On some enumeration questions concerning trees and tree-type networks

By A. ÁDÁM and J. BAGYINSZKI

To the memory of Dr Catherine Rényi and Professor Alfréd Rényi

Zusammenfassung. Im § 3 werden gewisse Fragen der Abzählung von Wurzel-Bäumen betrachtet. Sei T ein Wurzel-Baum mit der Wurzel R, bezeichnen wir durch k die Anzahl der Kanten von T. Teilen wir die Kanten in Klassen durch die folgende Relation ein: zwei Kanten sind äquivalent, wenn sie auseinander ohne Berühren von R erreichbar sind. Existieren genau $\varkappa_i$ Äquivalenzklassen, die aus je i Kanten bestehen (wobei i die Zahlen $1, 2, 3, \ldots, k$ durchläuft), so sagen wir, daß die Partition von T der Vektor $K=\langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k \rangle$ ist. Wir erhalten drei Formeln für die Anzahl $S_K(k)$ der numerierten Bäume von der Partition K unter die Annahme, daß die Nummer der Wurzel als 1 fixiert wird und die übrigen Punkte die Nummern $2, 3, \ldots, k+1$ (auf beliebige Weise) bekommen. Eine dieser Formeln stimmt im Wesentlichen mit einem (in verschiedener Weise bewiesenen) Resultat von J. Dénes überein. Aus unseren Ergebnissen ist auch die wohlbekannte Formel von Cayley ableitbar (Corollary 1).

In den Paragraphen 4—5 wird ein zeitliches Verhalten dem Wurzel-Baum T laut des Modells der früheren Arbeit [1] zugeordnet, so daß die Kanten in die Richtung der Wurzel gerichtet sind und jeder Punkt P_i einen im Intervall (0, 1) liegenden beliebigen Anfangswert $\beta(P_i)$ hat. Wir definieren fünf Typen von mit den Werten $\beta(P_i)$ versehenen Bäumen, die fünf charakteristischen Arten des Verhaltens entsprechen (Proposition 6). Im § 4 studieren wir die Wahrscheinlichkeit des Ereignisses, daß der Baum zu einem oder anderem Typ gehört, wenn sowohl der Baum (als ein Graph) wie die Werte $\beta(P_i)$ zufällig gewählt sind.

§ 1. Introduction

§ 3 is devoted to some enumeration questions of rooted trees. In Theorems 1, 2 and Corollary 2 several formulae for the number of labelled rooted trees having a fixed partition of the number k of edges are obtained, supposing that the root is labelled by 1 and the other vertices by $2, 3, \ldots, k+1$. From our results the well-known Cayley enumeration formula can be deduced, too (Corollary 1).

In §§ 4—5, a temporal behaviour is assigned to the rooted tree T in sense of the model exposed in the former paper [1], such that each edge is directed towards the root and any vertex P_i has an arbitrary initial value $\beta(P_i)$ lying in the interval $(0, 1)$. We define five types of trees, being supplemented with the values $\beta(P_i)$; these types correspond to five characteristic features of behaviour (Proposition 6). We study in § 4 the probability of the event that the tree belongs to one or another type provided that the tree (as a graph) and the values $\beta(P_i)$ are chosen randomly.

A large collection of results and methods concerning the enumeration questions of labelled trees is contained in the lecture note [4] of Moon. The articles of Dénes [3] and A. Rényi [6] deal with subjects closely connected with the present paper; especially, our Corollary 2 follows easily from Theorem 5 of [3] (by adding a new vertex as a root to the graph and by connecting the root to one vertex in each component). The publication [7] of C. and A. Rényi is devoted to the generalization of the questions of counting for the case of k-trees.

§ 2. Preliminaries

We suppose that the reader is familiar with the basic notions of graph theory. If the edge e and the vertex P are incident, then we say, equivalently, that P is a *terminal* of e.

Let H be a finite set and $H_1, H_2, \ldots, H_j$ be some pairwise disjoint non-empty subsets of H. If the union of $H_1, H_2, \ldots, H_j$ equals to H, then we say that $H_1, H_2, \ldots, H_j$ form a *set-partition* of H. (The ordering of the H_i's is indifferent.)

Let k be a natural number. If the members of the vector

$$K = \langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k \rangle$$

consisting of k non-negative integers satisfy the equality

$$1 \cdot \varkappa_1 + 2 \cdot \varkappa_2 + 3 \cdot \varkappa_3 + \cdots + k \cdot \varkappa_k = k, \tag{2. 1}$$

then we say that K is a *numerical partition* of the number k.

We speak about a partition simply if the context makes doubtless whether a numerical one or a set-partition is dealt with.

Let a set-partition $H_1, H_2, \ldots, H_j$ of the set H consisting of k elements be given. If, among the subsets $H_1, H_2, \ldots, H_j$,

there are $\varkappa_1$ subsets each consisting of 1 element,

there are $\varkappa_2$ subsets each consisting of 2 elements,

...

and there are $\varkappa_k$ subsets each consisting of k elements,

then ((2. 1) is obviously fulfilled and) we say that the numerical partition, assigned to the partition of H in question, is $\langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k \rangle$.

Denote by Ω_k the set of all numerical partitions of the number k. If we write $\sum_{\Omega_k}$, then the summation must be taken for all elements K of Ω_k.

Let a, b be real numbers such that $a \leqq b$. By the *closed interval* $[a, b]$ we mean the set of the real numbers x satisfying $a \leqq x \leqq b$. By the *open interval* (a, b) we understand the set of the real numbers fulfilling $a < x < b$. In analogy, we define $[a, b)$ and $(a, b]$ by the conditions $a \leqq x < b$ and $a < x \leqq b$, respectively.

We shall often write $\exp x$ instead of e^x where e is the base of natural logarithms (this is useful if a long expression occurs in the role of x). The Bürmann—Langrangeformula concerning the series expansion of inverse functions is supposed to be known (see [2]). We shall use the subsequent Proposition I, II (of analytic character):

Proposition I. *There holds the identity*

$$(f(x)=)\prod_{j=1}^{\infty}\left(\sum_{\varkappa_j=0}^{\infty}(a_{j-1}x^j)^{\varkappa_j}\frac{1}{\varkappa_j!}\right)=1+\sum_{k=1}^{\infty}A_kx^k \tag{2.2}$$

in the real interval (u, v) *where*

$$A_k=\sum_{\Omega_k}\prod_{j=1}^{k}a_{j-1}^{\varkappa_j}\frac{1}{\varkappa_j!}$$

if the power series on the right-hand side of (2. 2) *is uniformly convergent in* (u, v).

Proof. Let the expression on the left-hand side of (2. 2) be ordered according to the increasing powers of x. Then the coefficient of x^k gets an additive contribution from all the possible partitions of the number k; the contribution of any single partition is $\prod_{j=1}^{k}a_{j-1}^{\varkappa_j}\frac{1}{\varkappa_j!}$.

Before stating Proposition II, we introduce three notations z_m, $T_r(x)$, $Z(x)$ as follows:

$$T_r(x)=\cos(r\arccos x)$$

(i.e. $T_r(x)$ is the Chebyshev polynomial of degree r),

$$z_m=6\cdot 2^{1/2}\pi^{-3/4}m^{-1/4}e^{-4\sqrt{\pi m}}\left(1+O(m^{-1/2})\right)\quad\text{if}\quad m\to\infty,$$

$$Z(x)=\frac{1}{12x}\left(z_0+2\sum_{m=1}^{\infty}(-1)^m z_m T_{2m}\left(\frac{1}{x}\right)\right)\quad\text{where}\ x\geqq 1. \tag{2.3}$$

Proposition II. *There holds the identity*

$$n\cdot\Gamma(n)=n!=(2\pi)^{1/2}\cdot n^{n+1/2}\cdot\exp\left(-n+Z(n)\right),$$

consequently, the right-hand side of the definition (2. 3) *is convergent.*

The proof of Proposition II may be found in [5]. We note that the analogon of the convergence conclusion of this proposition does not hold for Stirling series.

§ 3. The enumeration of rooted trees

A *rooted tree* is a finite connected undirected graph without circuits in which a vertex is distinguished. The distinguished vertex is called the *root* of the tree. If R is the root and P is an arbitrary vertex in a rooted tree, then the distance of R and P is called also the *height* of P.[1]

[1] In §§ 4—5 we shall consider the rooted trees as *directed* graphs in such a manner that each edge is oriented toward the vertex of smaller height.

Let T be a rooted tree and R the root in it; suppose that the degree of R is 1. We say that the *partition* of the tree T is

$$K_0 = \langle \overset{1}{\breve{0}}, \overset{2}{\breve{0}}, \overset{3}{\breve{0}}, \ldots, \overset{k-1}{\breve{0}}, \overset{k}{\breve{1}} \rangle.$$

Denote by P the single vertex adjacent to R. If we delete R and the edge between P, R, then we get a tree T'; we agree that P should be the root of T'. The rooted tree T', defined in this manner, is called the *truncated tree* of T. (It is defined only if the degree of the root is one.) If the number of edges of T is k, then T' has $k-1$ edges (consequently, k vertices).

Now let T be a rooted tree (with the root R) such that the degree d of R is at least two. Denote the edges incident to R by $e_1, e_2, \ldots, e_d$, and their terminals, different from R, by $P_1, P_2, \ldots, P_d$, respectively. We define d new rooted trees $T_1, T_2, \ldots, T_d$ in the following four steps:

(i) we delete $R, e_1, e_2, \ldots, e_d$,

(ii) we introduce d new vertices $R_1, R_2, \ldots, R_d$ and d new edges $e'_1, e'_2, \ldots, e'_d$,

(iii) for each number i $(1 \leqq i \leqq d)$, let e'_i be incident to R_i and P_i,

(iv) for each i $(1 \leqq i \leqq d)$ let T_i be that connected component of the graph, built up in the previous steps, which contains R_i; let R_i be the root of T_i.

The process, described in (i), (ii), (iii), (iv), is called the *dismembering* of the tree T (having a root of degree >1) and every T_i is called a *branch* of T.

If, for each number j $(1 \leqq j \leqq k)$, there are exactly $\varkappa_j$ branches T_i such that the number of edges of any T_i equals to j, then we say that the *partition* of T is

$$\langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k \rangle.$$

Evidently, this expression is a partition of the number of edges of T.

Let T be a rooted tree with k edges. T has $k+1$ vertices. Let us assign $k+1$ different natural numbers to the vertices of T. The tree T together with such an assignment is called a *labelled rooted tree*. If we require, in addition, that the assigned numbers should be $1, 2, 3, \ldots, k, k+1$ and, especially, the root should have the number 1, then we speak on a *standardly labelled rooted tree*.

We denote by $N(k)$ the number of the (non-isomorphic) labelled rooted trees with k edges when the set of numbers, corresponding to the vertices, is fixed. Furthermore, we denote by $S(k)$ the number of the standardly labelled rooted trees with k edges. If K is a partition of k and only the trees having partition K are counted, then the analogous numbers are denoted by $N_K(k)$ and $S_K(k)$, respectively. Obviously,

$$N(k) = \sum_{\Omega_k} N_K(k) \quad \text{and} \quad S(k) = \sum_{\Omega_k} S_K(k).$$

In case $k=1$ we have evidently

Proposition 1. *For the single partition $K_0 = \langle 1 \rangle$ of 1*

$$N(1) = N_{K_0}(1) = 2 \quad \textit{and} \quad S(1) = S_{K_0}(1) = 1$$

hold.

Proposition 2. *If K is an arbitrary partition of k, then*

$$N_K(k) = (k+1) \cdot S_K(k). \tag{3.1}$$

Remark. We get from (3.1) $N(k) = (k+1)S(k)$ by summarizing for all partitions K.

Proof. We can suppose (without an essential restriction of the generality) that the vertices are labelled with the numbers $1, 2, \dots, k+1$ in the non-standard case too. Let the set $\mathfrak{S}$ of all the trees (with k edges) labelled with these numbers be considered. For any element T of $\mathfrak{S}$, let us consider the vertex P to which 1 corresponds. If we interchange the labelling of R and P, then we get a standardly labelled tree. In the mapping, defined by this interchanging, every standardly labelled tree is obtained exactly $k+1$ times.

Proposition 3. *For the partition* $K_0 = \langle 0, 0, \dots, 0, 1\rangle$ *of* k, *the equality*

$$S_{K_0}(k) = N(k-1)$$

is satisfied.

Proof. Let us consider the set of the standardly labelled trees T (with k edges) the partition of which is K_0. If we form the truncated trees of the T's, we get a one-to-one correspondence with the set of the trees with $k-1$ edges, being labelled with the numbers $2, 3, \dots, k+1$.

Theorem 1. *Let* $K = \langle \varkappa_1, \varkappa_2, \dots, \varkappa_k\rangle$ *be an arbitrary partition of the number* k. *Then*

$$S_K(k) = k! \prod_{i=1}^{k} \left(\left(\frac{N(i-1)}{i!} \right)^{\varkappa_i} \frac{1}{\varkappa_i!} \right). \tag{3.2}$$

Remarks. $N(0)$ is regarded to be 1. If $\varkappa_i = 0$, then the i-th factor of the product in (3.2) equals to 1.

Proof. Let us consider the set $\mathfrak{S}$ of all the standardly labelled rooted trees, with k edges, having the numerical partition K; moreover, all the set-partitions Λ of the set $\{2, 3, \dots, k+1\}$ to which the numerical partition K corresponds. To each element T of $\mathfrak{S}$, we assign a set-partition Λ as follows: two numbers i, j belong to a common class precisely if the vertices, labelled with i and j, are in the same branch of T. Let the set-partition Γ of $\mathfrak{S}$ be defined so that $T(\in \mathfrak{S})$ and $T'(\in \mathfrak{S})$ are in a common class when the same set-partition Λ is assigned to them.

It is easy to see that the number of the set-partitions Λ is

$$k! \Big/ \prod_{i=1}^{k} ((i!)^{\varkappa_i} \varkappa_i!).$$

Furthermore, for any fixed Λ, there exist

$$\prod_{i=1}^{k} (N(i-1))^{\varkappa_i}.$$

trees T lying in a common class modulo Γ (this can be pointed out if one considers the forest consisting of the truncated trees of the branches of T). The product of the obtained quantities yields the formula exposed in the theorem.

In the remaining parts of this §, we shall show that the well-known formula of Cayley may be deduced as a consequence of Theorem 1, moreover, two explicit formulae for the quantity $S_K(k)$ will be given.

Corollary 1 (the enumeration formula of Cayley).

$$S(k) = (k+1)^{k-1}.$$

Proof. Let us summarize both sides of (3. 2) for all the partitions of k. We get, by use of (3. 1), the recursion

$$S(k) = \sum_{\Omega_k} S_k(k) = k! \sum_{\Omega_k} \left(\prod_{j=1}^{k} \left(\frac{S(j-1)}{(j-1)!} \right)^{\varkappa_j} \frac{1}{\varkappa_j!} \right). \tag{3.3}$$

This recursion can be solved by the method of generating functions. Let the exponential generating function of $S(k)$ be defined as

$$G(x) = \sum_{k=0}^{\infty} \frac{S(k)}{k!} x^{k+1}$$

(the empty product $\prod_{j=1}^{0}$ is regarded to be 1). By utilizing (3. 3) and Proposition I, we get the functional equation

$$G(x) = \sum_{k=0}^{\infty} x \left[\sum_{\Omega_k} \left(\prod_{j=1}^{k} \left(\frac{S(j-1)x^j}{(j-1)!} \right)^{\varkappa_j} \frac{1}{\varkappa_j!} \right) \right] = x \prod_{j=1}^{\infty} \left[\sum_{\varkappa_j=0}^{\infty} \left(\frac{S(j-1)x^j}{(j-1)!} \right)^{\varkappa_j} \frac{1}{\varkappa_j!} \right] =$$

$$= x \prod_{j=1}^{\infty} \exp \left\{ \frac{S(j-1)x^j}{(j-1)!} \right\} = x \exp \left\{ \sum_{j=1}^{\infty} \frac{S(j-1)x^j}{(j-1)!} \right\} = x \cdot e^{G(x)}$$

for $G(x)$. Since the Bürmann-Lagrange series expansion formula (see [2], p. 22) implies that the single solution of the functional equation

$$x = G(x) e^{-G(x)} \tag{3.4}$$

is

$$G(x) = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} x^{k+1}, \tag{3.5}$$

the assertion is proved.

The next statement is essentially the same as a result of Dénes ([3], Theorem 5), proved by him with other methods.

Corollary 2.

$$S_K(k) = k! \prod_{i=1}^{k} \left(\left(\frac{i^{i-2}}{(i-1)!} \right)^{\varkappa_i} \frac{1}{\varkappa_i!} \right). \tag{3.6}$$

Proof. We get from (3. 2) the formula (3. 6) by substituting i^{i-2} for $S(i-1)$ (in sense of Corollary 1).

Corollaries 1, 2 imply at once

Corollary 3. *Denote the quotient $S_K(k)/S(k)$ by $F_k(K)$. Then*

$$F_k(K) = \frac{k!}{(k+1)^{k-1}} \prod_{i=1}^{k} \left[\left(\frac{i^{i-1}}{i!} \right)^{\varkappa_i} \frac{1}{\varkappa_i!} \right]. \tag{3.7}$$

Theorem 2. *For any partition $K=\langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k\rangle$ of k, we have*

$$S_K(k) = (2\pi)^{-\frac{\varkappa-1}{2}} k^{k+\frac{1}{2}} \left(\prod_{i=1}^{k} (i^{\frac{3\varkappa_i}{2}} \varkappa_i!)\right)^{-1} \exp\left(Z(k) - \sum_{i=1}^{k} \varkappa_i Z(i)\right)$$

where $\varkappa = \varkappa_1+\varkappa_2+\cdots+\varkappa_k$ *($Z(n)$ was defined in § 2).*

Proof. Let Corollary 2 be taken into account. Since

$$\frac{i^{i-2}}{(i-1)!} = \frac{i\cdot i^{i-2}}{i!},$$

we obtain the formula stated in the theorem in such a manner that Proposition II. is applied for $i!$ and $k!$, furthermore, the obvious possibilities for simplifying are performed. The proof is completed.

It remains an open problem to get a simpler formula being asymptotically equal to the quantity

$$\left(\prod_{i=1}^{k} (i^{\frac{3\varkappa_i}{2}} \varkappa_i!)\right)^{-1} \exp\left(Z(k) - \sum_{i=1}^{k} \varkappa_i Z(i)\right)$$

occurring in Theorem 2. We did not succeed in doing this.

§ 4. Some enumeration questions of networks with a rooted tree structure

By a *network*, we understand a finite directed graph G together with a function β defined on the vertex set of G, the range of β is the (real) open interval $(0, 1)$.[2] The number $\beta(P)$ is called the *state* of the vertex P.[3]

In what follows, we shall consider networks formed from rooted trees, any edge being oriented toward its terminal having smaller height. We suppose that the states are assigned randomly to the vertices. Hence, we can assume that $\beta(P)\neq\beta(Q)$ if $P\neq Q$ because the complementary event is (possible but) of probability 0.

The state of the root of a network is called the *state of the network,* too.

Assume that the root of the network G is of (in-)degree 1. Let the truncated tree G' be formed and, to the vertices of G', let the same states be attributed as their states in G. In this case the network G' is called the *truncated network* of G. — The term "*branch of a network*" is used in an analogous sense.

Let e be an edge going from P to Q. For the sake of the brevity, we say that e is a *red edge* or *green edge* according as $\beta(P)<\beta(Q)$ or $\beta(P)>\beta(Q)$, respectively.

We are going to introduce a partition of the set of networks into the types A, B, C, D, E. These types will be defined inductively by the twelve rules (i)—(xii)

[2] This definition (and the subsequent ones still more) has a certain formal character. The reasonable meaning of the notions introduced now will be explained in § 5 where we shall attribute a temporal behavior to the networks, starting with the states $\beta(P)$ assigned to the vertices.

[3] Now we have required that any state $\beta(P)$ must differ from 0 and 1. This was done for the simplicity's sake, because, on the one hand, the possibility when some values $\beta(P)$ are equal to 0 or 1 will be an event of probability 0, on the other hand, our treatment would be more lengthy and intricate if the states 0, 1 were allowed.

We emphasize that the numbers 0, 1 as states will *not* be excluded in § 5.

to be exposed. The root is denoted by R. In the rules (ii), (iii), (iv), (v), (vi), the degree of R is supposed to be 1; in the rules (vii), (viii), (ix), (x), (xi), (xii) the degree of R is supposed to be at least 2. If R is of degree 1, then let e_R be the single edge incident to R.

(i) If G has only one vertex (and no edge), then G belongs to the type A.

(ii) If the truncated network G' of G is either of type C or of type E, then G belongs to the type A.

(iii) If the edge e_R is red and G' is of type B or D, then G belongs to the type B.

(iv) If the edge e_R is green and G' is of type B or D, then G belongs to the type C.

(v) If the edge e_R is red and G' is of type A, then G belongs to the type D.

(vi) If the edge e_R is green and G' is of type A, then G belongs to the type E.

(vii) If G has a branch being of type E, then G belongs to the type E.

(viii) If G has two branches being of type C and D (respectively), then G belongs to the type E.

(ix) If G has no branch of type D or E but it has a branch being of type C, then G belongs to the type C.

(x) If G has no branch of type C or E but it has a branch being of type D, then G belongs to the type D.

(xi) If G has no branch of type C, D or E but it has a branch being of type B, then G belongs to the type B.

(xii) If every branch of G is of type A, then G belongs to the type A.

e_R \ G'	A	B	C	D	E
green	E	C	A	C	A
red	D	B	A	B	A

Table 1.

The rules (ii), (iii), (iv), (v), (vi) are illustrated in Table 1. The rules (vii), (viii), (ix), (x), (xi), (xii) can be summarized by saying that the strength of the five types is the partial ordering seen in Table 2.

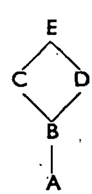

Table 2.

Let N be a network. We agree in some notations. The number of edges of N is k. The state of (the root of) N is β $(0<\beta<1)$. The partition of (the graph of) N is

$$K=\langle \varkappa_1, \varkappa_2, \ldots, \varkappa_k \rangle.$$

The partition

$$\langle \overset{1}{\breve{0}}, \overset{2}{\breve{0}}, \overset{3}{\breve{0}}, \ldots, \overset{k-1}{\breve{0}}, \overset{k}{\breve{1}} \rangle$$

is denoted by K_0. K_1 denotes an arbitrary partition of k different from K_0.

In what follows, we use a small letter p or a Capital one P according as the probability, to be denoted, does or does not depend on β (resp.). (In the latter case, β can vary in the interval (0, 1).) After p, the variable β will or will not be written out.

For a partition K of k, we denote by p_x^K the probability of the event that a randomly chosen network of partition K (with k edges), being of state β, belongs to the type X where X can be any of A, B, C, D, E (and, accordingly, the subscript of p is a small letter a, b, c, d or e). We write $p_x^{(k)}$ for the analogous probability when k is fixed but not K. We denote by $P_x^{(k)}$ the probability of the fact that a network, chosen randomly out of all networks having k edges, belongs to the type X.

We adopt three hypotheses (H1), (H2), (H3):

(H1) All the graph-theoretical structures of forming a rooted tree from k edges (distinghished from each other by the isomorphy of standardly labelled trees) are equiprobable.

(H2) The state of a vertex P is chosen from the real interval (0, 1) in sense of the uniform distribution.

(H3) The states of two different vertices P, Q are chosen independently of each other.

If these hypotheses are accepted, then the rules (i)—(xii) imply the following recursive system for the probabilities introduced above:

(4. 1) $$p_x^{(k)} = \sum_{\Omega_k} p_x^K F_k(K)$$

(the quantities $F_k(K)$ were determined in Corollary 3)

(4. 2) $$P_x^{(k)} = \int_0^1 p_x^{(k)} d\beta$$

(where x can be any of a, b, c, d, e)

(4. 3) $$p_a^{K_0} = P_c^{(k-1)} + P_e^{(k-1)}$$

(4. 4) $$p_b^{K_0} = \int_0^\beta \left(p_b^{(k-1)}(\beta') + p_d^{(k-1)}(\beta')\right) d\beta'$$

(4. 5) $$p_c^{K_0} = \int_\beta^1 \left(p_b^{(k-1)}(\beta') + p_d^{(k-1)}(\beta')\right) d\beta'$$

(4. 6) $$p_d^{K_0} = \int_0^\beta p_a^{(k-1)}(\beta') d\beta'$$

(4. 7) $$p_e^{K_0} = \int_\beta^1 p_a^{(k-1)}(\beta') d\beta'$$

$$(4.8)\qquad p_a^{K_1} = \prod_{j=1}^{k} (p_a^{(j)})^{\varkappa_j}$$

$$(4.9)\qquad p_b^{K_1} = \prod_{j=1}^{k} (p_a^{(j)}+p_b^{(j)})^{\varkappa_j} - p_a^{K_1}$$

$$(4.10)\qquad p_c^{K_1} = \prod_{j=1}^{k} (p_a^{(j)}+p_b^{(j)}+p_c^{(j)})^{\varkappa_j} - (p_a^{K_1}+p_b^{K_1})$$

$$(4.11)\qquad p_d^{K_1} = \prod_{j=1}^{k} (p_a^{(j)}+p_b^{(j)}+p_d^{(j)})^{\varkappa_j} - (p_a^{K_1}+p_b^{K_1})$$

$$(4.12)\qquad p_e^{K_1} = 1-(p_a^{K_1}+p_b^{K_1}+p_c^{K_1}+p_d^{K_1})$$

(in (4. 8)—(4. 12), $K_1=\langle \varkappa_1, \varkappa_2, \dots, \varkappa_k\rangle$). Indeed, equality (4. 1) follows from (H1); (4. 2) is implied by (H2), (H3). The equalities (4. 3)—(4. 7) are consequences of the rules (ii)—(vi), respectively. (4. 8)—(4. 12) follow by analyzing (vii)—(xii) if one takes into account that these rules do not contradict to each other and the premissa of them form a full system of events (if events having probability 0 are disregarded).

We are going to point out that the solution of the equation system (4. 3)—(4. 12) can be reduced to solving a recursive equation system such that the recursive system depends on the expressions

$$x_k = p_x^{(k)} \frac{k^{k-1}}{k!}$$

(where x may be any of a, b, c, d, e) and, of course, on the number k (but is independent of the partition K of k).

Proposition 4. *Let us introduce the simpler notation*

$$\Sigma\Pi_{y,z,\dots,u}$$

for the expression

$$\sum_{\Omega_k}\prod_{j=1}^{k}\left\{(y_j+z_j+\dots+u_j)^{\varkappa_j}\frac{1}{\varkappa_j!}\right\}.$$

The system (4. 3)—(4. 12) *implies the subsequent system of equations* (4. 13)—(4. 17):

$$(4.13)\qquad \left[1+\left(\frac{k+1}{k}\right)^{k-1}\right]a_k = \Sigma\Pi_a + \left(\frac{k}{k-1}\right)^{k-2}\int_0^1 (c_{k-1}+e_{k-1}^{*})\,d\beta'$$

$$(4.14)\qquad \left[1+\left(\frac{k+1}{k}\right)^{k-1}\right](a_k+b_k) = \\ = \Sigma\Pi_{a,b} + \left(\frac{k}{k-1}\right)^{k-2}\left\{\int_0^{\beta}(b_{k-1}+d_{k-1})\,d\beta' + \int_0^1 (c_{k-1}+e_{k-1})\,d\beta'\right\}$$

(4. 15)
$$\left[1+\left(\frac{k+1}{k}\right)^{k-1}\right](a_k+b_k+c_k) = = \Sigma\Pi_{a,b,c}+\left(\frac{k}{k-1}\right)^{k-2}\int_0^1 (b_{k-1}+c_{k-1}+d_{k-1}+e_{k-1})d\beta'$$

(4. 16)
$$\left[1+\left(\frac{k+1}{k}\right)^{k-1}\right](a_k+b_k+d_k) = = \Sigma\Pi_{a,b,d}+\left(\frac{k}{k-1}\right)^{k-2}\left\{\int_0^\beta (a_{k-1}+b_{k-1}+d_{k-1})d\beta'+\int_0^1 (c_{k-1}+e_{k-1})d\beta'\right\}$$

(4. 17)
$$\frac{k^{k-1}}{k!} = a_k+b_k+c_k+d_k+e_k.$$

Proof. We shall use the following terminology: if two equations of form $X=Y$, $Z=W$ are given, then the equation $XZ=YW$ is called the product of them.

Let us form the product of any of (4. 3), (4. 4), (4. 5), (4. 6) with (3. 7), applied for K_0; similarly, let the product of any of (4. 8)—(4. 12) with (3. 7), applied for K_1, be formed. Furthermore, let the sums corresponding to (4. 1) be formed for each of the subscripts a, b, c, d, e (for x), concerning all the partitions of k. This equation system can be deduced by use of (3. 7) to the system (4. 13)—(4. 17).

We did not succeed in solving the system (4. 13)—(4. 17) completely. However, we can prove some partial results:

Theorem 3. *The following assertions hold:*

(A) *Any of a_k, b_k+c_k, d_k+e_k is a rational expression of k, independent of β.*

(B) *b_k and d_k are polynomials of β with degree exactly k, with non-negative (rational) coefficients, without a term of degree zero.*

(C)
$$a_k+b_k+c_k+d_k+e_k = \frac{k^{k-1}}{k!}.$$

(D) *Each of a_k, b_k+c_k, d_k+e_k, b_k, d_k is contained in the interval* $\left[0, \frac{k^{k-1}}{k!}\right]$.

(E) *Each of a_k, b_k, c_k, d_k, e_k is a polynomial of β with coefficients being rational in k.*

Proof. First we verify the independence statements of the assertion (A). The last term of (4. 13) does not depend on β, because the limits of the integration concerning β are constant. Subtract a_k from both sides; we get $\left[\frac{k+1}{k}\right]^{k-1} a_k$ on the left-hand side, and a sum on the right-hand one each term of which is a product of expressions a_j (with $j<k$) (the summation is taken over all partitions of the number k except the one-member partition $j=k$ that was subtracted). Hence the independence of a_k can be obtained by induction with respect to k.

(4. 13) implies by an analogous deduction that $a_k+b_k+c_k$ is independent on β. Since a_k proved to be independent, the same holds for b_k+c_k, too.

The independence of d_k+e_k follows from (4. 17) and the previous parts of the proof.

The rationality statements of (A) can be obtained as consequences of (E) (to be proved later). Now we are going to prove (B).

Let (4. 13) be subtracted from (4. 14). We get by the independence of a_k:

$$\left[1+\left(\frac{k+1}{k}\right)^{k-1}\right]b_k = \Sigma\Pi_{a,b}-\Sigma\Pi_a+ \tag{4. 18}$$
$$+\left(\frac{k}{k-1}\right)^{k-2}\int_0^\beta (b_{k-1}+d_{k-1})d\beta' =$$
$$=\sum_{\Omega_k}\left(\prod_{j=1}^{k}\frac{1}{\varkappa_j!}\right)\left(\prod_{j=1}^{k}(a_j+b_j)^{\varkappa_j}-\prod_{j=1}^{k}a_j^{\varkappa_j}\right)+\left(\frac{k}{k-1}\right)^{k-2}\int_0^\beta (b_{k-1}+d_{k-1})d\beta'.$$

The binomial theorem enables the subsequent transformation:

$$\prod_{j=1}^{k}(a_j+b_j)^{\varkappa_j}-\prod_{j=1}^{k}a_j^{\varkappa_j} = \prod_{j=1}^{k}\left\{a_j^{\varkappa_j}+\sum_{l=0}^{\varkappa_j-1}\binom{\varkappa_j}{l}a_j^l b_j^{\varkappa_j-l}\right\}-\prod_{j=1}^{k}a_j^{\varkappa_j}$$

(where the empty sum of type $\sum_{l=0}^{-1}$ or $\sum_{l=0}^{0}$ is regarded to be 0). If we multiply out in the first product, then an expression is yielded every term of which contains a power of b_j (with a positive exponent) as a factor, because precisely that term $\prod_{j=1}^{k}a_j^{\varkappa_j}$ is subtracted which does not contain such a power. It is clear that subtraction cannot occur in the remaining terms, furthermore, if b_k has been subtracted from both sides of (4. 13), every subscript j of a b_j on the right-hand side of the resulting equality satisfies $j<k$. This implies the statement, to be proved, by induction, with regard to the following remarks. The right-hand side is a sum each term of which is a polynomial of degree $\Sigma\varkappa_j\cdot j = k$ with non-negative coefficients without a term of degree zero (by the induction hypothesis). The latter term containing the integral is the integral of a polynomial with non-negative coefficients on the interval $[0, \beta]$, the degree of this polynomial is exactly $k-1$; hence the integration yields a polynomial exactly of degree k with non-negative coefficients, without a term of degree zero.

Thus the assertion of (B) concerning b_k is proved. By the analogy, we give the proof for d_k only in outlines. We subtract (4. 14) from (4. 16); afterwards, we calculate with a_j+b_j, d_j, $a_{k-1}\beta$ instead of a_j, b_j, $\int_0^\beta (b_{k-1}+d_{k-1})d\beta'$ (resp.) occurring in the above proof concerning b_k.

(C) coincides with (4. 17).

In order to prove (D), first we note that the definition of x_k implies that each of a_k, b_k, c_k, d_k, e_k is contained in the interval $\left[0, \frac{k^{k-1}}{k!}\right]$. Since the values $p_b^{(k)}+p_c^{(k)}$ and $p_d^{(k)}+p_e^{(k)}$ are probabilities, also b_k+c_k, d_k+e_k belong to $\left[0, \frac{k^{k-1}}{k!}\right]$.

Finally also the assertions of (E) will be proved by induction (with respect to k). Suppose that (E) is true with $k-1$ (instead of k). (4. 13) implies that the assertion (with k) holds for a_k; (4. 14) implies that it is valid for a_k+b_k; hence it is true for b_k, too. Similarly, the assertion follows from (4. 15) and (4. 14) for c_k, from (4. 16) and (4. 14) for d_k, thus (by (4. 17)) for e_k as well.

The inductive proof is completed by Tables 3, 4. Table 4 contains the values of $p_x^{(k)}$ and x_k if k is 0, 1, 2, 3, 4; similarly, Table 3 gives the values of p_x^K when $0 \leq k \leq 4$.

k	0	1	2		3		
K	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 0, 1\rangle$	$\langle 2, 0\rangle$	$\langle 0, 0, 1\rangle$	$\langle 1, 1, 0\rangle$	$\langle 3, 0, 0\rangle$
$F_k(K)$	1	1	2/3	1/3	9/16	3/8	1/16
p_a^K	1	0	1/2	0	4/9	0	0
p_b^K	0	0	$\beta^2/2$	0	$2\beta^3/9$	0	0
p_c^K	0	0	$(1-\beta^2)/2$	0	$(2-2\beta^3)/9$	0	0
p_d^K	0	β	0	β^2	$\beta/3$	$(\beta+2\beta^2)/3$	β^3
p_e^K	0	$1-\beta$	0	$1-\beta^2$	$(1-\beta)/3$	$(3-\beta-2\beta^2)/3$	$1-\beta^3$

k	4			
K	$\langle 0, 0, 0, 1\rangle$	$\langle 1, 0, 1, 0\rangle$	$\langle 0, 2, 0, 0\rangle$	$\langle 2, 1, 0, 0\rangle$
$F_k(K)$	64/125	36/125	12/125	12/125
p_a^K	89/192	0	1/9	0
p_b^K	$(30\beta^2+16\beta^3+9\beta^4)/192$	0	$(2\beta^2+\beta^4)/9$	0
p_c^K	$(55-30\beta^2-16\beta^3-9\beta^4)/192$	0	$(3-2\beta^2-\beta^4)/9$	0
p_d^K	$\beta/4$	$(4\beta+5\beta^2+4\beta^3+3\beta^4)/16$	$(2\beta^2+3\beta^4)/9$	$(\beta^2+2\beta^4)/3$
p_e^K	$(1-\beta)/4$	$(16-4\beta-5\beta^2-4\beta^3-3\beta^4)/16$	$(5-2\beta^2-3\beta^4)/9$	$(3-\beta^2-2\beta^4)/3$

Table 3.

Proposition 5. *Let us introduce the notations m_k, u_k, v_k, w_k, z_k by*

$$m_k = (2\pi)^{-\frac{1}{2}} k^{-\frac{3}{2}} e^k,$$

$$u_k = a_k+b_k, \quad v_k = a_k+b_k+c_k,$$

$$w_k = a_k+b_k+d_k, \quad z_k = d_k+e_k.$$

k	0	1	2	3	4
$p_a^{(k)}$	1	0	1/3	1/4	31/125
a_k	0	0	1/3	3/8	248/375
$P_a^{(k)}$	1	0	1/3	1/4	31/125
$p_b^{(k)}$	0	0	$\beta^2/3$	$\beta^3/8$	$(38\beta^2+16\beta^3+13\beta^4)/375$
b_k	0	0	$\beta^2/3$	$3\beta^3/16$	$(304\beta^2+128\beta^3+104\beta^4)/1125$
$P_b^{(k)}$	0	0	1/9	1/32	289/5625
$p_c^{(k)}$	0	0	$(1-\beta^2)/3$	$(1-\beta^3)/8$	$(67-38\beta^2-16\beta^3-13\beta^4)/375$
c_k	0	0	$(1-\beta^2)/3$	$(3-3\beta^3)/16$	$(536-304\beta^2-128\beta^3-104\beta^4)/1125$
$P_c^{(k)}$	0	0	2/9	3/32	716/5625
$p_d^{(k)}$	0	β	$\beta^2/3$	$(5\beta+4\beta^2+\beta^3)/16$	$(300\beta+215\beta^2+108\beta^3+237\beta^4)/1500$
d_k	0	β	$\beta^2/3$	$(15\beta+12\beta^2+3\beta^3)/32$	$(600\beta-430\beta^2-216\beta^3-474\beta^4)/1125$
$P_d^{(k)}$	0	1/2	1/9	49/192	4441/22500
$p_e^{(k)}$	0	$1-\beta$	$(1-\beta^2)/3$	$(10-5\beta-4\beta^2-\beta^3)/16$	$(860-300\beta-215\beta^2-108\beta^3-237\beta^4)/1500$
e_k	0	$1-\beta$	$(1-\beta^2)/3$	$(30-15\beta-12\beta^2-3\beta^3)/32$	$(1720-600\beta-430\beta^2-216\beta^3-474\beta^4)/1125$
$P_e^{(k)}$	0	1/2	2/9	71/192	8459/22500

Table 4.

If $k\to\infty$, then the following equation system (4. 19)—(4. 23) *is asymptotically vaild for the polynomials u_k, w_k and the constants a_k, v_k, z_k:*

(4. 19) $$(1+e)a_k = \Sigma\Pi_a+m_k-e\int_0^1 w_{k-1}d\beta'$$

(4. 20) $$(1+e)u_k = \Sigma\Pi_u+m_k-e\left\{a_{k-1}\beta+\int_\beta^1 w_{k-1}d\beta'\right\}$$

(4. 21) $$(1+e)v_k = \Sigma\Pi_v+m_k-e\cdot a_{k-1}$$

(4. 22) $$(1+e)w_k = \Sigma\Pi_w+m_k-e\int_\beta^1 w_{k-1}d\beta'$$

(4. 23) $$z_k+v_k = m_k$$

Proof. Let us apply the substitution

$$c_{k-1}+e_{k-1} = \frac{(k-1)^{k-2}}{(k-1)!}-w_{k-1}$$

and term-wise integration in (4. 13), using (4. 17). Let analogous transformations be performed for (4. 14), (4. 15), (4. 16) (e.g., in case of (4. 14), the substitution

$$b_{k-1}+c_{k-1}+d_{k-1}+e_{k-1} = \frac{(k-1)^{k-2}}{(k-1)!}-a_{k-1}$$

is to be applied). Taking the asymptotic equalities

$$\left(\frac{k+1}{k}\right)^{k-1} \sim \left(\frac{k}{k-1}\right)^{k-2} \sim e,$$

$$\frac{k^{k-1}}{k!} \sim (2\pi)^{-\frac{1}{2}} k^{-\frac{3}{2}} e^k$$

into account, we get the system (4. 19)—(4. 23).

Remark. For the particular choice $\beta=0$,

$$w_k(0)=u_k(0)=a_k$$

holds (this follows from Theorem 3 as well from (4. 19)—(4. 23)).

§ 5. The connection between the type and the behavior of a tree-structure network

The types A, B, C, D, E were distinguished in § 4 in a formal way so that the reader should feel the lack of a convincing motivation. Now (as this was promised in Footnote 2) we are going to point out that the fact that a network N is contained in one or other of the types A, B, C, D, E implies entirely unlike consequences if the behaviour of the network is studied, as it was introduced in Section 3 of the former article [1], starting with the values $\beta(P)$.

We suppose that the reader is familiar with Sections 1—3 of [1]. Let N be a tree-type network, let us denote the vertices of N by $P_1, P_2, \ldots, P_{k+1}$ (where k is the number of edges of N) such that the subscripts constitute a standard labelling. To any P_i, let us assign a function $\alpha_i(t)$ by the method explained in Sect. 3 of [1] such that the initial values are determined by $\alpha_i(0)=\beta(P_i)$ (where $1 \leq i \leq k+1$). Especially, to the root P_1 the function $\alpha_1(t)$ is attributed. We have

Proposition 6. *If the assumptions, exposed previously, are accepted, then the following six statements are valid for the network N:*

(I) *If N belongs to one of the types* A, B, C, D, E, *then the functions $\alpha_i(t)$ are defined at least in the interval $[0, \tau]$ (where $1 \leq i \leq k+1$).*[4]

(II) *If N belongs to the type* A, *then $\alpha_1(\tau)=1$.*

(III) *If N belongs to the type* B, *then $0<\alpha_1(\tau)<1$ and there exists a t such that $0<t<\tau$ and $\alpha_1(t)=1$.*

(IV) *If N belongs to the type* C, *then $0<\alpha_1(\tau)<1$ and $\alpha_1(t)<1$ for every t lying in the interval $[0, \tau]$.*

(V) *If N belongs to the type* D, *then $\alpha_1(\tau)=0$ and there exists a t such that $0<t<\tau$ and $\alpha_1(t)=1$.*

(VI) *If N belongs to the type* E, *then $\alpha_1(\tau)=0$ and $\alpha_1(t)<1$ for every t lying in the interval $[0, \tau]$.*

[4] The words "at least" mean that the α_i's may also be defined for some (possibly all) values t fulfilling $t>\tau$.

Remark. Since the conclusions of (II)—(VI) exclude each other, each of (II)—(VI) holds with the formulation "if and only if" provided that N is contained in some of the five types.

Proof. (I) does not require a separate treatment (it follows from the other five assertions). To prove (II)—(VI), we use induction with respect to the number of vertices of N. The type of a network was defined in § 4 by the rules (i)—(xii) recursively; now twelve cases can be distinguished corresponding to these rules.

If N has a single vertex, then, on the one hand, it is of type A by (i); on the other hand, evidently $\alpha_1(t)=1$ if $t \geqq \tau(1-\beta(P_1))$, especially, $\alpha_1(\tau)=1$.

Assume that the number of vertices of N is $k+1$ and the assertions (II)—(VI) hold for the networks having at most k vertices. We distinguish eleven cases corresponding to (ii)—(xii).

Suppose that N is of type A by virtue of (ii). Denote (by P_1 the root of N and) by P_2 the root of the truncated network N'. There exists the edge $\overrightarrow{P_2P_1}$ and no other edge is incident with P_1 (in N). By the induction hypothesis, the conclusion of (IV) or (VI) holds for N, thus $\alpha_2(t)<1$ is valid in the whole interval $[0, \tau]$. Hence $\alpha_1(t)=1$ in the interval $[\tau(1-\beta(P_1)), \tau]$.

Assume that N belongs to the type B in consequence of (iii). Either the conclusion of (III) or that of (V) holds for N'; in both cases, $\alpha_2(t)=1$ is satisfiable with some t in $(0, \tau)$. Let t_0 be the minimal t such that $t \leqq t' \leqq \tau$ implies $\alpha_2(t')<1$ (it exists since $\alpha_2(\tau)<1$ and the functions α are continuous from right); it is clear that the value of α_1 grows from 0 to $(\tau-t_0)/\tau$ in the interval $[t_0, t]$. Because $\overrightarrow{P_2P_1}$ is a red edge, $\beta(P_2)<\beta(P_1)$, hence $\alpha_1(\tau(1-\beta(P_1)))=1$.

If N is of type C in sense of (iv), then $\beta(P_2)>\beta(P_1)$, thus α_1 grows in the interval $[0, \tau(1-\beta(P_2)))$ from $\beta(P_1)$ towards $1-\beta(P_2)+\beta(P_1)$ (<1) (without reaching it), furthermore $\alpha_2(\tau(1-\beta(P_2)))=1$ and $\alpha_1(\tau(1-\beta(P_2)))=0$. $\alpha_1(t) \leqq \beta(P_2)<1$ whenever $\tau(1-\beta(P_2)) \leqq t \leqq \tau$.

Still we have to prove $0<\alpha_1(\tau)$. If N' is of type B, then this is obviously valid. If N' is of type D and there exists a t' such that $0<t'<\tau$ and the implication

$$t' \leqq t \leqq \tau \Rightarrow \alpha_2(t)<1$$

is true, then evidently $\alpha_1(\tau) \geqq (\tau-t')/\tau>0$. If N' is of type D and no t' (with the mentioned property) exists, then it is clear that some α_i grows in the interval $[0, \tau]$ from 0 to 1; however, $\alpha_i(0)(=\beta(P_i))=0$ was excluded (cf. the hypothesis (H2)).

If the type of N is determined by (v) or (vi), then the proof can be carried out by similar ideas.

If one of (vii)—(xii) decides the type of N, then the conclusion of the corresponding statement of Proposition 6 can be proved by use of the subsequent principle (following from the behaviour defined in [1]): if the out-degree of P_1 is at least two, then the value $\alpha_1(t)$ (at any instant t) equals to the minimum of the values that result if the values assigned to P_1 (at t) are calculated for the several branches of N.

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY

CENTRAL RESEARCH INSTITUTE FOR PHYSICS
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY

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