On the semigroup of automaton mappings with finite alphabet

By P. Dömösi

Let $F(X)$ denote the free semigroup generated by a (non-empty) finite set X, and consider the set K_x of all automaton mappings of $F(X)$ into itself. It has been shown (see [4] and [6]) that K_x is a semigroup under the usual multiplication of mappings. It is also known that the subgroup A_x consisting of all one-to-one mappings from K_x has cardinality of continuum provided X has at least two elements (see [1]). This implies that neither A_x nor K_x has any finite generating system.

Let G_x and L_x denote the group and semigroup, respectively, of all automaton mappings of $F(X)$ into itself induced by finite automata (see [4] and [6]). It has been proved in [2] that G_x and L_x have no finite generating system (except for the trivial case). In this paper we show that neither K_x nor L_x has any minimal generating system provided X has more than one element. It is an unsolved problem whether A_x and G_x have any minimal generating system.

Before proving our statement, we introduce some notions and notations.

First of all we assume that $F(X)$ has the identity element e. By the *length* $|p|$ of a word $p \in F(X)$ we mean the number of all occurences of elements from X. (Thus $|e|=0$.) We say that a word q is an *initial part* of p if there exists an $r \in F(X)$ such that $qr=p$, this situation is denoted by $q \subseteq p$. If q is a proper initial part of p, i.e. $q \subseteq p$ and $|q|<|p|$ then we use the notation $q \subset p$.

Take two non-empty sets X and Y. A mapping φ of $F(X)$ into $F(Y)$ is called *automaton mapping* if for any $p \in F(X)$, $|p|=|\varphi(p)|$ and $\varphi(pq)=\varphi(p)r$ hold where r is a suitable word in $F(X)$ (see [3]). It is well-known that every automaton mapping can be induced by automaton and conversely.

Consider an arbitrary automaton mapping $\varphi : F(X) \to F(Y)$ and let $p \in F(X)$. If for a $q \in F(X)$, $\varphi(pq)=\varphi(p)r$ hold then let us denote this r by $\varphi_p(q)$. Let $\psi : F(X) \to F(Y)$ be a mapping for which $\psi(q)=\varphi_p(q)$ $(q \in F(X))$ holds. This φ_p is called a *state* of φ induced by p. It should be noted that every state of an automaton mapping is an automaton mapping.

We say that $\varphi : F(X) \to F(Y)$ is an automaton mapping *with finite alphabet* if φ is an automaton mapping and X and Y are finite. An automaton mapping with finite alphabet is *finite* if it has finitely many different states. It is known from [3] that an automaton mapping is finite if and only if it can be induced by a finite automaton. Thus the semigroup L_x consists of all finite automaton mappings.

Let X be an arbitrary non-empty finite set and consider the semigroup K_x of all automaton mappings of $F(X)$ into itself. Take $\varphi \in K_x$ and let $I(\varphi)$ denote the set of all $p\in F(X)$ for which there exist $x_1, x_2\in X$ with $x_1\neq x_2$ such that $\varphi_p(x_1)=\varphi_p(x_2)$. Consider the set $J(\varphi)$ of all words p from $I(\varphi)$ whose each proper initial part q satisfies the condition $q\notin I(\varphi)$. If $e\in I(\varphi)$ then let $J(\varphi)=\langle e\rangle$.

The following holds.

Lemma. If $p\notin J(\varphi^{(1)})$ and $\varphi^{(1)}(p)\notin J(\varphi^{(2)})$ then $p\notin J(\varphi^{(1)}\varphi^{(2)})$ for any $\varphi^{(1)}, \varphi^{(2)}\in K_x$ and $p\in F(X)$.

Proof. By the definition of $I(\varphi)$, it can easily be seen that $p\in I(\varphi^{(1)}\varphi^{(2)})$ if and only if $p\in I(\varphi^{(1)})$ or $\varphi^{(1)}(p)\in I(\varphi^{(2)})$. Therefore, if $p\notin I(\varphi^{(1)})$ and $\varphi^{(1)}(p)\notin I(\varphi^{(2)})$ then $p\notin I(\varphi^{(1)}\varphi^{(2)})$, i.e. in this case our Lemma is valid because $J(\varphi^{(1)}\varphi^{(2)})\subseteq I(\varphi^{(1)}\varphi^{(2)})$.

Assume that $p\in I(\varphi^{(1)})\setminus J(\varphi^{(1)})$. Then, by the definition of $J(\varphi^{(1)})$, p has a proper initial part q such that $q\in I(\varphi^{(1)})$. Therefore, $q\in I(\varphi^{(1)}\varphi^{(2)})$, i.e. taking into consideration $q\subset p$, we get $p\notin J(\varphi^{(1)}\varphi^{(2)})$.

It remains to be shown that our Lemma is valid in the case of $\varphi^{(1)}(p)\in I(\varphi^{(2)})\setminus J(\varphi^{(2)})$. Let $r\subset\varphi^{(1)}(p)$ denote a proper initial part of $\varphi^{(1)}(p)$ for which $r\in I(\varphi^{(2)})$. (By the definitions of $J(\varphi^{(2)})$ and $I(\varphi^{(2)})$ there exists such r.) Thus there exists a proper initial part q of p such that $\varphi^{(1)}(q)=r$. Therefore, by $\varphi^{(1)}(q)\in I(\varphi^{(2)})$ we have $q\in I(\varphi^{(1)}\varphi^{(2)})$. Since $q\subset p$ this means that $p\notin J(\varphi^{(1)}\varphi^{(2)})$ which completes the proof of the Lemma.

We have the following

Theorem. If X is a finite set having at least two elements then neither K_x nor L_x has any minimal generating system.

Proof. Let K be a generating system of K_x or L_x. First we show the existence of a $\varphi\in K$ for which $J(\varphi)$ has at least two elements.

Let L denote the set of all elements φ from K for which $J(\varphi)$ has only one element. Take arbitrary elements $\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(n)}\in L$. Using our Lemma it can be proved by induction that $J(\varphi^{(1)}\varphi^{(2)}\dots\varphi^{(n)})$ has at most n elements.

Let $x_i\in X$ be fixed. We define a mapping $\psi\in L_x$ as follows:

$$\psi_p(x)=\begin{cases} x & \text{if } p\in F(\langle x_i\rangle),\\ x_i\text{-otherwise.} \end{cases}$$

Since $J(\psi)$ is infinite thus ψ cannot be given as a product of mappings from L. Therefore, $K\setminus L$ is not empty, i.e. there exists a $\varphi\in K$ such that $J(\varphi)$ has at least two elements.

Let $p_1, p_2\in J(\varphi)$ different words such that

$$|p_1|=\min_{q\in J(\varphi)}|q| \quad\text{and}\quad |p_2|=\min_{q\in J(\varphi)\setminus\langle p_1\rangle}|q|. \tag{1}$$

Take two mappings $\varphi^{(1)}, \varphi^{(2)}$ from K_x defined as follows. For any $p\in F(X)$ and $x\in X$, let

$$\varphi_p^{(1)}(x)=\begin{cases} \varphi_p(x) & \text{if } p_1\subseteq p,\\ x\text{-otherwise} \end{cases} \tag{2}$$

and

$$\varphi_p^{(2)}(x)=\begin{cases} x & \text{if } p_1\subseteq p,\\ \varphi_p(x)\text{-otherwise.} \end{cases} \tag{3}$$

Let us show that a) $\varphi^{(1)}\varphi^{(2)}=\varphi$, b) $\varphi^{(1)}$ and $\varphi^{(2)}$ can be given as products of elements from $K\setminus\langle\varphi\rangle$ and c) if $\varphi\in L_x$ then $\varphi^{(1)}$ and $\varphi^{(2)}$ are in L_x.

To prove our theorem, by the choice of φ, it is enough to show that a)—c) are valid.

By the definition of automaton mappings it is obvious that a) holds.

In order to prove b) it is enough to show that whenever φ is among $\varrho^{(1)}, \varrho^{(2)}, \ldots$ $\ldots, \varrho^{(n)} (\in K)$ then $\varrho^{(1)}\varrho^{(2)}\ldots\varrho^{(n)}\notin\langle\varphi^{(1)}, \varphi^{(2)}\rangle$. In other words, for any pair $\psi^{(1)}$, $\psi^{(2)}(\in K_x)$,

$$\psi^{(1)}\varphi\psi^{(2)}\neq\varphi^{(1)} \tag{4}$$

and

$$\psi^{(1)}\varphi\psi^{(2)}\neq\varphi^{(2)}. \tag{5}$$

By (1), for each word $q(\in F(X))$ with $|q|<|p_1|$ we have $q\notin I(\varphi)$. Thus, using (2) and (3) we get $q\notin I(\varphi^{(1)})\cup I(\varphi^{(2)})$ provided $|q|<|p_1|$. Therefore, if there exists a $q\in F(X)$ with $|q|<|p_1|$ and $q\in I(\psi^{(1)}\varphi\psi^{(2)})$ then (4) and (5) holds. If such q does not exist then for arbitrary $p\in F(X)$ with $|p|=|p_1|$ there is an $r\in F(X)$ such that $\varphi_{\psi^{(1)}(r)}=\varphi_p$.

For a given $\psi\in K_x$, let us denote by $I(k, \psi)$ the number of all elements from $I(\psi)$ of length k. Then, taking into consideration the fact that $p\in I(\varphi^{(1)}\varphi^{(2)})$ if and only if $p\in I(\varphi^{(1)})$ or $\varphi^{(1)}(p)\in I(\varphi^{(2)})$ we get $I(|p_1|, \psi^{(1)}\varphi)\geqq I(|p_1|, \varphi)$. In the same way we get

$$I(|p_1|, \psi^{(1)}\varphi\psi^{(2)})\geqq I(|p_1|; \varphi). \tag{6}$$

By (3) it is obvious that $p_1\notin I(\varphi^{(2)})$. On the other hand, by (1), p_1 and p_2 are in $J(\varphi)$, i.e. $p_1\nsubseteq p_2$. Thus, taking into consideration (2) we get $p_2\notin I(\varphi^{(1)})$.

If $|p_1|=|p_2|$ then $I(|p_1|, \varphi^{(1)}), I(|p_1|, \varphi^{(2)})<I(|p_1|, \varphi)$ because of $p_1, p_2\in$ $\in J(\varphi)(\subseteqq I(\varphi))$. This, by (6), means that $I(|p_1|, \varphi^{(1)}), I(|p_1|, \varphi^{(2)})<I(|p_1|, \psi^{(1)}\varphi\psi^{(2)})$. Therefore, in this case (4) and (5) hold.

Let $|p_1|<|p_2|$. Then, by (1) $I(\varphi)$ has no word of length $|p_1|$ except for p_1. Since $p_1\notin I(\varphi^{(2)})$ thus, by (3), $I(|p_1|, \varphi^{(2)})=0$, i.e. $I(|p_1|, \varphi^{(2)})<I(|p_1|, \varphi)$. Therefore, by (6), (5) holds in this case too.

We now show that (4) holds if $|p_1|<|p_2|$. As has been shown it can be assumed that $q\notin I(\psi^{(1)}\varphi\psi^{(2)})$ if $|q|<|p_1|$ because in the opposite case (4) holds. Thus $q\notin I(\psi^{(1)})$ holds as well, that is, for every word $r\in F(X)$ of length less than or equal to $|p_1|$ there exists a $t\in F(X)$ such that $\psi^{(1)}(t)=r$. Therefore, $\psi^{(1)}(p_1)\neq p_1$ implies $\psi^{(1)}(s)=p_1$ for a suitable $s\in F(X)$ with $s\neq p_1$. In this case $s\in I(\psi^{(1)}\varphi\psi^{(2)})$ because of $p_1\in I(\varphi)$. On the other hand, by $|p_1|=|s|$ and (2), $p_1\neq s$ implies $r\notin I(\varphi^{(1)})$ from which (4) follows.

Now suppose that $\psi^{(1)}(p_1)=p_1$. Let us write p_2 in the form $p_2=pr$ where $|p|=$ $=|p_1|$. We can assume that there exists a word $q\in F(X)$ such that $\psi^{(1)}(q)=p$ (because, as has been shown, in the opposite case (4) holds). Moreover, by (1), $p_1\nsubseteq p_2$, that is, $p\neq p_1$. Since $\psi^{(1)}(p_1)=p_1$ thus $p\neq\psi^{(1)}(p_1)$. This, by $\psi^{(1)}(q)=p$, means that $q\neq p_1$. Therefore, for arbitrary $s\in F(X)$ we have $\varphi_q^{(1)}(s)=s$, i.e. $qs\notin I(\varphi^{(1)})$. Thus if for $p_2(=pr)$ there exists no word $r_1\in F(X)$ such that $\psi_q^{(1)}(r_1)=r$ then (4) holds, because in this case there is a word $r_2\in F(X)$ with $qr_2\in I(\psi^{(1)}\varphi\psi^{(2)})$. Now assume that $\psi_q^{(1)}(r_1)=$ $=r(r_1\in F(X))$. Then $qr_1\in I(\psi^{(1)}\varphi\psi^{(2)})$ because of $p_2\in I(\varphi)$. Therefore, (4) holds.

Thus we have got that (4) and (5) are valid in all possible cases, i.e. b) holds.

It remains to be shown that c) is valid. It is clear that the number of all states of φp_1 is less than or equal to that of all states of φ. Therefore, using (2) and (3) we get that both $\varphi^{(1)}$ and $\varphi^{(2)}$ have finitely many different states. Thus $\varphi \in L_x$ implies $\varphi^{(1)}, \varphi^{(2)} \in L_x$. This completes the proof of our Theorem.

References

[1] Чакань, Б., Ф. Гечег, О группеав томатных подстановок, *Кибернетика*, 1:5, 1965, pp. 14—17.
[2] Гечег, Ф., О группе взаимно однозначных преобразований, определенные конечными автоматами, *Кибернетика*, 1:1, 1965, pp. 37—39.
[3] Глушков, В. М., Абстрактная теория автоматов, *Успехи Мат. Наук.*, v. 16, 1961, pp. 3—62.
[4] Хорейщ Й., Преобразования, определенные конечнымиавто матами, *Проблемы кибернетики*, v. 9, 1963, pp. 23—26.
[5] Raney, G. N., Sequential functions, *J. Assoc. Comput. Mach.*, v. 5, 1968, pp. 177—180.
[6] Заровный, В. П., Автоматные подстановки и сплетения групп, *Кибернетика*, 1:1, 1965, pp. 29—36.

(Received October 30, 1971)