Algorithm for constructing of university timetables and criterion of consistency of requirements

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1. Introduction

The construction of timetable by means of a computer is the subject of numerous publications. In all these papers two similar problems are investigated:

(1) constructing a school timetable,

(2) constructing a timetable for university department.

In the first case there are given three sets: a set of classes, a set of teachers and a set of time periods. One lesson can be interpreted as a meeting of a teacher and a class for one period. The problem is to schedule all lessons so that no teacher and no class has two or more different lessons at the same hour. Moreover, we must also take into consideration the problem of the so-called preassignments, it means that lessons are not available at every period of time.

The second case is more complicated. We shall indicate below three requirements which will be the subject of further investigations.

(a) University department consists of years, sections groups etc. which can have certain common jobs.

(b) One lecture can last more than one time period.

(c) Every lecture must take place in a given room; therefore apart from sets just defined there is given a fourth set, a set of rooms.

In the present paper we shall give a condition necessary and sufficient for existence of university timetable and an algorithm of constructing of it. We shall use some basic notions of the theory of graphs such as; an independent set, a chromatic number or a colouring of a graph whose definitions the reader can find in [1].

2. Two definitions of timetable

For the first time the timetable problem was defined by Gotlieb [2] as follows.

Let $T=\{t_i\}$ $(i\leqq n)$ be the set of teachers, $C=\{c_j\}$ $(j\leqq n)$ the set of classes and $H=\{h_k\}$ $(k\leqq p)$ the set of time periods.

Let us consider two matrices: $A=\{a_{ij}\}$ $(i\leqq m, j\leqq n)$ where a_{ij} is an integer pointing out how many times a teacher t_i must meet with a class c_j and $B=\{b_{ijk}\}$ $(i\leqq m, j\leqq n, k\leqq p)$ where element b_{ijk} is 1 if teacher t_i can meet class c_j at hour h_k and 0 in the opposite case. A pair $\langle A, B\rangle$ defines the set of all requirements.

Definition 1. The matrix $S=\{s_{ijk}\}$ $(i\leqq m, j\leqq n, k\leqq p)$ fulfilling the conditions:

$$\sum_{i=1}^{m} s_{ijk} \leqq 1 \qquad (1) \qquad\qquad \sum_{j=1}^{n} s_{ijk} \leqq 1 \qquad (2)$$

$$\sum_{k=1}^{p} s_{ijk} = a_{ij} \qquad (3) \qquad\qquad \text{If } s_{ijk}=1 \text{ then } b_{ijk}=1 \qquad (4)$$

for arbitrary $i\leqq m$, $j\leqq n$, $k\leqq p$ is called a timetable for the requirements $\langle A, B\rangle$.

Gotlieb describes in his paper an algorithm of constructing the timetable S for given requirements $\langle A, B\rangle$. The method used by him is based on theorem of P. Hall [3] on distinct representatives of subsets. Unfortunatly this algorithm does not answer the questions whether timetable exists and whether solutions attained are all which satisfy conditions (1)—(4).

In order to introduce our method of reducing the timetable problem to the colouring of graph we must change a little the definition of timetable. In 2. 1 we shall show that this new definition is an extension of the first one.

Now, let $L=\{l_i\}$ $(i\leqq q)$ be the set of all lessons. With every l_i $(i\leqq q)$ we associate the set $g_i\subset H$, of time periods at which lesson l_i is admissible. The interference condition between lessons is described by the relation $\varrho\subset L\times L$ fulfilled if the lessons can not be scheduled at the same our.

Definition 2. A sequence $x=\langle h^1, ..., h^q\rangle$ of elements of H will be called a timetable for the family $G=\{g_i\}$ $(i\leqq q)$ and the relation ϱ iff

$$h^i\in g_i \qquad i=1, ..., q \qquad (5)$$

$$\text{if } \; l_i\varrho l_j \; \text{ then } \; h^i\neq h^j \quad i, j\leqq q. \qquad (6)$$

In fact, these conditions say that if lesson l_i is scheduled at hour h^i then from (5) l_i is admissible at h^i and from (6) lessons never interfere.

Now we shall show that definition 1 can be replaced by the other.

2. 1. For arbitrary requirements $\langle A, B\rangle$ there exist set L, family G and relation ϱ so that there is a one-to-one correspondence between timetables S and x.

Proof. For the given matrix A we can easily define L as a set of corresponding pairs $\langle t_i, c_j\rangle$. The relation ϱ is given by the following equivalence:

$$\langle t_i, c_j\rangle\varrho\langle t_u, c_w\rangle\equiv(i=u)\wedge(j=w).$$

Next

$$g_{ij}=\{h_k: b_{ijk}=1\}$$

is a set of time periods admissible for $\langle t_i, c_j\rangle$. By a direct verification we see that equivalence

$$s_{ijk}=1\equiv h_k\in g_{ij}$$

determines demanded correspondence.

Let us observe that definition 2 is an essential extension of the first one. In this definition we can take into account the condition of type (a) and many others not mentioned here, by appropriate determination of ϱ. So, if two lessons l_i, l_j for some reason or other cannot be scheduled at the same hour we put $l_i \varrho l_j$, and $\neg\, l_i \varrho l_j$ if this is not the case.

To compare requirements $\langle A, B\rangle$ to these described by G and ϱ we shall consider an example due to Cisma and Gotlieb [4].

In their example $n=m=p=3$, $A=\{a_{ij}\}$ $(i\leqq 3,\ j\leqq 3)$ where $a_{ij}=1$ and the matrix B is following:

$$b_{1jk}=\begin{matrix}1&1&0\\0&1&1\\1&0&1\end{matrix}\quad b_{2jk}=\begin{matrix}1&0&1\\1&1&1\\1&1&0\end{matrix}\quad b_{3jk}=\begin{matrix}0&1&1\\1&1&0\\1&1&0\end{matrix}$$

For these requirements Hall's conditions are fulfilled but a timetable S does not exist.

In the new definition the set L contains all pairs $\langle t_i, c_j\rangle$ $i\leqq 3$, $j\leqq 3$. Subsets g_{ij} are following:

$$g_{11}=\{h_1, h_2\}\ g_{12}=\{h_2, h_3\}\ g_{13}=\{h_1, h_3\}$$

$$g_{21}=\{h_1, h_3\}\ g_{22}=\{h_1, h_2, h_3\}\ g_{23}=\{h_1, h_3\}$$

$$g_{31}=\{h_2, h_3\}\ g_{32}=\{h_1, h_2\}\ g_{33}=\{h_1, h_2, h_3\}$$

then $G=\{g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{23}, g_{31}, g_{32}, g_{33}\}$. The relation ϱ can be displayed as a matrix:

$$\varrho=\begin{matrix} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9\\ l_1 & 0&1&1&1&0&0&1&0&0\\ l_2 & 1&0&1&0&1&0&0&1&0\\ l_3 & 1&1&0&0&0&1&0&0&1\\ l_4 & 1&0&0&0&1&1&1&0&0\\ l_5 & 0&1&0&1&0&1&0&1&0\\ l_6 & 0&0&1&1&1&0&0&0&1\\ l_7 & 1&0&0&1&0&0&0&1&1\\ l_8 & 0&1&0&0&1&0&1&0&1\\ l_9 & 0&0&1&0&0&1&1&1&0\end{matrix}$$

where $\varrho_{ij}=1\equiv l_i \varrho l_j$ (see also figures 1 and 2).

3. Graph of a timetable

We denote by F the set $\{l_1, ..., l_q, h_1, ..., h_p\}$ and by $\pi\subset F\times F$ the binary relation defined as follows:

$$l_i \pi l_j \equiv l_i \varrho l_j \tag{7}$$

$$l_i \pi h_j \equiv h_j \notin g_i \qquad h_j \pi l_i \equiv l_i \pi h_j \tag{8}$$

$$h_i \pi h_j \equiv i \neq j \tag{9}$$

The graph $E=\langle F, \pi\rangle$ where F is a set of vertices and π a set of edges will be called the graph of a timetable. Since a relation π is symmetric and antireflexive then there exists the unique chromatic number of graph E.

Now we can establish the main result of the present paragraph.

3. 1. A timetable $x=\langle h^1, \ldots, h^q\rangle$ exists iff a chromatic number of graph $E=\langle F,\pi\rangle$ is equal to the number of elements of $H(E$ is p-chromatic).

Proof. Let $x=\langle h^1, \ldots, h^q\rangle$ be a timetable fulfilling (5) and (6) and let $D_k=\{h_k\}\cup\{l_i\colon h_k=h^i\}$ $(k=1, \ldots, p)$. We shall show that the sets $D_1, \ldots, D_p$ form a family of independent sets which covers the graph E.

Really, if $l_i, l_j\in D_k$ then $h^i=h^j=h_k$ and from (6) $\neg(l_i \varrho l_j)$. Next if $l_i\in D_k$ then $h_k=h^i$ and from (5) $h_k\in g_i$. By (7) $\neg(l_i\pi l_j)$ and by (8) $\neg(l_i\pi h_k)$ so D_k are independent. Since for every l_i exists D_k such that $l_i\in D_k$, sets $D_1, \ldots, D_p$ cover the graph E, it means E is at least p-chromatic. On the other hand the chromatic number of E cannot be less than p, because there is a complete subgraph of the order p containing all vertices h_k $(k=1, \ldots, p)$.

Thus necessity is proved.

Now, let the family $D_1, \ldots, D_p$ denote a covering of graph E. As all D_k $(k=1, \ldots, p)$ are independent and every h_k must belong to some D_k we can associate with every D_k one element h_k.

Now for every l_i $(i=1, \ldots, q)$ we choose an arbitrary h_k such that $l_i\in D_k$. If h^i stands for this h_k then a sequence $x=\langle h^1, \ldots, h^q\rangle$ is a timetable.

In fact, $l_i, h^i\in D^i$ so $\neg(h^i\pi l_i)$ and by (8) $h^i\in g_i$. If for some l_i, l_j $(i\neq j)$ $h^i=h^j$ then l_i, l_j belong to the same D_k, it means $\neg(l_i\pi l_j)$ and by (7) $\neg(l_i\varrho l_j)$.

It ends the proof of sufficiency.

Immediatly from 3. 1. we have

3. 2. There is an effective procedure of constructing for arbitrary p-colouring of graph E a timetable x if it exists.

The constructing procedure was given in the proof of sufficiency in 3. 1.

So far as can be seen 3. 1 establishes the condition necessary and sufficient for the existence of timetable. In 4. it will be shown how to obtain all p-colourings of graph E and due to 3. 2 we shall be able to obtain all sequences satisfying (5) and (6).

4. Algorithm 1

Efficient methods for graph colouring were investigated by many authors ([5], [6]) and any of them may be used here.

In this paragraph we shall present a simple idea of J. Wiessman [6] who applied boolean transformations to this problem.

Let us consider a graph $E=\langle F, \pi\rangle$ for requirements given in 2. (see figure 1). We treat an ordered set of all vertices as a set of boolean variables. A boolean polynomial:

$$f_1=(l_2+l_1)\ (l_3+l_1l_2)\ (l_4+l_1)\ (l_5+l_2l_4)$$

$$(l_6+l_3l_4l_5)\ (l_7+l_1l_4)\ (l_8+l_2l_5l_7)\ (l_9+l_3l_6l_7l_8)$$

$$(h_1+l_2l_7)\ (h_2+l_3l_4h_1)\ (h_3+l_1l_6l_8h_1h_2)$$

where every disjunction contains a negation of successive vertex and conjunction of negations of all precedent coincident vertices with this one, is transformed into the disjunctive-conjunctive normal form $DC(f_1)$.

Complements of the set of vertices which occur in successive conjunctions of $DC(f_1)$ are maximal independent sets ([6]). Thus

$$D_1=\{l_3, l_5, h_1\} \quad D_2=\{l_3, l_4, l_8, h_1\} \quad D_3=\{l_1, l_6, l_8, h_1\}$$

$$D_4=\{l_4, l_9, h_1\} \quad D_5=\{l_1, l_5, l_9, h_1\} \quad D_6=\{l_5, l_7, h_2\}$$

$$D_7=\{l_2, l_6, l_7, h_2\} \quad D_8=\{l_2, l_9, h_2\} \quad D_9=\{l_1, l_6, l_8, h_2\}$$

$$D_{10}=\{l_1, l_5, l_9, h_2\} \quad D_{11}=\{l_3, l_5, l_7, h_3\} \quad D_{12}=\{l_2, l_4, l_9, h_3\}$$

$$D_{13}=\{l_3, l_4, h_3\} \quad D_{14}=\{l_2, l_7, h_3\} \quad D_{15}=\{l_5, l_9, h_3\}.$$

In order to obtain all p-colourings of E let us observe that,

$$l_1 \in D_3 \quad \text{or} \quad l_1 \in D_5 \quad \text{or} \quad l_1 \in D_9 \quad \text{or} \quad l_1 \in D_{10},$$

$$l_2 \in D_7 \quad \text{or} \quad l_2 \in D_8 \quad \text{or} \quad l_2 \in D_{12} \quad \text{or} \quad l_2 \in D_{14} \text{ etc.}$$

Then a boolean polynomial

$$f_2 = (D_3+D_5+D_9+D_{10})\,(D_7+D_8+D_{12}+D_{14})\,(D_1+D_2+D_{11}+D_{13})$$
$$(D_2+D_4+D_{12}+D_{13})\,(D_1+D_5+D_6+D_{10}+D_{11}+D_{15})$$
$$(D_3+D_7+D_9)\,(D_6+D_7+D_{11}+D_{14})\,(D_2+D_3+D_9)$$
$$(D_4+D_5+D_8+D_{10}+D_{12}+D_{15})\,(D_1+D_2+D_3+D_4+D_5)$$
$$(D_6+D_7+D_8+D_9+D_{10})\,(D_{11}+D_{12}+D_{13}+D_{14}+D_{15})$$

transformed into the disjunctive-conjunctive normal form $DC(f_2)$ determines all coverings of graph E. In fact, if a conjunctive $D_{i_1}, \ldots, D_{i_k}$ occurs in $DC(f_2)$ then every vertex must belong to a certain D_{i_j} $(j \leq k)$. Since we search only p-colourings in every step of transformation those conjunctions which have more than p elements must be removed. In our example there is no conjunction in $DC(f_2)$ which has 3 elements then in virtue of 3. 1 a timetable x for these requirements does not exist.

But if the number of the edges of E is reduced by deleting an edge between l_6 and h_3, in the polynomial f_1 we obtain $h_3+l_1l_8h_1h_2$ instead of $h_3+l_1l_6l_8h_1h_2$, then $D_{14}=\{l_2, l_6, l_7, h_3\}$ and next in f_2 there is $(D_3+D_7+D_9+D_{14})$ instead of $(D_3+D_7+D_9)$. Thus in $DC(f_2)$ occurs the conjunction $D_2\, D_{10}\, D_{14}$ which gives a unique timetable $x=\langle h_2, h_3, h_1, h_1, h_2, h_3, h_3, h_1, h_2\rangle$.

An interesting problem arises in the case of inconsistency of requirements: What is a minimal number of edges whose removing decreases a chromatic number of graph E?

This problem is strictly connected with the notion of the critical graph which was investigated by G. A. Dirac ([7], [8]).

5. Multiperiod jobs

In the case of condition (b) apart from sets L, H, G and relation ϱ there is given a function $n: L \to N$ (set of integers) the value of which $n(l_i) = n_i$ defines how many consecutive time periods l_i must last. So, $n_i = 1$ defines a single period, $n_i = 2$ a double period etc.

We denote by $\langle h_k, n \rangle$ a time interval beginning at h_k and lasting n time periods. It means that

$$\langle h, n \rangle = \{h_k, h_{k+1}, \ldots, h_{k+n-1}\}$$

provided that h_k, h_{k+1} $k=1, \ldots, p-1$ are consecutive periods.

Now, for requirements with function n we must introduce a new definition of timetable.

Definition 3. A sequence $x = \langle h^1, \ldots, h^q \rangle$ will be called a timetable for requirements with function n iff

$$\langle h^i, n_i \rangle \subset g_i \quad i=1, \ldots, q \tag{10}$$

$$\text{If } l_i \varrho l_j \text{ then } \langle h^i, n_i \rangle \cap \langle h^j, n_j \rangle = \varnothing \text{ (empty set).} \tag{11}$$

These two condition correspond with (5) and (6) where one time period h_i is changed by a whole interval $\langle h^i, n_i \rangle$.

5. 1. A timetable $x = \langle h^1, \ldots, h^q \rangle$ exists iff there is a covering $D = \{D_1, \ldots, D_p\}$ of graph $E = \langle F, \pi \rangle$ such that D_k, $k=1, \ldots, p$ are independent sets and

$$h_k \in D_k \quad k=1, \ldots, p \tag{12}$$

for every $i=1, \ldots, q$ exists $k_i \leqq p - n_i + 1$ such that (13)

$$l_i \in \bigcap_{j=k_i}^{k_i+n_i-1} D_j \text{ (} l_i \text{ belongs to the successive } n_i \text{ independent sets)}$$

Proof. Let $x = \langle h^1, \ldots, h^q \rangle$ be a timetable and let $D_k = \{h_k\} \cup \{l_i : h_k \in \langle h^i, n_i \rangle\}$. The proof of independence of D_k is analogous as in 3. 1. The condition (12) is immediate. Let k_i stand for an index of h^i in the set H. Thus $l_i \in D_{k_i} \cap D_{k_i+1} \cap \ldots \ldots \cap D_{k_i+n_i-1}$ which proves the condition (13).

Let us assume that independent sets $D_1, \ldots, D_p$ satisfy (12) and (13). We can define a timetable x as a sequence $\langle h_{k_1}, h_{k_2}, \ldots, h_{k_q} \rangle$. For $h_k \in \langle h_{k_i}, n_i \rangle$ by (12) $h_k \in D_k$ and by (13) $l_i \in D_k$ which is equivalent $\neg(h_k \pi l_i)$. From (8) $\neg(h_k \pi l_i)$ iff $h_k \in g_i$ thus $\langle h_k, n_i \rangle \subset g_i$. In order to prove (11) let us assume that $\langle h_{k_i}, n_i \rangle \cap \langle h_{k_j}, n_j \rangle \neq \varnothing$. It means that for $h_k \in \langle h_{k_1}, n_i \rangle \cap \langle h_{k_j}, n_j \rangle$ in virtue of (9) and (13) $l_i \in D_k$, $l_j \in D_k$. Thus l_i, l_j belong to the same D_k which implies $\neg(l_i \varrho l_j)$.

6. Algorithm 2

The theorem 5. 1 establishes the condition necessary and sufficient for the existence of a timetable with multiperiod jobs. First, so as in algorithm 1 all maximal independent sets $D = \{D_j\}$ of graph E must be achieved.

The second part of procedure we exemplify by colouring the graph from figure 2. This graph we obtain from the graph displayed on figure 1 by adding one vertex h_4, five edges h_4h_1, h_4h_2, h_4h_3, h_4l_9 and removing one edge h_3l_6. The function n is determined in this example as follows: $n_2=n_6=n_7=2$, $n_1=n_3=n_4=n_5=n_8=n_9=1$.

The family of maximal independent sets for this graph is increased by five sets

$$D_{16}=\{l_1, l_5, h_4\} \quad D_{17}=\{l_1, l_6, l_8, h_4\} \quad D_{18}=\{l_2, l_6, l_7, h_4\}$$

$$D_{19}=\{l_3, l_5, l_7, h_4\} \quad D_{20}=\{l_3, l_8, h_4\}.$$

Since $n_1=1$ the vertex l_1 satisfies condition $l_1 \in D_3 \cup D_5 \cup D_9 \cup D_{11} \cup D_{16} \cup D_{17}$. Next, for the vertex l_2 $n=2$, so

$$l_2 \in (D_7 \cap D_{12}) \cup (D_7 \cap D_{14}) \cup (D_8 \cap D_{12}) \cup (D_8 \cap D_{14}) \cup (D_{12} \cap D_{18}) \cup (D_{14} \cap D_{18})$$

Similarly for l_6 and l_7. In the analogous way as in algorithm 1 we verify that a boolean polynomial:

$$\begin{aligned} f_3 = {} & (D_3+D_5+D_9+D_{10}+D_{16}+D_{17})\ (D_7D_{12}+D_7D_{14}+D_8D_{12}+D_8D_{14}+D_{12}D_{18}+D_{14}D_{18}) \\ & (D_1+D_2+D_{11}+D_{13}+D_{19}+D_{20})\ (D_2+D_4+D_{12}+D_{13})\ (D_1+D_5+D_6+ \\ & +D_{10}+D_{11}+D_{15}+D_{16}+D_{19})\ (D_3D_7+D_3D_9+D_7D_{14}+D_9D_{14}+D_{14}D_{17}+D_{14}D_{18}) \\ & (D_6D_{11}+D_6D_{14}+D_7D_{11}+D_7D_{14}+D_{11}D_{18}+D_{11}D_{19}+D_{14}D_{18}+D_{14}D_{19})\ (D_2+D_3+ \\ & +D_9+D_{17}+D_{20})\ (D_4+D_5+D_8+D_{10}+D_{12}+D_{15})\ (D_1+D_2+D_3+D_4+D_5) \\ & (D_6+D_7+D_8+D_9+D_{10})\ (D_{11}+D_{12}+D_{13}+D_{14}+D_{15})\ (D_{16}+D_{17}+D_{18}+D_{19}+D_{20}) \end{aligned}$$

transformed into the disjunctive-conjunctive normal form gives all coverings which satisfy (12), (13). In this case we obtain only one covering containing 4 elements: $D_2D_{10}D_{14}D_{18}$ and $x=\langle h_2, h_3, h_1, h_1, h_2, h_3, h_3, h_1, h_2\rangle$.

If for some l_i $n_i>2$ a correspondent boolean expression consists of all conjunctions which have n elements $D_{k_1}, D_{k_2}, \ldots, D_{k_q}$, such that l_i belongs to every D_{k_j} and $h_{k_1}, h_{k_2}, \ldots, h_{k_q}$ are consecutive time periods.

Obviously, in this expression conjunctions in which time periods belong to two different days or contain a lunch break must be omited.

7. Room problem

In the extension of timetable problem taking into account the condition (c) there is given *a* set $R=\{r_j\}$ $j\leqq s$ of rooms. As in the case of lectures with every r_j we associate a set $f_j\subset H$, time periods at which room r_j is available. Moreover, there are rooms not fitting to every lecture. This condition is described by a relation $\sigma\subset L\times R$ fulfilled if lecture l_i can take place in room r_j.

Definition 4. A pair $\langle x, y\rangle$ where x is a timetable for the set L and y is a sequence $\langle r^1, r^2, \ldots, r^q\rangle$ rooms will be called a timetable for sets L and R iff

$$l_i\sigma r^i \quad i=1, \ldots, q \tag{14}$$

$$\langle h^i, n_i\rangle\subset f_i \quad i=1, \ldots, q \tag{15}$$

$$\text{if } r^i=r^j \text{ then } \langle h^i, n_i\rangle\cap\langle h^j, n_j\rangle=\emptyset. \tag{16}$$

The condition (14) says that a lecture l_i can take place in a room r^i, (15) that this room is available at hours $\langle h^i, n_i \rangle$ and finally (16) assures that no room is used simulteneously for two lectures.

Now, if there is given a timetable $x=\langle h^1, \ldots, h^q \rangle$ we can define a new graph $E=\langle I, \pi_x \rangle$ where a set of vertices $I=\{l_1, \ldots, l_q, r_1, \ldots, r_s\}$ and the relation π_x is following:

$$r_i \pi_x r_j \equiv i \neq j \tag{17}$$

$$l_i \pi_x r_j \equiv \neg (l_i \sigma r_j) \vee \neg (\langle h^i, n_i \rangle \subset f_j) \tag{18}$$

$$l_i \pi_x l_j \equiv \langle h^i, n_i \rangle \cap \langle h^j, n_j \rangle \neq \varnothing . \tag{19}$$

Of course, $r_j \pi_x l_i \equiv l_i \pi_x r_j$.

7. 1. If x is a timetable for L then a timetable $\langle x, y \rangle$ exists iff graph E_x is s-chromatic.

Proof. If $y=\langle r^1, \ldots, r^q \rangle$ fulfills (14)—(16) then sets $D_j=\{r_j\} \cup \{l_i \colon r^i=r_j\}$ are independent. In fact for $l_i \in D_j$ from (14) $l_i \sigma r_j$ and from (15) $\langle h^i, n_i \rangle \subset f_j$ thus by (18) $\neg l_i \pi_x r_j$. On the other hand if $l_i, l_k \in D_j$ then $r^i=r^k=r_j$ and by (16) $\langle h^i, n_i \rangle \cap \langle h^k, n_k \rangle = \varnothing$ which gives in virtue of (19) that $\neg l_i \pi_x l_k$.

Since sets D_j $j=1, \ldots, s$ are independent and cover the graph E_x, its chromatic number is equal s.

Now, let a family $D_1, \ldots, D_s$ denotes a covering of E. By (17) we can assume that $r_j \in D_j$ $j=1, \ldots, s$. Let us define $y=\langle r^1, \ldots, r^q \rangle$ where r^i is an arbitrary room belonging to the same set D_j as l_i. So, $\neg l_i \pi_x r^i$ gives by (18) that $l_i \sigma r^i$ and $\langle h^i, n_i \rangle \subset f^i$. If $\langle h,^i n_i \rangle \cap \langle h^j, n_j \rangle \neq \varnothing$ then by (19) $l_i \pi_x l_j$ and l_i, l_j cannot belong to the same D_k. This proves that $r^i \neq r^j$.

8. Algorithm 3

The algorithm consists of two phases. First, all timetables x by the help of algorithm 2 are generated. The second phase is concerned with assignment of rooms. In the analogous way as in 4. the problem is reduced to the colouring of the graph. Since two timetables $\langle x, y \rangle$ and $\langle x, z \rangle$ where $y \neq z$ may be treated as equivalent we break the realization of Wiessman's method after an achievement of first colouring. If a graph E is not s-chromatic a timetable $\langle x, y \rangle$ for the given sequence x does not exist (theorem 7. 1).

We must investigate the next sequence x. A choice of this sequence can depend on desirable features of timetable such as the distribution of lectures over the days and the week, the maximal possibility of choice in the case of facultative jobs etc.

Let us end the presentation of methods hitherto described by an example considered in 6 with following room requirements:

$$R=\{r_1, r_2, r_3, r_4\}$$

$$f_1=\{h_1, h_3, h_4\} \quad f_2=\{h_1, h_2, h_3, h_4\}$$

$$f_3=\{h_1, h_2\} \quad f_4=\{h_1, h_2, h_3, h_4\}$$

$$\sigma = \begin{array}{c|ccccccccc} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 \\ r_1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ r_2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ r_3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ r_4 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}$$

For the sequence $x = \langle h_2, h_3, h_1, h_1, h_2, h_3, h_3, h_1, h_2\rangle$ the graph $E_x = \langle I, \pi_x\rangle$ (see figure 3) has eleven maximal independent sets:

$D_1 = \{l_2, l_3, r_1\}$ $D_2 = \{l_3, l_7, r_1\}$ $D_3 = \{l_2, l_4, r_1\}$ $D_4 = \{l_4, l_7, r_1\}$

$D_5 = \{l_1, l_6, l_8, r_2\}$ $D_6 = \{l_5, l_6, l_8, r_2\}$ $D_7 = \{l_1, l_7, l_8, r_2\}$

$D_8 = \{l_5, l_7, l_8, r_2\}$ $D_9 = \{l_5, l_8, r_3\}$ $D_{10} = \{l_8, l_9, r_3\}$

$D_{11} = \{l_3, l_7, l_9, r_4\}$.

Two 4-colourings are determined by the conjunction $D_3 D_5 D_9 D_{11}$, thus there are two equivalent timetables

$$\langle x, \langle r_2, r_1, r_4, r_1, r_3, r_2, r_4, r_2, r_4\rangle\rangle \quad \text{and} \quad \langle x, \langle r_2, r_1, r_4, r_1, r_3, r_2, r_4, r_3, r_4\rangle\rangle.$$

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$$\pi = \begin{array}{c|ccccccccc|ccc} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & h_1 & h_2 & h_3 \\ l_1 & & & & & & & & & & 0 & 0 & 1 \\ l_2 & & & & & & & & & & 1 & 0 & 0 \\ l_3 & & & & & & & & & & 0 & 1 & 0 \\ l_4 & & & & & & & & & & 0 & 1 & 0 \\ l_5 & & & & & \varrho & & & & & 0 & 0 & 0 \\ l_6 & & & & & & & & & & 0 & 0 & 1 \\ l_7 & & & & & & & & & & 1 & 0 & 0 \\ l_8 & & & & & & & & & & 0 & 0 & 1 \\ l_9 & & & & & & & & & & 0 & 0 & 0 \\ h_1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ h_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ h_3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

$$F = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, h_1, h_2, h_3\}$$

Figure 1

$$
\pi = \begin{array}{c|ccccccccc|cccc}
 & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & h_1 & h_2 & h_3 & h_4 \\
\hline
l_1 & & & & & & & & & & 0 & 0 & 1 & 0 \\
l_2 & & & & & & & & & & 1 & 0 & 0 & 0 \\
l_3 & & & & & & & & & & 0 & 1 & 0 & 0 \\
l_4 & & & & & & & & & & 0 & 1 & 0 & 0 \\
l_5 & & & & & \varrho & & & & & 0 & 0 & 0 & 0 \\
l_6 & & & & & & & & & & 0 & 0 & 0 & 0 \\
l_7 & & & & & & & & & & 1 & 0 & 0 & 0 \\
l_8 & & & & & & & & & & 0 & 0 & 1 & 0 \\
l_9 & & & & & & & & & & 0 & 0 & 0 & 1 \\
\hline
h_1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
h_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
h_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
h_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

$$F = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, h_1, h_2, h_3, h_4\}$$

Figure 2

$$
\pi_x = \begin{array}{c|ccccccccc|cccc}
 & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & r_1 & r_2 & r_3 & r_4 \\
\hline
l_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
l_2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
l_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
l_4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
l_5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
l_6 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
l_7 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
l_8 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
l_9 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
r_1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
r_2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
r_3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
r_4 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

$$I = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, r_1, r_2, r_3, r_4\}$$

Figure 3

Алгоритм для получения расписания университета и критерий согласования с требованиями

В первой части приводим формальное определение расписания учебных занятий, в котором появляется только очень простая модель [2]. Эквивалентное определение в терминах раскраски графов позволяет сформулировать необходимые и достаточные условия существования расписания занятий. Предлагается алгоритм построения расписания и приводится пример, который неразрешим комбинаторными методами (взят из [4]).

Далее приводятся более сложные модели с учетом неравнодлительных занятий и проблемой залов. Все они записаны терминами проблемы раскраски графов. Приводятся соответствующие критерии существования и алгоритмы построения расписания учебных занятий.

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