On measure-theoretic problems involving retrospective sequential functions

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1. Introduction

The present paper can be regarded as self-contained inasmuch as it does not rely on outside repositories of references to an extent we would think underisable, yet, we think it should, in a proper setting, be considered as a continuation of, or an addendum to, L. Klukovits's paper [6] in the first issue of these Acta.[1] Otherwise it might be questionable whether the present paper, investigating pure measure-theoretic properties of certain types of functions, should appear in a periodical of *cybernetics.* Though these researches might have some potential applications to cybernetics and to the theory of automata, this aspect of the problem will not be elaborated here in detail. Perhaps some additional research in this area may be useful.

Yet, from a cybernetical angle, our study can be viewed as an investigation, on a theoretical level, of the relation between the behaviours of an automaton, firstly, if an arbitrarily large, but only a finite, number of input signs is successively fed into it and, secondly, if the feeding of input signs is repeated infinitely many times.

The approach to the characterization of the behaviours of automata is achieved through studying measure-theoretic properties of retrospective sequential functions, the precise definition of which, along with other definitions, may be found below. We shall point out that under certain natural conditions such functions are measurable, or, in more specific circumstances, they are even continuous. They map Borel sets onto sets which, in a natural sense, can be called Lebesgue-measurable; we shall give an example which illustrates that the image of a Borel set may be a non-Borel set, even in a very simple case.

2. Preliminary notions

Since the sections that follow depend to a considerable extent on different sets of notions we think it undesirable to accumulate here all the necessary definitions,

[1] The cited paper contains some inaccuracies and a considerable number of proofs in it are presented in an unnecessarily complicated way. Our observations concerning this matter are presented on p. 89.

and we collect here only the concepts that play a rôle throughout the whole of these notes.

The very concept around which all that follows centres is that of the *retrospective sequential function,* shortly RS function. The domain of such a function is the Cartesian product

(2. 1) $$X = \bigtimes_{n=1}^{\infty} X_n,$$

where X_n is intuitively interpreted as the set of *input signs* that can be fed into a given automaton at the nth stage. The range is a subset of the Cartesian product

(2. 2) $$Y = \bigtimes_{n=1}^{\infty} Y_n,$$

where Y_n is, intuitively, the set of *output signs* that can be emitted immediately after the digestion of the input sign absorbed at the nth stage. The automaton in question is to be imagined as having a fixed initial state that completely determines its reactions to sequences of input signs. The RS function assotiated with this automaton makes correspond to an infinite sequence of input signs the sequence of output signs the automaton emits while receiving the former.

This intuitive description of RS functions may easily be put in the form of a precise definition: a function f mapping the set X into Y is called an RS function if, under f, the first n signs of the image sequence are uniquely determined by the first n signs of the argument sequence for every positive integer n. This specific property of an arbitrary RS function f enables us to consider its restrictions to finite sequences. In notations, for every positive integer n put

(2. 3) $$X|n = \bigtimes_{k=1}^{n} X_k, \quad Y|n = \bigtimes_{k=1}^{n} Y_k;$$

the function $f|n$ sends, by definition, all sequences in $X|n$ to sequences in $Y|n$ in the same way as f handles these sequences as finite segments of infinite sequences. The notations

(2. 4) $$n|X = \bigtimes_{k=n+1}^{\infty} X_k, \quad n|Y = \bigtimes_{k=n+1}^{\infty} Y_k \qquad (n \geqq 0)$$

will sometimes prove useful, too.

In all our considerations, each of the sets X_n and Y_n will be vested with a *measurability structure*, by which we mean an ordered pair consisting of a set, the underlying space, and a σ-ring defined on this set, this latter being usually suppressed in the notational framework. The spaces $X|n$, $Y|n$, X and Y will be endowed with the measurability structures that are the products of their respective factors. The σ-rings determining these structures are the *minimal* ones generated by the sets of all rectangular sets or, in case of an infinite number of factors, by the sets of all *cylindrical* sets; here a subset of e.g. X is said to be cylindrical if, for some n, it is the Cartesian product of a set measurable in $X|n$ with the whole set $n|X$. (As seen, the notion of cylindrical sets already depends on the concept of measurability in products of finite numbers of spaces.)

A part of our study depends only on this measurability structure, without the need of actually considering measures. In other parts we have also to assume that certain measures are given on the described σ-rings. Sometimes we shall also consider the completions of these measures; these are, in general, defined on larger σ-rings, and this fact should carefully be kept in mind since, unless specifically mentioned our results may not hold for these extensions of the measures involved.

Another point to be stressed is that, up to Section 5, when measures are considered the measures on the product measurability structure of X and Y are never assumed to be the products of the measures on the respective factors; on the other hand, all our counter-examples are so constructed that, when measures on X and Y are at all considered, these are the products of the measures defined on the respective factors.

3. Measurability of RS functions

A very simple necessary and sufficient condition in order that an RS function be measurable in the sense that the whole inverse image of any measurable set is measurable is provided by

Theorem 3.1. An RS function f is measurable if and only if the functions $f|n$ are measurable for all positive integers n.

Proof. The "only if" part of the assertion is quite obvious and needs no comment whatsoever. Not much more complicated is the reverse implication, either. Indeed, observing that the inverse of a function does not spoil set-theoretical operations such as union and difference, the desired result immediately follows from the minimality restrictions, as imposed in Section 2, on product spaces.

Here, of course, the question might be raised how far these minimality restrictions are indispensable. The situation is, perhaps, illuminated by

Counter-example 3.1. The tacit assumption that in Theorem 3.1 measurability cn Y means belonging to the minimal σ-ring generated by cylindrical sets cannot be omitted even in the simplest case.

This assertion is intended to be a vague intuitive description of the situation rather than a precise mathematical statement.

To consider a σ-ring, larger than the minimal one, of measurable sets in Y is senseless unless motivated in some suggestive way. Thus, what we are going to do will be to introduce measures on X_n and Y_n and consider the σ-rings that are the domains of the completions of the product measures on X and Y.

Now we actually set to describe the counter-example in question. Choose X_n and Y_n as coinciding with the two-element discrete space, containing the integers 1 and 2, such that the measure of each of its one-element subsets is 1/2. Let μ be the product measure on $X=Y$, and $\bar{\mu}$ its completion.

Define the RS function f mapping X into itself by the stipulation that for an arbitrary sequence $x=\{x_n\}_{n=1}^{\infty}$ the image $f(x)=y=\{y_n\}_{n=1}^{\infty}$ be such that $y_{2n-1}=1$ and $y_{2n}=x_n$. In compliance with the clause in Theorem 3.1, $f|n$ is clearly measurable with respect to the (minimal) measurability structure on $X|n=Y|n$, this being the discrete structure.

Now the function f being one-to-one, for any set $Z \subseteq X$ we have $Z = f^{-1}(f(Z))$. Here obviously $\mu^*(f(Z)) = 0$,[2] so $f(Z)$ is always measurable with respect to $\bar{\mu}$; thus, providing Z is chosen nonmeasurable with respect to $\bar{\mu}$, this set is an example for a measurable set whose full inverse image under f is not measurable. To make our considerations complete, we only have to point out that X has a subset which is not measurable with respect to $\bar{\mu}$; this, however, follows from the fact that X endowed with the measure $\bar{\mu}$ is essentially identical as a measure space to the interval $(0,1)$ with the usual Lebesgue measure on it.

Finally, we remark that if the measurability structures of the spaces occurring here are coupled with certain topological ones then some simple conditions ensure the completion measurability of an RS function. These conditions and the proofs are analogous as in the cases of Lemma 4.2 and Theorem 4.3; the proofs in this case are even slightly simpler. We do not formulate these results here since they do not seem as natural as well as have no such a consequence as their counterparts in the next section (see Theorem 5.1 below).

4. Questions concerning the transportation of measurability

The question studied here, a much more difficult one than that envisaged in the previous section, concerns the transportation of measurability. More exactly, the problem to which we try to find an answer here is under what circumstances it is guaranteed that the image of a measurable set under an RS function is measurable again. This problem seems to depend much more on the topological structures of the spaces involved and on measures rather than on measurabilities than we experienced it in connection with the question studied in the previous section. Thereby we are forced to impose further restrictions on the spaces X_n and Y_n, and it will be convenient to do this along with a short description of the related concepts.

Throughout the rest of the paper we assume that, for each positive integer n, the spaces X_n and Y_n are endowed with topologies induced by metrics under which these spaces are complete and separable metric spaces. The topologies on $X|n$, $Y|n$, X, and Y are defined as the products of the topologies on their respective factors. As is well known, it is possible also on these spaces to introduce metrics with respect to which they are complete and separable metric spaces. For example, if we denote the distance function on X_n by ϱ_n then the function

$$\varrho(x, x') = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho_n(x_n, x'_n)}{1+\varrho_n(x_n, x'_n)} \tag{4.1}$$

serves as such a metric on X. Since our main concern is the possibility of the introduction of such metrics rather than the particular distance functions chosen, we shall suppress these latter in the notational framework; nevertheless, we might refer to the spaces involved as metric when it were enough to say metrizable in a certain way.

Measures on these spaces will also be considered. μ and ν will denote two Borel measures on X and Y, respectively; here a Borel measure, by definition, is a

[2] The asterisk * in superscript indicates outer measure.

σ-finite measure explained on the σ-ring of all Borel sets, this being the smallest σ-ring generated by e.g. all closed sets. $\bar{\mu}$ and $\bar{\nu}$, called Lebesgue measures, will denote the completions of μ and ν. In an obvious way we can also define the restrictions μ_n, ν_n, $\mu|n$ and $\nu|n$, of the measures μ and ν, to the spaces X_n, Y_n, $X|n$ and $Y|n$, respectively; e.g. for a Borel set $H \subseteq X|n$ put $\mu|n(H) = \mu(H \times n|X)$. It is usually not assumed that μ and ν are the products of the measures μ_n and ν_n.

A simple condition in order that an RS function under the circumstances specified above be in a sense measurability transporting is

Theorem 4.1. If the RS function f is such that $f|n$ is Borel-measurable for any positive integer n then f maps all Borel sets onto Lebesgue-measurable ones.

Here the Borel measurability of a function means that the whole inverse image under it of a Borel set is again a Borel set.

Proof. It follows from Theorem 3.1 that, under the given assumption, f is itself Borel-measurable; so it maps Borel sets onto analytic or, by another name, Suslin sets (see e.g. [2, 2.2.14 on p. 70]). As is well known, every analytic set is Lebesgue-measurable (see [2, 2.2.1.2. Theorem on p. 68]), which completes the proof.

To illustrate how far the assumption in this theorem is necessary and whether the conclusion goes far enough we give several counter-examples. The assumption that $f|n$ is Borel-*measurable* when we want to prove that f is *measurability transporting* may seem artificial; Counter-example 4.1, however, shows that it is not enough to suppose that $f|n$ is measurability transporting. Counter-example 4.2 shows that the given assumption does not ensure that f maps every Lebesgue-measurable set onto a Lebesgue-measurable set. It is not certain, either, that, under this assumption, the image of every Borel set is a Borel set; this will be shown later, in Counter-example 5.1.

Counter-example 4.1. The assumption that, for any positive integer n, the function $f|n$ maps every subset of $X|n$ onto a Borel set of $Y|n$ does not imply the conclusion of Theorem 4.1.

In the example we are going to give, the validity of the assumption that $f|n$ maps every set onto a Borel set will be ensured by choosing as $Y|n$ a finite discrete space, every subset of which is, of course, a Borel set. To elaborate, choose the spaces $X_2, X_3, \dots$ and $Y_1, Y_2, \dots$ as identical to a two-element discrete space, with points 1 and 2, such that either of its one-element subsets is of measure 1/2. Explain the Borel measure on Y as the product of those defined on the spaces Y_n; define X_1 as identical to Y, with the same topology and measure defined on it. Finally, choose the Borel measure on X as identical to the product of the measures explained on the spaces X_n.

Now choose as f_1 an arbitrary function from X_1 into Y whose range is not Lebesgue-measurable. Then the function f that makes correspond to every $x = \{x_n\}_{n=1}^{\infty}$ the sequence $f_1(x_1)$, independently of x_n for $n \geqq 2$, is an RS function that satisfies our requirements, yet it does not map the whole set X onto a Lebesgue-measurable set.

Counter-example 4.2. The assumption of Theorem 4.1 does not assure that the image under f of a Lebesgue-measurable set is Lebesgue-measurable.

Using the same spaces X and Y as in the counter-example just before, define the RS function f from X into Y so that it send a sequence $x=\{x_n\}_{n=1}^{\infty}$ to a sequence y that is identical to $x_1 \in X_1 = Y$, independently of the values of x_n for $n \geqq 2$. It is obvious that the function $f|n$ is Borel-measurable for each positive integer n. If we select an arbitrary set $X_1' \subseteqq X_1 = Y$ that is not Lebesgue-measurable, then the image under f of the set $X' = X_1' \times \{1\} \times \{1\} \times \cdots \subseteqq X$ is X_1'; now the set X' is Lebesgue-measurable, since its outer Borel measure is zero; yet its image is not so.

It seems to be a rather difficult problem to give conditions that subtly differentiate between cases when Lebesgue-measurable sets are mapped onto this same kind of sets and when they are, possibly, not. Nevertheless, the following two results, however rough they are, point in this direction.

Lemma 4.2. Assume that the space X is locally compact and that the measure of every compact set in X is finite.[3] *Suppose, furthermore, that the RS function f is such that, n running over all positive integers, the function $f|n$ is Borel-measurable, and moreover, with some positive constant C,*

$$(4.2) \qquad (\nu|n)^*(f|n(G_n)) \leqq C\mu|n(G_n)$$

holds for any open set G_n in $X|n$.[2,4] *Then f maps all Lebesgue-measurable sets onto Lebesgue-measurable ones.*

We remind that the local compactness of X is an additional assumption and, as said at the beginning of this section, all the spaces considered here are assumed to be complete and separable metric spaces. We also recall that in order for the product of topological spaces to be locally compact it is necessary and sufficient that all factors, with the possible exception of a finite number of them, be compact and the non-compact factors be locally compact (see [1, Proposition 11 on p. 65]). Taking this into account, we can reformulate the lemma accordingly.

The point in adopting (4.2) as an assumption of the lemma is that it ensures that the mapping f does not increase the outer measure of any set more than C times; thus, in particular, it maps sets of zero outer measure onto sets also of zero outer measure, and this implies the assertion of the lemma.

Proof. Since every Lebesgue-measurable set can be represented as the union of two sets of which one is Borel-measurable and the other is of zero outer Borel measure, the assertion will follow from the previous Theorem if we show that f maps every set of zero outer Borel measure onto a set also of zero outer Borel measure. To accomplish this, let Z be an arbitrary subset of zero outer Borel measure of X. Since in a locally compact and separable metric space every Borel set is a Baire set, and a Baire measure on a locally compact space is always regular, pro-

[3] Usually, Borel measures are considered on locally compact spaces and it is traditionally included in their definition that they are finite on compact sets. Here we cannot conform to this tradition since it would involve some unnecessary restrictions on the measures considered.

[4] It is enough to require the conditions depending on n in this lemma and in the next theorem only for large enough integers, though the statements so obtained are not real generalizations since they easily follow from the assertions, analogous to the given ones, arrived at by grouping the factors of X as $(X_1 \times \cdots \times X_k) \times X_{k+1} \times \ldots$, and those of Y similarly. Moreover, it does not represent any real change to require only for large n's that $f|n$ is Borel-measurable since then the same follows for every positive integer n.

vided it is finite on compact sets (see [3, Theorem E on p. 218 and Theorem G on p. 228]), for an arbitrarily small positive ε there exists an open subset G of X with $\mu(G)<\varepsilon$ such that $Z \subseteqq G$.

n being an arbitrary positive integer, let U run over all the open subsets of $X|n$, and write

$$G_n = \bigcup\{U\colon\ U\times n|X \subseteqq G\}. \tag{4.3}$$

Then G_n is an open subset of $X|n$, and, obviously, we have

$$G_n\times n|X \subseteqq G_{n+1}\times(n+1)|X, \tag{4.4}$$

and

$$G = \bigcup_{n=1}^{\infty} G_n\times n|X; \tag{4.5}$$

moreover, on account of (4.2), we obtain

$$\begin{aligned} \nu^*\big(f(G_n\times n|X)\big) &\leqq \nu^*\big(f|n(G_n)\times n|Y\big) \leqq^{5} \\ &\leqq (\nu|n)^*\big(f|n(G_n)\big) \leqq C\mu|n(G_n) = C\mu(G_n\times n|X). \end{aligned} \tag{4.6}$$

Here all the sets are actually Lebesgue-measurable; for the first one this is stated by the previous theorem. For the second and the third one this fact will not be used, so we do not go into details and only note that in the proofs similar arguments involving analytic sets may be used. So, writing $\bar{\nu}$ instead of ν^*, the last four centred lines imply

$$\bar{\nu}\big(f(G)\big) = \bar{\nu}\left(\bigcup_{n=1}^{\infty} f(G_n\times n|X)\right) \leqq C\mu(G). \tag{4.7}$$

Since G, by its choice, includes Z, we obtain

$$\nu^*\big(f(Z)\big) \leqq \bar{\nu}\big(f(G)\big) \leqq C\mu(G)<C\varepsilon. \tag{4.8}$$

Since ε can be selected arbitrarily small we have $\nu^*\big(f(Z)\big)=0$, which completes the proof of the lemma.

Though in the proof of this lemma we made a relevant use of the local compactness of X, this assumption can actually be dispensed with if we stipulate that μ is totally finite, and we can derive

Theorem 4.3. Assume that the measure μ is totally finite and the RS function f mapping X into Y is such that, n running over all positive integers, $f|n$ is Borel-measurable, and, moreover, with some positive constant C,

$$(\nu|n)^*\big(f|n(B_n)\big) \leqq C(\mu|n)(B_n) \tag{4.9}$$

holds for any Borel set B_n in $X|n$.[6] *Then f maps all Lebesgue-measurable sets onto Lebesgue-measurable ones.*

[5] An easy argument invoking analytic sets shows that here actually equality holds, and would continue to hold even if the set $f|n(G_n)$, which could easily be shown to be Lebesgue-measurable, were replaced by any subset of $Y|n$. This is, however, irrelevant for our purposes.

[6] See footnote [4].

Proof. Any complete and separable metric space is either countable or of continuum cardinality (see [4, IV on p. 320]), therefore to each X_n there is a compact and separable metric space X'_n of the same cardinality. Since a compact metric space is necessarily complete, on account of a well-known result (see [7, 2° on p. 358]), there exists a one-to-one function g_n mapping X'_n onto X_n that is Borel-measurable in both ways.[7]

Now define the function g from X', this being the product space $X'_1 \times X'_2 \times \cdots$, onto X componentwise, i.e. put

$$g(x'_1 x'_2 \ldots) = g_1(x'_1) g_2(x'_2) \ldots \tag{4.10}$$

Then g is obviously one-to-one and Borel-measurable in both ways. Determine the Borel measure μ' on X' so that g be also measure-preserving. Instead of the RS function $f(x)$ we may consider the RS function $f(g(x))$ mapping X' into Y and the assertion of the theorem directly follows from the previous lemma on account of the compactness of X'.

5. The problem of transportation of measurability in a special case[8]

In this concluding section we shall be concerned only with the following special case: $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are all identical discrete spaces, with a finite number $N \geqq 2$ of points, and the Borel measures μ_n and ν_n on X_n and Y_n, respectively, are such that the measure of a one-point set is $1/N$; finally we determine μ and ν as the products of the measures μ_n and ν_n, respectively. It is easy to see that in this case all RS functions are continuous. Moreover, the assumptions of Theorem 4. 3 are satisfied for any RS function f. Indeed, $f|n$ is Borel-measurable for many reasons, e.g. since it is defined on a discrete space. The assumption (4. 9) is also satisfied with $C=1$. The argument showing this is simply that the measure of a set in $X|n$ is a constant multiple of the number of the (finite) sequences contained in it; this measure may only decrease by performing the mapping $f|n$, as a consequence of the phenomenon that two different sequences may have a common image. So in this case we have

Theorem 5. 1. Every RS function maps all Lebesgue-measurable sets onto Lebesgue-measurable sets.

The proof of this theorem does not, in fact, need such sophisticated tools as have been used to accomplish it. Namely, cylindrical sets being compact, their image is also compact, and the considerations based on (4. 9) that establish the full strength of the theorem are largely simplified by the fact that the measure μ is the product of the measures μ_n.

[7] Actually, the phrase "in both ways" need not be added; namely, it is easy to show that if a one-to-one function which maps a complete and separable metric space onto another is Borel-measurable then its inverse is so, too.

[8] The more ambitious reader is advised also to consult L. Kalmár's paper [5], where a case with generality lying between that of the cases dealt with in this and the previous sections is studied from a somewhat different angle.

The paper of Klukovits [6], which spurred us to investigate measure-theoretic problems involving RS functions, considered only the particular case studied in this section. We take now a closer look at the relationship between some of his results and some of our considerations here. It will turn out, in particular, that many of the proofs in his paper can be radically shortened by using some simple devices of topology.

Theorem 1 in the paper in question says that two RS functions differing only on a set of Lebesgue measure zero coincide. The functions in question being continuous, this is naturally true, since in this case the measure of any non-empty open set is positive, and thus the set of coalescence is dense.

Lemma 1 claims that if the range of an RS function is Lebesgue-measurable then the image under it of any Lebesgue-measurable set is so, too. This is a consequence of Theorem 5.1 of ours, though the assumption on the range is superfluous. Independently of our result just referred to, the fact that the range of f is measurable is obvious since, being a continuous image of a compact set, it is in fact compact. In the proof of the cited lemma, the author leaves to the reader the verification of the assertion that, under the assumption of the measurability of the range, the image of every cylindrical set is Lebesgue-measurable. Cylindrical sets being compact, the task of the reader in proving this is indeed not difficult. He may, however, be annoyed by not finding a way to weave the measurability of the range into his considerations.

Theorem 2 states that an RS function f is measure-preserving if and only if it is an onto mapping. Here the proof of the necessity can be contraced into a few lines as follows: the range of an RS function f being compact, its complement is open. The stipulation that f is measure-preserving implies that the measure of this open set is zero; so it is empty, which means that f is indeed onto.

Lemma 2 asserts that the range of a "finite-state RS function without one-to-one state" is Lebesgue-measurable. (The phrase is not an exact quotation; the author writes fsrsf for what we called a finite-state RS function.) In whatever way the above attributes may specify the notion of RS function, the range is a continuous image of a compact set, therefore it is compact, and so measurable.

Theorem 4 announces that the range of any "fsrsf" is Lebesgue-measurable. Actually, the range is again compact.

The concluding result of these notes is

Counter-example 5.1. There exists an RS function under which the image of a certain Borel set is not a Borel set.

In order to give such an example, for every positive integer n, identify the spaces X_n and Y_n with the discrete space consisting of the points 1 and 2 and choose N_n as the discrete space containing exactly the positive integers; let N be the topological product of the spaces N_n.

Decompose the space $X = Y$ as

$$X = Z_1 \times Z_2 \times Z_3, \tag{5.1}$$

where

$$(5.2) \qquad Z_j = \mathop{\times}_{k=0}^{\infty} X_{3k+j} \qquad (j=1,2,3).$$

Let Z'_j be the subspace of Z_j which consists precisely of the sequences that contain an infinite number of ones. It is easy to see that Z'_j is homeomorphic to N. Indeed, a homeomorphism between these spaces can be described as follows: for an arbitrary element z of Z'_j form groups of consecutive elements constituting z so that each group consist purely of 2's except that it end with a 1. The numbers of elements in each group, in turn, form a sequence of positive integers which, if considered as the image of z, determines a homeomorphism between Z'_j and N. Denote this homeomorphism from Z'_3 onto N by h; for a sequence $z \in Z'_3$ denote by $h_n(z)$ the nth integer forming the sequence $h(z)$.

Now, following closely the lines of the example for an analytic set that is not a Borel set given in [2, 2.2.11 on p. 68], our example can be described as follows:

Choose a countable open base $U(n)$ of $Z'_1 \times Z'_2$ and define a closed subset of $Z'_1 \times Z'_2 \times Z'_3$ by

$$(5.3) \qquad C = \left\{(z_1, z_2, z_3)\colon\ (z_1, z_2) \notin \bigcup_{n=1}^{\infty} U\big(h_n(z_3)\big)\right\}.$$

It is obvious that all the closed subsets of $Z'_1 \times Z'_2$ occur among the slices

$$(5.4) \qquad C_{z_3} = \{(z_1, z_2)\colon\ (z_1, z_2, z_3) \in C\}.$$

Now, on the one hand,

$$(5.5) \qquad S = \{(z_1, z_3)\colon\ (z_1, z_2, z_3) \in C \quad \text{for some} \quad z_2\}$$

is an analytic subset of $Z'_1 \times Z'_3$; and, on the other hand, the slices

$$(5.6) \qquad S_{z_3} = \{z_1\colon\ (z_1, z_3) \in S\} = \{z_1\colon\ (z_1, z_2) \in C_{z_3} \quad \text{for some} \quad z_2\}$$

run over all the analytic subsets of Z'_1, since Z'_3 is homeomorphic to N (see [2, 2. 2. 10 on p. 65]).

Finally, the intersection of S with the diagonal of $Z'_1 \times Z'_3$, the latter being a set closed in the relative topology, is an analytic subset of $Z'_1 \times Z'_3$. The projection of this set into Z'_1,

$$(5.7) \qquad T = \{z_1\colon\ (z_1, z_3) \in S \quad \text{and} \quad z_1 = z_3\},$$

is therefore analytic; now the complement of T, $Z'_1 - T$, is not analytic since it does not occur among the sets S_{z_3}. Indeed, the assumption $Z'_1 - T = S_{z_3}$ is equivalent to saying that for any $z_1 \in Z'_1 = Z'_3$

$$(5.8) \qquad (z_1, z_1) \notin S \quad \text{holds if and only if} \quad (z_1, z_3) \in S.$$

This is, however, certainly not true for $z_1 = z_3$, implying that $Z' - T$ is not analytic, as asserted. Thus, since the complement of a Borel set is again a Borel set, and so a fortiori an analytic set, we may conclude that T is not a Borel set.

To complete our example, we shall determine an RS function that maps a Borel subset of $Z'_1 \times Z'_2 \times Z'_3 \subseteq X$ essentially onto T. To this end, define a diagonal plane of the set C:

(5. 9) $$D=\{(z_1, z_2, z_3): (z_1, z_2, z_3)\in C \text{ and } z_1=z_3\},$$

and consider the function f from $Z_1\times Z_2\times Z_3$ (without accents $'$!) into itself such that

(5. 10) $$f(z_1, z_2, z_3)=(z_1, c, c),$$

where c is an arbitrary but fixed sequence in $Z_2=Z_3$, e.g. $c=111\ldots$. In view of (5. 1), f can be rewritten as an RS function mapping $X=Y$ into itself.

Now the set D, being a set closed in the relative topology on the Borel set $Z_1'\times Z_2'\times Z_3'$, is itself a Borel set, and its image under f, the set $T\times\{c\}\times\{c\}$, was proved to be a non-Borel set just before.

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