=1$. Any case is subdivided to some subcases with respect to the smallest integer h satisfying $P_{i+h}=\varphi(P_i, 0)$. In every discussed case, the following statement will be always true: whenever $\alpha_j(t)=0$ and there exists a positive number ε such that $\alpha_j(t')>0$ holds for every t' fulfilling $t-\varepsilon < t' < t$, then $\alpha_{j+1}(t)=1$. We shall apply this method of inference (in a number of steps) without being mentioned explicitly.

Case 1: $0<\beta_i<1$. We distinguish three subcases.

Case 1/a: $h > 2k+1$, in other words, each of $P_{i+1}, P_{i+2}, \ldots, P_{i+2k+1}$ differs from $\varphi(P_i, 0)$. This assumption implies (by the definition of the regular state)

$$\beta_i>\beta_{i+1}>\cdots>\beta_{i+k}>\beta_{i+k+1} \geqq \beta_{i+k+2} \geqq \cdots \geqq \beta_{i+2k+1}.$$

The behaviour of α_i in $[0, \tau]$ can be described as follows:

(i) in the interval $[0, \tau(1-\beta_i)]$ the value of α_i grows linearly from β_i to 1,

(ii) in the interval $[\tau(1-\beta_i), \tau(1-\beta_{i+1}))$ α_i is constantly 1,

(iii) in the interval $[\tau(1-\beta_{i+1}), \tau(1-\beta_{i+k+1})]$ α_i is constantly 0,

(iv) in the interval $[\tau(1-\beta_{i+k+1}), \tau]$ (of length $\tau\beta_{i+k+1}$) the value of α_i grows linearly from 0 to $\tau\cdot\beta_{i+k+1}/\tau = \beta_{i+k+1}$.

Indeed, P_i gets edges exactly from the vertices $P_{i+1}, P_{i+2}, \ldots, P_{i+k}$. None of $\alpha_{i+1}, \ldots, \alpha_{i+k}$ can be 1 in the interval $[0, \tau(1-\beta_{i+1}))$. However, at every instant t of the interval $[\tau(1-\beta_{i+1}), \tau(1-\beta_{i+k+1}))$, (exactly) one of $\alpha_{i+1}(t), \ldots, \alpha_{i+k}(t)$ is 1. In the interval $[\tau(1-\beta_{i+k+1}), \tau)$ α_{i+k+1} is constantly 1, thus each of $\alpha_{i+1}, \ldots, \alpha_{i+k}$ is constantly 0. We have also $\alpha_{i+1}(\tau)=\cdots=\alpha_{i+k}(\tau)=0$, hence α_i may grow in $[\tau(1-\beta_{i+k+1}), \tau]$.

Case 1/b: $k+2 \leqq h \leqq 2k+1$. Then

$$\beta_i>\beta_{i+1}>\cdots>\beta_{i+h-k-1} \geqq \beta_{i+h-k} = \beta_{i+h-k+1} = \cdots = \beta_{i+h-1} = 0,$$

$$1 = \beta_{i+h}>\beta_{i+h+1} \geqq \beta_{i+h+2} \geqq \cdots \geqq \beta_{i+h+k}.$$

The condition of the case implies the inequalities

$$i+2 \leqq i+h-k \leqq i+k+1 \leqq i+h-1 \leqq i+2k,$$

thus $\beta_{i+k+1}=0$. The behaviour of α_i satisfies the assertions (i), (ii) of Case 1/a, moreover,

(iii) in the interval $[\tau(1-\beta_{i+1}), \tau]$ α_i is constantly 0. Indeed, since $\alpha_{i+k+1}(t')<1$ at each instant t' of the interval $[0, \tau)$, the behaviour of $\alpha_{i+1}, \ldots, \alpha_{i+k}$ is similar to Case 1/a (with τ instead of $\tau(1-\beta_{i+k+1})$).

Case 1/c: $h = k+1$. Then

$$\beta_i > \beta_{i+1} = \beta_{i+2} = \ldots = \beta_{i+k} = 0,$$
$$1 = \beta_{i+k+1} > \beta_{i+k+2} \geqq \beta_{i+k+3} \geqq \ldots \geqq \beta_{i+2k+2}.$$

The behaviour of α_i can be described as follows:

(i) in the interval $[0, \tau(1-\beta_i)]$ the value of α_i grows linearly from β_i to 1,

(ii) in the interval $[\tau(1-\beta_i), \tau]$ α_i is constantly 1.

Indeed, none of $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{i+k}$ can reach 1 in the interval $[0, \tau(2-\beta_{i+k+2}))$, furthermore $\tau < \tau(2-\beta_{i+k+2})$.

Case 2: $\beta_i = 0$. We distinguish four subcases:

Case 2/a: $h = k+1$. We can prove by ideas similar to Case 1/c that α_i grows linearly from 0 to 1 in the whole interval $[0, \tau]$.

Case 2/b: $h = k$. Then

$$\beta_i = \beta_{i+1} = \ldots = \beta_{i+k-1} = 0,$$
$$1 = \beta_{i+k} > \beta_{i+k+1} \geqq \beta_{i+k+2} \geqq \ldots \geqq \beta_{i+2k+1}.$$

The behaviour of α_i is as follows:

(i) in the interval $[0, \tau(1-\beta_{i+k+1})]$ α_i is constantly 0,

(ii) in the interval $[\tau(1-\beta_{i+k+1}), \tau]$ α_i grows linearly from 0 to

$$(\tau - \tau(1-\beta_{i+k+1}))/\tau = \beta_{i+k+1}.$$

Case 2/c: $1 \leqq h \leqq k-1$ and $\beta_{i+k+1} = 0$. Then

$$\beta_i = \beta_{i+1} = \ldots = \beta_{i+h-1} = 0,\ 1 = \beta_{i+h} > \beta_{i+h+1} > \ldots$$
$$\ldots > \beta_{i+k+1} > \beta_{i+k+2} \geqq \beta_{i+k+3} \geqq \ldots \geqq \beta_{i+2k+2}.$$

The same conclusions (i), (ii) are true as in Case 2/b.

Case 2/d: $1 \leqq h \leqq k-1$ and $\beta_{i+k+1} = 0$. Then

$$\beta_i = \beta_{i+1} = \cdots = \beta_{i+h-1} = 0,$$
$$1 = \beta_{i+h} > \beta_{i+h+1} \geqq \beta_{i+h+2} \geqq \cdots \geqq \beta_{i+k+1} = 0.$$

In this case α_i is constantly 0 in the whole interval $[0, \tau]$.

Case 3: $\beta_i = 1$. This case can be discussed similarly to Case 1. The single modification is that $\tau(1-\beta_i) = 0$, thus the conclusions (i) do not occur in the subcases.

§ 3. Propositions on the behaviour of a network starting with a regular state

We are going to expose some statements which summarize the discussion performed in the preceding paragraph. Let g be the least common multiple of $k+1$ and n.

Proposition 2. Any regular state is cyclic; $g\tau/(k+1)$ *is a suitable period.*

Proof. If we apply Proposition 1 $g/(k+1)$ times, then we get

$$\alpha_i(0)=\alpha_{i+(k+1)}(\tau)=\alpha_{i+2(k+1)}(2\tau)=\dots=\alpha_{i+g}(g\tau/(k+1))=\alpha_i(g\tau/(k+1))$$

for every i.

Proposition 3. If $\mathfrak{A}$ is a regular state, then the state $\mathfrak{A}[+t]$ is regular for each non-negative t.

Proof. Assume that the instant of $\mathfrak{A}$ is denoted by 0. Let d be the greatest integer so that $d\tau\leqq t$. We get by successive application of Proposition 1 that the conclusion of the present proposition is true for $d\tau$. By analyzing § 2, we obtain that it holds for t too (because $t-d\tau<\tau$). The proof is completed.

An easy consequence of our former investigations is

Proposition 4. If $\mathfrak{A}$ is a regular state and t is a non-negative number, then the number of arcs of $\mathfrak{A}$ equals to the number of arcs of $\mathfrak{A}[+t]$.

Let us fix a vertex P_i, let us consider the sequence

$$P_i,\ P_{i+(k+1)},\ P_{i+2(k+1)},\ P_{i+3(k+1)},\ \dots,\ P_{i-(k+1)} \tag{2}$$

consisting of $g/(k+1)$ (distinct) vertices and the sequence

$$P_{i+1},\ P_{i+(k+1)+1},\ P_{i+2(k+1)+1},\ P_{i+3(k+1)+1},\ \dots,\ P_{i-(k+1)+1} \tag{3}$$

which consists likewise of $g/(k+1)$ vertices. Either $n, k+1$ are relatively prime to each other (thus $g=n(k+1)$ and both of (2), (3) contain all the vertices) or (2), (3) are disjoint.[4] Let us define the instants v_h and w_h by

$$v_h=\tau(h-\beta_{i+(h-1)(k+1)}) \quad\text{and}\quad w_h=\tau(h-\beta_{i+(h-1)(k+1)+1})$$

(where h can be $1, 2, \dots, g/(k+1)$). This definition implies immediately

Lemma 1. For any h,

$$\tau(h-1)\leqq v_h\leqq \tau h \quad\textit{and}\quad \tau(h-1)\leqq w_h\leqq \tau h.$$

Lemma 2. For any h we have one of the three possibilities

(a$_1$) $v_h<w_h$

(a$_2$) $v_h=w_h=\tau h$

(a$_3$) $w_h=\tau(h-1)$ *and* $v_h=\tau h$

(*according as*

(b$_1$) $\beta_{i+(h-1)(k+1)}>\beta_{i+(h-1)(k+1)+1}$

(b$_2$) $\beta_{i+(h-1)(k+1)}=\beta_{i+(h-1)(k+1)+1}=0$

(b$_3$) $\beta_{i+(h-1)(k+1)}=0,\ \beta_{i+(h-1)(k+1)+1}=1$).

[4] For, if (2), (3) contain a vertex in common, then some multiple of $k+1$ is congruent to 1 modulo n, hence n and $k+1$ are relatively primes.

Proof. The equivalence of (a_i) and (b_i) can be shown easily (for all the three values of i), the proof is completed by the remark either (b_1) or (b_2) or (b_3) is true since the state is regular.

Lemma 3. If $v_{h-1} < w_{h-1}$ and $v_h < w_h$ for some $h(\geqq 2)$, then either $w_{h-1} = v_h = \tau(h-1)$ or $w_{h-1} < v_h - \tau$.

Proof. The supposition implies

$$\beta_{i+(h-2)(k+1)} > \beta_{i+(h-2)(k+1)+1},$$

$$\beta_{i+(h-1)(k+1)} > \beta_{i+(h-1)(k+1)+1}.$$

The sequence (consisting of $k+1$ numbers)

(4) $$\beta_{i+(h-2)(k+1)+1}, \beta_{i+(h-2)(k+1)+2}, \beta_{i+(h-2)(k+1)+3}, \dots, \beta_{i+(h-1)(k+1)}$$

is monotonically decreasing unless $\beta_{i+(h-1)(k+1)} = 1$ (by the regularity of the state), thus we can distinguish two cases.

Case 1: (4) is monotonically decreasing. Then the number

$$\beta_{i+(h-2)(k+1)+1} - \beta_{i+(h-1)(k+1)} \left(= (v_h - \tau - w_{h-1})/\tau\right)$$

is positive, hence $w_{h-1} < v_h - \tau$.

Case 2: $\beta_{i+(h-1)(k+1)} = 1$. Then, on the one hand, $v_h = \tau(h-1)$; on the other hand, $\beta_{i+(h-2)(k+1)+1} = 0$, this implies $w_{h-1} = \tau(h-1)$.

By use of the numbers v_h, w_h we can explicitly characterize the behaviour of α_i in the interval $[0, g\tau/(k+1))$:

Proposition 5. Let us consider a regular state at the instant 0. *The function α_i, assigned to a vertex P_i, satisfies the following four assertions:*

(A) *If* $(1 \leqq h \leqq g/(k+1)$ *and*$)$ $v_h < w_h$, *then α_i is constantly* 1 *in the interval* $[v_h, w_h)$.[5]

(B) *If* $(2 \leqq h \leqq g/(k+1)$ *and*$)$ $w_{h-1} < v_h < w_h$, *then α_i grows linearly in the interval* $[v_h - \tau, v_h]$ *from* 0 *to* 1.

(C) *If* $v_1 < w_1$, *then α_i grows linearly in the interval* $[0, v_1]$ *from* $1 - v_h/\tau$ *to* 1.

(D) *The value of α_i is* 0 *at all the instants of the interval* $[0, g\tau/(k+1))$ *which are not referred to in* (A), (B) *and* (C).

Proof. Let an instant t lying in $[0, g\tau/(k+1))$ be considered. There exists a number h such that $\tau(h-1) \leqq t < \tau h$ (where $1 \leqq h \leqq g/(k+1)$). By using Proposition 1 successively $h-1$ times (with $t-\tau, t-2\tau, t-3\tau, \dots, t-\tau(h-1)$ instead of 0), we get

$$\alpha_i(t) = \alpha_{i+(k+1)}(t-\tau) = \alpha_{i+2(k+1)}(t-2\tau) = \cdots$$

$$\cdots = \alpha_{i+(h-2)(k+1)}(t-\tau(h-2)) = \alpha_{i+(h-1)(k+1)}(t-\tau(h-1)),$$

i.e. the behaviour of α_i in the interval $[\tau(h-1), \tau h)$ is the same as the behaviour of $\alpha_{i+(h-1)(k+1)}$ in $[0, \tau)$ (with the appropriate translation).

[5] Since $w_h = v_{h+1}$ may occur, two or more intervals of this character can be joined.

First we show (A). The function $\alpha_{i+(h-1)(k+1)}$ takes the value 1 exactly in the sub-interval

$$[\tau(1-\beta_{i+(h-1)(k+1)}),\ \tau(1-\beta_{i+(h-1)(k+1)+1}))$$

of $[0, \tau)$ by Cases 1/a, 1/b, 1/c, 3/a, 3/b, 3/c of the discussion in § 2 (even if at least one of

$$\beta_{i+(h-1)(k+1)}=1, \quad \beta_{i+(h-1)(k+1)+1}=0$$

is true).

In order to verify (B), let $t(\geqq\tau)$ be such an instant that $\alpha_i(t)=1$ but, for every positive ε, there exists a t^* fulfilling $\alpha_i(t^*)<1$ and $t-\varepsilon<t^*<t$. Then $\alpha_{i+(h-2)(k+1)}$ has the analogous property at the instant $t-(h-2)$, and $\tau\leqq t-\tau(h-2)<2\tau$. By analyzing the discussion and by Proposition 1, we get that $\alpha_{i+(h-2)(k+1)}$ grows linearly in $[t-\tau(h-1), t-\tau(h-2)]$ from 0 to 1, consequently α_i behaves in $[t-\tau, t]$ analogously.

(C) follows from the discussion immediately.

(D) is equivalent to the subsequent statement: any function α_i is 0 at t unless t is contained in an interval $(t', t'+\tau]$ such that $\alpha_i(t'+\tau)=1$. This statement follows easily from the discussion and Proposition 1 in the interval $[0, 2\tau]$, it can be extended for any non-negative t by Proposition 1.

The last assertion we state relying upon § 2 is the evident

Proposition 6. The following three statements are equivalent for a regular state:
(A) *The state is steady.*
(B) *Every arc of the state consists of exactly $k+1$ vertices.*
(C) *$k+1$ is a divisor of n and the number of arcs in the state is $n/(k+1)$.*

§ 4. Study of non-regular states

The purpose of this paragraph is to show that only the regular states are cyclic. First we define the *irregularity indices* of an arbitrary permitted state[6] $\mathfrak{A}$ by the following three rules:

(i) if $\beta_{i-1}<\beta_i<1$, then i is an irregularity index,
(ii) if $\beta_{i-1}=\beta_i>0$, then i is an irregularity index,
(iii) if $\beta_{i-1}=\beta_i=0$ and each of $\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{i+k}$ is <1, then i is an irregularity index.

(The conditions in (i), (ii), (iii) exclude each other.) We agree that no remaining number (out of the set $\{1, 2, \ldots, n\}$) is an irregularity index. The *irregularity number* of the state $\mathfrak{A}$ is the number of its irregularity indices.

If (i) or (iii) holds for i, then i is called a *strong irregularity index*, the number of strong irregularity indices is the *strong irregularity number* of $\mathfrak{A}$. If (ii) holds for i then we call i a *weak irregularity index.*

Lemma 4. The irregularity number of $\mathfrak{A}$ is 0 if and only if $\mathfrak{A}$ is a regular state.

Proof. It is obvious that the definition of the regular state does not admit any of the possibilities (i), (ii), (iii). — Conversely, assume that no vertex fulfilling

[6] A state is permitted if $\alpha_i=1$, $P_j\in\chi(P_i)$ imply $\alpha_j=0$.

the condition of either (i) or (ii) or (iii) occurs in $\mathfrak{A}$; let P_i be an arbitrary vertex. If $\beta_i=1$, then

$$\beta_{i-k}=\beta_{i-k+1}=\beta_{i-k+2}=\cdots=\beta_{i-1}=0$$

(since the state is permitted). If $0<\beta_i<1$, then $\beta_{i-1}>\beta_i$ (since otherwise (i) or (ii) would be violated). If $\beta_i=0$, then either one of $\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{i+k}$ is 1 or $\beta_{i-1}>\beta_i$ (in consequence of (iii)). Thus $\mathfrak{A}$ is a regular state.

Lemma 5. Let $\mathfrak{A}$ be a state at the instant 0 *and t be a positive instant such that the functioning of the network is defined (at least) in the interval* $[0, t]$. *If i is not a strong irregularity index at* 0, *then i is a strong irregularity index nor at t.*

Proof. Let t^* be the (possibly non-existing) least real number such that $0\leqq t^*\leqq t$ and none of $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{i+k}$ takes the value 1 in the interval $[t^*, t]$. Either $t^*=0$ or there exists a number q such that $1\leqq q\leqq k$ and to every positive ε there exists a t' satisfying both $t^*-\varepsilon<t'<t^*$ and $\alpha_{i+q}(t')=1$.

Case 1: $t^*>0$ and $q<k$. We have

$$\alpha_{i-1}(t^*)=\alpha_i(t^*)=0,$$

the functions α_{i-1}, α_i are equal and increase linearly in the whole interval $[t^*, t]$ from 0 to $(t-t^*)/\tau$. (Necessarily $t-t^*<\tau$; if the contrary were true, we should get a contradiction to the hypothesis that the functioning is defined in $[0, \tau]$.)

Case 2: $t^*>0$ and $q=k$. We have

$$\alpha_{i-1}(t^*)\geqq\alpha_i(t^*)=0.$$

Three subcases are possible:

Case 2/a: $\alpha_{i-1}(t^*)=0$. This subcase can be treated similarly to Case 1.

Case 2/b: $\alpha_{i-1}(t^*)>0$ and $t-t^*<\tau$. Then α_i increases linearly in the whole interval $[t^*, t]$ from 0 to $(t-t^*)/\tau$. α_{i-1} increases linearly from

$$\alpha_{i-1}(t^*) \text{ to } \begin{cases} \alpha_{i-1}(t^*)+(t-t^*)/\tau & \text{in } [t^*, t] \text{ if } \alpha_{i-1}(t^*)+(t-t^*)/\tau\leqq 1, \\ 1 & \text{in } \left[t^*, t^*+\tau\left(1-\alpha_{i-1}(t^*)\right)\right] \text{ if } \alpha_{i-1}(t^*)+(t-t^*)/\tau>1. \end{cases}$$

In the second of these cases α_{i-1} is constantly 1 in $\left[t^*+\tau\left(1-\alpha_{i-1}(t^*)\right), \tau\right]$.

Case 3: $t^*=0$ and $\beta_{i-1}>\beta_i$. Let us assume that t is so large that all the intervals to be discussed are in $[0, t]$. (If this assumption is not fulfilled, then the subsequent discussion is altered so that it breaks off at the instant t.) In the interval $[0, \tau(1-\beta_{i-1})]$ both α_{i-1} and α_i increase linearly. In $\left[\tau(1-\beta_{i-1}), \tau(1-\beta_i)\right)$ α_{i-1} is constantly 1 and α_i increases linearly. In $[\tau(1-\beta_i), t]$ α_i is constantly 1 and α_{i-1} is constantly 0.

Case 4: $t^*=0$ and $\beta_{i-1}=\beta_i$. Then $t<\tau$, furthermore α_{i-1}, α_i are equal and increase from 0 to t/τ similarly as in Case 1.

Case 5: t^* does not exist. Then there is at least one number q such that $1\leqq q\leqq k$ and $\alpha_{i+q}(t)=1$, thus $\alpha_i(t)=0$. i fulfils the conditions of neither (i) nor (iii) at t.

Lemma 6. If the strong irregularity number of a state $\mathfrak{A}$ *at the instant* 0 *is positive and the functioning of the network in the interval* $[0, \tau]$ *is defined, then the strong irregularity number of the state* $\mathfrak{A}[+\tau]$ *is* 0.

Proof. Let i be an arbitrary index. If i is not a strong irregularity index, then we can apply Lemma 5. Otherwise, let us define t^* and q as in the proof of Lemma 5. If $t^*>0$ then Cases 1, 2 of the preceding proof remain valid; if t^* does not exist, then the inference of Case 5 can be applied. We have still to study the cases when $t^*=0$ and i fulfils (i) or (iii).

If (i) is true, then

$$\alpha_i(\tau(1-\beta_i)) = 1 \quad \text{and} \quad \alpha_{i-1}(\tau(1-\beta_i)) = 0.$$

i is not a strong irregularity index at $\tau(1-\beta_i)$ consequently nor at τ (by Lemma 5).

If (iii) holds, then it is easy to see that the functioning of the graph is defined at most in the interval $[0, \tau)$; this contradicts the supposition of Lemma 6.

Lemma 7. Let $\mathfrak{A}$ *be a state at the instant* 0 *such that the strong irregularity number of* $\mathfrak{A}$ *is* 0. *If the functioning of the network in the interval* $[0, \tau]$ *is defined, then the irregularity number of* $\mathfrak{A}[+\tau]$ *is* 0.

Proof. Whenever j is an arbitrary index and t' is an instant such that $0 \leqq t' \leqq \tau$, then j cannot be a strong irregularity index at t' (by Lemma 5). We shall study a function α_i in $[0, \tau]$. Let us define t^* and q in the same manner as at beginning of the proof of Lemma 5.

Case 1: $t^*>0$. Necessarily $q=k$ (since now the value 1 "steps" from j to $j+1$, similarly to the case of a regular state, discussed in § 2). Hence $\alpha_{i-1}(t^*)>\alpha_i(t^*)=0$. In the interval

$$[t^*, t^*+\tau(1-\alpha_{i-1}(t^*))]$$

α_{i-1}, α_i increase parallel (i.e. $\alpha_{i-1}-\alpha_i$ remains constant). In the interval

$$[t^*+\tau(1-\alpha_{i-1}(t^*)), \tau]$$

(provided that it exists) α_{i-1} is constantly 1 and α_i continues its growth.

Case 2: $t^*=0$. We distinguish two subcases.

Case 2/a: $\beta_{i-1}=\beta_i$. This assumption implies that the functioning of the network is defined only in $[0, \tau(1-\beta_i))$, i.e. it contradicts the supposition of Lemma 7.

Case 2/b: $\beta_{i-1}>\beta_i$. In the interval $[0, \tau(1-\beta_{i-1})]$, α_{i-1} and α_i increase parallel. In

$$[\tau(1-\beta_{i-1}), \tau(1-\beta_i))$$

α_{i-1} is constantly 1 and α_i continues its growth. In $[\tau(1-\beta_i), \tau]$ α_i is constantly 1 and α_{i-1} is constantly 0.

Case 3: t^* does not exist. We get $\alpha_i(\tau)=0$ similarly to Case 5 of the proof of Lemma 5, hence i does not fulfil the condition of (ii).

Proposition 7. If the state $\mathfrak{A}$ *(at the instant* 0*) is non-regular, then either* $T'_{\max}$ *is defined for* $\mathfrak{A}$ *and* $0<T'_{\max}<2\tau$ *or* $\mathfrak{A}[+2\tau]$ *is regular.*[7]

[7] $T'_{\max}$ was introduced in [2].

Proof. Assume that the states $\mathfrak{A}[+t]$ are definable whenever $0 \leqq t \leqq 2\tau$. The state $\mathfrak{A}[+\tau]$ cannot have a strong irregularity index (by Lemma 6), hence the state $\mathfrak{A}[+2\tau]$ is regular (by Lemmas 7 and 4).

Proposition 8. Any non-regular state is acyclic.

Proof. Let $\mathfrak{A}$ be a non-regular state (at the instant 0). If the state $\mathfrak{A}[+t]$ is not definable for every positive t (i.e. if $T'_{\max}$ does exist), then $\mathfrak{A}$ is obviously acyclic. Assume that $\mathfrak{A}[+t]$ is defined for every t. Let $\mathfrak{A}$ be cyclic and p be a period of it, we shall get a contradiction. Let d be the least integer such that $dp \geqq 2\tau$ holds. On the one hand,

$$\mathfrak{A} = \mathfrak{A}[+p] = \mathfrak{A}[+2p] = \dots = \mathfrak{A}[+dp],$$

thus $\mathfrak{A}[+dp]$ is non-regular. On the other hand, $\mathfrak{A}[+2\tau]$ is regular by Proposition 7, hence also $\mathfrak{A}[+dp]$ is regular by Proposition 3.

§ 5. On some possibilities for future researches

Let us consider a graph. Denote by A the set of its permitted states (i.e. all the mappings of the vertex set into the interval [0, 1] such that the restriction mentioned in Footnote 6 is satisfied), by $A_r (\subset A)$ the set of its regular states. We define two partitions π_1, π_2 of A and a further partition π_3 of A_r in the following manner:

$\mathfrak{A}(\in A)$, $\mathfrak{A}'(\in A)$ are in a common class mod π_1 if there exists an integer s such that $0 \leqq s \leqq n-1$ and

$$\alpha_1 = \alpha'_{1+s},\ \alpha_2 = \alpha'_{2+s},\ \dots,\ \alpha_{n-1} = \alpha'_{s-1},\ \alpha_n = \alpha'_s$$

where $\mathfrak{A} = \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$, $\mathfrak{A}' = \langle \alpha'_1, \alpha'_2, \dots, \alpha'_n \rangle$.

$\mathfrak{A}(\in A)$, $\mathfrak{A}'(\in A)$ are in a common class mod π_2 if the inequalities $\alpha_i < \alpha_j$ and $\alpha'_i < \alpha'_j$ are equivalent to each other for every index pair i, j.

$\mathfrak{A}(\in A_r)$, $\mathfrak{A}'(\in A_r)$ are in a common class mod π_3 if there exists a non-negative real number t such that $\mathfrak{A}[+t] = \mathfrak{A}'$.

The partitions π_1 and π_2 generate a sublattice of the lattice of all partitions of A; similarly, π_1, π_2 and π_3 generate a sublattice in the partition lattice of A_r. Various questions (concerning both the lattice-theoretical properties and numerical problems) can be raised on the lattices generated in this manner.

Finally, we mention a problem of this character. Let A_h be the set of the states $\mathfrak{A} = \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$ fulfilling the three requirements:

(i) $\alpha_i = 1$ holds for exactly one index i,

(ii) the state is permitted,

(iii) whenever l and l' are two indices such that $1 \leqq l < l' \leqq n$, $P_l \notin \{P_i\} \cup \chi(P_i)$, $P_{l'} \notin \{P_i\} \cup \chi(P_i)$, then the inequalities $0 < \alpha_l < 1$, $0 < \alpha'_l < 1$, $\alpha_l \neq \alpha'_l$ hold.

It is easy to see that a randomly chosen element $\mathfrak{A}' = \langle \alpha'_1, \alpha'_2, \dots, \alpha'_n \rangle$ of A satisfies $\mathfrak{A}'[+t] \in A_h$ with probability 1 where $t = \tau(1 - \max(\alpha'_1, \alpha'_2, \dots, \alpha'_n))$.

Let us consider the graphs $G(3; 2)$, $G(4; 3)$, $G(5; 4)$, $\dots$, $G(n; n-1)$, $\dots$. Starting with the general member $G(n; n-1)$ of this sequence, we denote by Ω_n the factor set $A_h^{(n)}/\pi_2$ where $A_h^{(n)}$ denotes the set A_h with respect to the graph $G(n; n-1)$. Ω_n is a finite set. On the other hand, let us define the subsets $A_h^{(n,x)}$ of $A_h^{(n)}$ so that $\mathfrak{A} \in A_h^{(n,x)}$ if and only if the regular state $\mathfrak{A}[+t]$ (with the least possible $t(\geqq 0)$)

(exists and) consists of x arcs ($x \leqq n/2$). The sets $A_h^{(n,x)}$ are pairwise disjoint (for varying x), moreover, $\mathfrak{A} \in A_h^{(n,x)}$, $\mathfrak{A}' \in A_h^{(n,x')}$, $\mathfrak{A} \equiv \mathfrak{A}' \pmod{\pi_2}$ imply $x = x'$. Let $\Omega_n^{(x)}$ be the subset of Ω_n which consists of the classes whose elements are in $A_h^{(n,x)}$. It is interesting to examine the asymptotical behaviour of the numerical function

$$f(n, x) = \frac{|\Omega_n^{(x)}|}{|\Omega_n|}.$$

(Evidently, $\sum_{x=1}^{[n/2]} f(n, x) \leqq 1$.) A discussion shows that the first values of $f(n, x)$ are:

x \ n	2	3	4	5	6
1	1	1	1/2	1/6	1/24
2			1/2	5/6	17/24
3					1/4

We conjecture that $f(n, [(n-1)/2])$ converges to 1 if n tends to the infinity.

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