Logical foundations for a general theory of systems

By T. GERGELY and I. NÉMETI

1. Introduction

Science has now reached a turning-point in its development at which it is becoming increasingly urgent for us to achieve a systematization and reduction to some common denominator of the vast body of knowledge that has been accumulated in its various disciplines. The need is to construct a unifying theory with the capacity to override the barriers which at present divide and compartmentalize specialist investigations in order that we may be able to free the disciplines to interact with and reinforce one another. While unification on this scale necessarily entails raising the level of abstraction of the theories with which each discipline works, the language and concepts of this higher, more general level must retain the precision and explicitness in their interrelations that are found at their lower level. These are the objectives and constraints which a General Systems Theory (GST) must fulfil.

More exactly, the features that a General Systems Theory must display are the following:

— it must give a method in which the structural and functional aspects of the system* form a dialectical unity;

— it should be extendable in order to cater for any new aspects that may emerge in the future;

— it should be general only to the extent that it does not lose the property of reversibility, that is, of applicability to the disciplines on which it is founded;

— it should provide not only an approach but also a method of analysis whereby a sufficient body of knowledge can be acquired to permit intervention in the system (e.g. to control it);

— it should provide a mathematical apparatus which enables only the investigation and analysis but also the synthesis of systems.

For the time being there is no theory that completely satisfies all these requirements, a deficiency, which means that on many occasions only verbal methods are at our disposal. A verbal theory of systems too — essentially an initial stage of the

* The basic ideas of Systems Theory are assumed to be known already and so they are not defined here; the particular terminology with which they are expressed is not of importance for the present purposes.

GST — can give considerable help in the analysis of systems (biological, economic, social, etc.), provided it satisfies certain logical conditions to be discussed later. Nevertheless there are already certain systems theories, oriented to particular branches of science which satisfy the requirements of the last two points and among these cybernetics has achieved the most significant results. Indeed on the strength of this some branches of cybernetics (e.g. information theory, automata theory) have proclaimed themselves the base of the GST, though they give only some aspects of the GST, and their applicability depends upon the concrete problem. In problem solving, it is essential to select the level of abstraction adequate to the problem.

Let us consider these ideas more closely:

The interpretations of GST, the various specific systems theories (SST) correspond to the different types of system (and not to the different types of theoretical apparatus).

Theories of biological, psychological, economic, technical systems etc. are instances of Specific Systems Theories. Within a given SST — depending on the apparatus applied — different aspects can be distinguished, for instance, aspects of information theory, automata theory, control theory, etc. Independently of this, it is possible to distinguish various levels of abstraction within an SST. This concept can be illustrated in the following way.

On the highest level of abstraction we are restricted to the study of the relations between the system and its environment, the system being regarded as a single indifferentiated unit or a *black box*. On the second level the system may be broken down to its immediate component parts, that is, the system is analysed as the ensemble of its immediate subsystems. On the lower levels the system is seen in a more and more detailed analysis down to an apparently arbitrary degree of refinement, to whatever number of levels of abstractions we wish. The selected level of abstraction can said to be adequate to the task, if it allows to solve the problem with the minimum effort (i.e. without having to go into unnecessary details). Naturally we can speak of the adequacy of the level of abstractions only if the aspect selected is appropriate. The concepts of level and aspect are orthogonal to one other; that is to say, any level may be combined with any aspect.* To sum up, the main tasks that must be in developing and applying a GST are:

— elaboration of a mathematical base that allows the GST to satisfy the above four points;

— the elaboration of a method of applying the GST on the appropriate level of abstraction.

To these ends it is necessary first to create the logical foundations *a)* for establishing the GST; *b)* for applying the GST.

After the logical foundations have been laid it will become possible to create the theory itself. The logical foundations can of course only provide guide-lines for the creation of the theory, but then it is hardly conceivable that the GST could be built up in a single step.

Properly based logical foundations are necessary to ensure that the set of

* Selection of the adequate level and aspect is inseparable from the process of problem solving (optimalization). In its course the level and the aspect are modified alternately, until the ensemble of the two suits the given purpose.

SST-s is such that the GST which is built on it forms a dialectical unity with the SST-s themselves.

The aim of the present work is to clarify the logical foundations needed to establish the GST, taking Curry's book [1] as a starting point. The logical foundations for the application of the GST will be dealt with in a subsequent paper.

2. The language

As in building a theory the first steps are taken with the aid of natural language, it is necessary to start with the analysis of the natural language. After the basic definitions have been set out we can go on to examine the relation of language to the sphere of phenomena and then the manner in which a formal language can be produced.

From the semantical point of view any language, including a natural language, is produced by the combination of *nouns*, statements* and *functors* (phrases).

The set of nouns will be denoted N, the set of statements E. Both sets are inductive classes** and their definitions — see later — are given by a grammar employing the auxiliary concept of functors.

Functors. Transformers with one or more arguments. The argument may be: noun, statement, functor. The value of the functor (result) may be: noun, statement, functor.

Generally all combinations are possible, which means that nine main functor-types can be distinguished. It is conceivable that a specification defining the inductive class (N) of the objects (nouns) uses all nine functor types. In the specification of N the concepts of "statement" and "functor" are auxiliary ideas.

The set E (i.e. the set of statements to be generated from the nouns and functors) is called the language. The aggregate of the (inductive) laws defining set E is called the grammar (for instance the set of nouns and functors together with the rules for their combination).

The sphere of phenomena (J) is taken here to mean the entirety of research objects. We start from the supposition that the properties of the individual objects to be analysed are recognised in the course of their interaction with other objects.

Let be J a space (set). The relation between the elements of J and the elements of N the set of some nouns gives the representation of the objects in the sphere of phenomena in language: for short we shall call this the $J \to N$ representation.

Let us look at a basic property of this representation. Let us constitute the product set $J \times N$ and map it on the closed interval $[0, 1]$. $\mu: J \times N \to [0, 1]$. The value $\mu(j, n)$ will be called *the measure of validity* of the $j-n$ relation. In other words, the value $\mu(j, n)$ specifies the degree to which noun n is congruent with the phenomenon j.

Set N is a precise representation of the sphere of phenomena only if the function μ has the values 0 or 1. It is called an imprecise representation if μ takes away value between 0 and 1.

* *Noun* denotes an expression specifying some object and corresponds to the linguistic concept *noun-phrase.* (Note that a noun may be composed of more than one word.)

** A definition of the inductive class is given later.

An imprecise function μ can be made to correspond to a precise function μ_k by giving the threshold value $0<k\leqq 1$, where μ_k is defined as $\mu_k(j, n)=1\Leftrightarrow\mu(j, n)\geqq k$. Accordingly, any precise representation may be coordinated to imprecise representation by giving the threshold limit k.

The precise function μ_1 is called the kernel of the imprecise function μ, and its representation is called the precise kernel of the original imprecise representation.

Note: if N proves to be precise then the $J\rightarrow N$ relationship can also be represented as a relation.

The precision of the $J\rightarrow N$ representation depends upon the extent to which the structure of objects in the sphere of phenomena (J) corresponds to, or how homomorphic* it is to with the structure of the set N which is determined by the grammar.

As already explained, the set N and a grammar G together define a set of expressions (E), called the language. The relationship between the elements of J and E comprises the linguistic representation of the sphere of phenomena, in short the $J\rightarrow E$ representation.

Note, that if N happens to be imprecise, so too will E**. This is also true in a more general way: as it was shown, E is built on N, but we shall see later that a certain T is built on E, T_1 on T, and so on.

It therefore holds for this arbitrarily long series that if any of its terms becomes imprecise, the imprecision will be transferred to all terms that are constructed directly or indirectly upon the term in question. More exactly, the imprecision will not be transferred, provided only the precise parts (kernels) are employed in construction.

The language E is a means of describing our experiences connected with the phenomenal sphere.

The criterion of truth of a statement $e\in E$ is found in the sphere of phenomena. Let us map the set E by the function γ in to the closed interval [0, 1] of the numerical axis. $\gamma: E\rightarrow[0, 1]$. $\gamma(e)$ is now the *measure of truth* of the statement e.

It should be noted that the value $\gamma(e)$ can also be interpreted as the aprobability of the statement being true. The measure of validity μ in contrast does not admit such an interpretation. Values of the function γ lying between 0 and 1 will have significance in complete theories (in the Gödel), when we are unable to demonstrate or refute certain statements but can render them probable (and likewise their consequences) by repeated heuristic attempts. If, for instance a true but indemonstrable statement is in question, the mesure of truth of that statement converges to 1.

A subset (L) of the set E to the elements of which values of the function γ have been given is called a description.

The description L is said to be precise if none of the γ values pertaining to its elements lies between 0 and 1. Similarly to the imprecise μ function, an imprecise function γ can be coordinated with a precise function γ_k by giving a threshold value k. The function γ_1 gives the kernel of the description (L_1). If L is precise, then $L=L_1$.

* See later.

** The grammar also may be imprecise, as when we have a sentence the grammatical correctness of which lies between zero and one [2]: if G is imprecise, then plainly E will be imprecise too.

3. The theory and its interpretation

The results of completed observations can be recorded by the aid of some description L. However, L is not capable of predicting an event that will be observed in the future, based on the results of the completed observations. In order to permit predictions we must introduce the concept of rules of inference.

Rules of inference serve to produce from given true statements new, true statements.

A rule of inference r is a $n+1$ place relation defined on the set $E(r \subset E^{n+1})$ by means of which a new, true statement can be obtained from n true statements; that is, $r\colon E^n \to E$ or $(e_1, e_2, \dots, e_n) \overset{r}{\to} e$. The set of rules of inference is denoted by R.

Where there is imprecision after applying the rule of inference, the measure of truth of the inferred statement is obtained from the measure of truth of the initial statements.

The possibility also exists — at least in principle — of obtaining a precise inferential statement from a large but finite number of imprecise statements.

Definition 1. We define a *theory* as a description $(T \subseteqq E)$ and a set of rules of inference (R) whereby T is *closed* under R; that is $(\forall r \in R) r(T^n) \subseteqq T$. For the sake of simplicity T will be called a theory, though the existence of some R is to be understood the same time.

T is said to be *consistent* if $T \neq E$.

If T is based on an imprecise description then it is called an imprecise theory.* It is evident that the above task can be solved by a carefully selected T theory.

However there is a need for a communication between some sphere of phenomena and the theory appropriate for its analysis, on the one hand for theoretical prediction of experimental results and, on the other hand for development of the theory based on the experimental results.

Definition 2. A theory T is said to be *extendable* within the language E, if there is some $K \subseteqq E$ for which $T' = T \cup K$ is consistent and is closed under the rules of inference

$$T' \neq E \quad \& \quad (\forall r \in R) r(T'^{n_r}) \subseteqq T'.$$

If the theory is to be extended through extension of the language E, means must be provided for, ensuring the extendability of the measure of truth.

Definition 3. A description L_1 is said to be an interpretation of the theory T_2 if there exists a mapping $i\colon E_2 \to E_1$ such that $L_1 \subset E_1$ and $T_2 \subset E_2$ but no stipulation is made that E_2 should differ from E_1 nor that i should be defined for every element of E_2.

The interpretation is *complete* if it is everywhere defined in E_2. The interpretation is *valid,* if $L_1 \supseteqq i(T_2)$. The interpretation is *adequate*, if $L_1 \subseteqq i(T_2)$.

* The condition of consistence for imprecise theories is $(\forall t \in T) \gamma(t) + \gamma(\neg t) = 1$. In general *every condition* defined on T is true in the case of imprecision provided the condition holds for T_k with an arbitrary k. Note that, in predicate calculus, the definition of consistence $(T \neq E)$ is equivalent to the following condition $t \in T \Leftrightarrow \neg t \notin T$ [1, 3].

If the interpretation is valid and L_1 is a theory, then T_2 can also be regarded as an extension of L_1.

Note: A theory may be have more than one interpretation.

Any theory T may be formalized by formalizing the language E (the statement of which are now obtained with the aid of *formal objects* and *predicates* in such a way that formal objects or statements are substituted for the arguments of the predicates) and the formal notation of the rules R. Nevertheless, no particular use can be made of this formalization unless T forms an inductive class, that is, unless the set cannot be generated finitely by means of R.

Definition 4. A theory T is said to be *deductive* if it contains such a finite subset A from which any element of T can be derived by a finite number of repeated applications of the rules R. T is then said to be generated by A with the aid of R, and the set A is called the set of *axioms* of the theory.

If T is deductive and extendable, then an extension of T always follows from the extension of A. (This of course refers only to extensions within the language E.)

When a deductive theory is formalized, the product is a *formal theory*.

In the following we present the concepts needed in the definition of a formal theory.

An *inductive class* is an enumerable set (E) which is defined by an algorithm.* This algorithm is called the (constructive) *specification* of the inductive class E. By this we mean that e is an element of E, iff there exists a finite n such that e will be produced by the algorithm in n steps.

In detail: By an inductive class we mean an inductively defined set. By an inductive definition we mean the listing of a finite number of laws and statements by a finite number of applications of which any element of the set can be formed.

a) Thus we can list a finite number of statements

e.g.,

$$\begin{aligned} a, b, c &\in A_1 \\ \alpha, \beta &\in A_2 \\ &\vdots \\ b, c, \beta &\in A_8 \\ &\vdots \end{aligned}$$

b) We can also list a finite number of rules. We shall do this in terms of *variables* X and Y. (In using the rules anything can be substituted for the variables.)

$$X, Y \in A_1 \Rightarrow XYX \in A_1$$

$$X \in A_1 \quad \& \quad Y \in A_2 \Rightarrow YYX \in A_3$$

$$X, Y \in A_8 \Rightarrow XY \in A_8$$

c) We can specify the set to be defined e.g., $I = A_8 \cap A_3$.

Thus, for instance, $bccbccb \in I$.

* By algorithm a constructive procedure in the Hilbertien sense is meant, that is, a specification which unambigously defines the series of transformations to be executed on some objects. This series of transformations may be either finite or indefinite, but the specification must naturally always be finite. The execution of a single transformation is called a step, and the stipulation is made that the transformations should be realizable.

Note: This is usually called a closure condition; the elements defined above for instance are the elements of the set I only if they are elements of A_8 and A_3.

Definition 5. The minimum *base* of a language is defined as the finite set of some symbols from which the language can be generated. By a *string* we mean a series of the elements of a base (a string is therefore a series of symbols).

Definition 6. The number of the base elements included in the string x is termed the *string length* and will be denoted by $|X|$. A string of zero length is denoted by λ.

Let A, B and X be a set of strings of symbols. By a language we mean a set or strings on which the following operations hold true:

1. $$AB \overset{\text{def}}{=} \{ab|a \in A, b \in B\}$$

2. $$X^n \overset{\text{def}}{=} XX^{n-1}$$
$$X^0 \overset{\text{def}}{=} \{\lambda\}$$

Note: If no set of strings is involved then X^n represent a Cartesian product.

3. $$X^* \overset{\text{def}}{=} \bigcup_{i=0}^{\infty} X^i \overset{\text{def}}{=} X^0 \cup X \cup X^2 \cup \ldots$$

If X is a base, then X^* is the set of strings definable on the base X.

4. The formal theory

Let introduce the following concepts:

Definition 7. The set of *abstract objects* is denoted by O. The inductive class of strings defined on some base K is $(O \subset K^*)$.

Set of predicates $(F \subset G^*)$. This is an inductive class of strings defines on a base G, on which some mapping $r\colon F \to N$ is defined where N is the set of natural numbers. An $r(\varphi)$ value allocated to some predicate $\varphi \in F$ is called the order of the predicate (also denoted r_φ).

Note: $K \cap G = \emptyset$.

Set of statements (E)

$$E \overset{\text{def}}{=} \left\{\varphi(\{,\}O)^{r_\varphi}|\varphi \in F\right\}$$

E.g., $$\varphi O_1, O_2, \ldots, O_{r_\varphi} \in E$$

Definition 8. A *formal theory* consists of the following components:

a) Inductive class of abstract objects (O).
b) Inductive class of predicates (F).
c) Set of statements (E). (This is produced from the first two.)
d) Inductive class of true statements $(T \subseteq E)$.

To get T the following are needed: 1) A finite subset of T, called the set of axioms (A). 2) A set of rules of inference (R), giving together with A the inductive definition of T.

The set E is called a formal language. (It is on this that the theory T is defined.) The specification of the inductive class E is called the grammar. The grammar consists of the specification of the inductive classes O and F and of the laws of substitution for the arguments of the predicates.

Quite obviously the properties of the verbal theories defined so far (for instance, extensibility, imprecision, etc.) refer by definition to formal theories too.

In formalizing a verbal theory T_V, a formal theory T_F is sought of which T_V is a valid, adequate interpretation.

From the foregoing it follows that, since every theory can be formed as the valid, adequate interpretation of a deductive extensible theory, every theory can be formalized in the above sense.

The function γ is defined also to the set E_F, thus the imprecision of the formal theory T_F can be handled similarly to the precision of the verbal theory T_V. For this purpose the rules of inference must be defined accordingly. Since a formal theory in the above form is a syntactic system, it is necessary to define the precision of the syntax and the concept of an imprecise syntax. A simple example of this is given by Zadeh [2].

Let us see how a natural language E_V can be formalized, in other words, how a formal language E_F can be made to correspond to a language E_V in such a way that the correspondence between the statements should be bijective and also isomorphic. The establishment of these conditions will be symbolized by $E_V \cong E_F$. In many cases it is sufficient that the relation be homomorphic, for which we use the notation $E_V \simeq E_F$.

Let us now look at what is meant by homomorphic and isomorphic correspondence. Let E and E' be two formal languages to which belong the factors O, O'; F, F', etc.

Definition 9. The mapping $i: E \to E'$ is said to be homomorphic (denoted $i: E \simeq E'$) if the mappings* $i: O \to O'$ and $i: F \to F'$ — with regard to the structure of the inductive classes — are homomorphisms,** and the condition $i(\varphi O_1 \ldots O_n) = i(\varphi) i(O_1), \ldots, i(O_n)$ holds.

The relation is isomorphic if its inverse is also homomorphic: i.e. $E \cong E' \Leftrightarrow E \simeq E' \& E' \simeq E$. Where imprecision is encounted it is necessary to define the *measure* of homomorphism.

It should be remembered that, while the grammar of some formal language E_F consists of an inductive class of abstract objects (O) and an inductive class of predicates (F), the grammar of a natural language E_V consists of the inductive class of nouns (N) and the inductive class of functors.

A formal language E_F can be made to correspond to the language E_V be the following two steps:

* The mappings $i: O \to O'$, etc. are taken to mean those correspondences between objects, etc. forming the base of the correspondence between the sentences $i: E \to E'$.

** The mapping $i: O \to O'$ is homomorphic if, when forming the elements of the sets there is no difference between making an inductive step first followed by interpretation and making the interpretation before the corresponding step. For instance let $O_1, O_2, \ldots O_n \in O$ and $\omega O_1 \ldots O_n \in O$, where O is one of the inductive steps used in the definition. If $i(\omega O_1, \ldots O_n) = i(\omega) i(O_1) \ldots i(O_n)$ is satisfied generally, then the correspondence is homomorphic.

1. By making the inductive class O congruent with the inductive class N. The measure of isomorphism between the sets of N and O with respect to the inductive definition gives the precision of the congruence. (It is conceivable that all the functor types of the language to be defined have to be considered when O is defined.)

2. By making the set of predicates congruent with the potentialities ensuing from the use of functors (singly, combined or repeated) in obtaining statements from objects.* If arbitrary combinations of the functors are allowed in the language being formalized, then the set of predicates will be an inductive class.

If the formal language E_F proves to be a properly selected formalization of the language E_V, then it is possible to find for a verbal theory $T_V \subset E_V$ a consistent theory $T_F \subset E_F$ in such a way that T_V is valid, adequate interpretation of the formal theory T_F.

The congruence i: $E_F \simeq E_V$ (giving the interpretation i: $T_F \simeq T_V$) may also be imprecise in which case it is necessary to define a function μ: $E_F \times E_V \to [0, 1]$ where the value $\mu(e_F, e_V)$ gives the measure of validity of the congruence $e_F \to e_V$, e_F and e_V being statements from the theories T_F and T_V ($e_F \to e_V$ means that e_V is the interpretation of e).

Imprecision in a formal theory T_F can derive from the theory's own logical structure, from imprecision in the relation between T_F and T_V, from the structure of T_V, or from imprecision in the relation $T_V - J$. (And if a theory T'_F is constructed of which T_F is an interpretation, this chain is continued.)

5. Simple examples of a formal theory

I. 1. formal objects: $O = \{a, b, c\}$

2. predicates: let $o \in O$;

$$\varphi_1 o,$$

$$\varphi_2 o,$$

$$\varphi_3 o.$$

3. rules of inference:

$$\varphi_1 X \Rightarrow \varphi_1 Xb,$$

$$\varphi_1 X \& \varphi_1 Y \Rightarrow \varphi_2 XcY,$$

$$\varphi_3 XcY \Rightarrow \varphi_3 XbcYb.$$

(X and Y are used here as variables; that is, any object can be substituted for them.)

4. axioms:

$$\varphi_1 a,$$

$$\varphi_3 aca.$$

Interpretation. The object of the form $abb\ldots b$ corresponds to a natural number the value of which is the length of the series. Object c corresponds to the sign of

* Note that this represents only a fraction of the possibilities presented by the functors.

equality. Predicate $\varphi_1 o$ corresponds to the statement o is a number; $\varphi_2 o$ denotes that o is a statement and $\varphi_3 o$ that o is a true statement (theorem).

II. 1. set of formal objects O, starting object: $a \in O$
law of generation: $X \in O \Rightarrow Xb \in O$, or in short $X \Rightarrow Xb$.

2. predicates: let $o_1, o_2, \in O$,

$$\varphi o_1 o.$$

3. rules of inference:

$$\varphi XY \Rightarrow \varphi Xb Yb.$$

4. axioms:

$$\varphi aa.$$

Interpretation. O corresponds to the set of natural numbers, while $\varphi o_1 o_2$ means that the number corresponding to o_1 equals the number corresponding to o_2.

6. Epitheory

6.1. Each statement e which refers to the theory T but for which $e \notin T$ is an *epistatement* of the theory T. By an epitheorem is meant a true epistatement, and by the set of epistatements, an epitheory. The construction of an epitheory will be examined here by means of an extension of the theory.

An *extension* of a theory T is a procedure involving the complete (or partial) execution of the following steps:

1. Extension of language E to language $E' \supset E$ (by the addition of new statements).
   a) Extension of the set of objects to the set $O' \supset O$.
   b) Extension of the set of predicates to set $F' \supset F$.

2. Extension of theory T to the theory $T' \supset T$.
   a) Extension of the set of axioms to the set $A' \supset A$.
   b) Extension of the set of rules of inference to the set $R' \supset R$.

From here onwards, after the introduction of each type of epitheory it will be shown how it is reducible to the concept of extension defined above. Conversely, all general statements made about extensions.

In each of the above steps (in the case of imprecise theories) care must be taken that the measure of truth be continuable to new statements of the extended language. (In precise cases we are careful to ensure that the extended theory remains consistent.)

Let us look more closely at the various possible kinds of extension.

6.2. Inductive extension. Suppose we have several theories $T_1, T_2, \ldots, T_n$. These theories — in the course of development — may become so complicated that the need arises to achieve some clarification on a higher abstraction level. For this purpose those theorie should be selected (let us say $T_1, \ldots, T_m$) for which the following procedure seems to be efficiently performable. The resultant theory T we shall form in the way $T = T_1 \cup T_2 \cup \ldots \cup T_m$. Upon this we can define an equivalence relation the equivalence classes pertaining to which give the elements of the theory T'. The theories $T_1, \ldots, T_m$ are now interpretations of the theory T'.

In carrying out this abstraction cover* could be used instead of partition,* but is this case special attention must be paid to the continuability of the measure of truth. (The above process will vary in different cases, depending on the aims and aspects that must be considered. For instance, contradictory aspects would be an increase in m, minimalization of the complexity of T' and maximalization of the useful information content of the specific $T' \to T_i$ interpretations.)

This process can be reduced to extension inasmuch as when we have formed theory $\hat{T}$ (defined in the language $\hat{E} = E_1 \cup \ldots \cup E_m$), we then extend it to a theory $T \cup T'$ (by extension of the language to $E \cup E'$) and admit appropriate statements (predicates with two arguments) to fix the interpretational relations $T' \to T_i$.

A particular case of the above is when, in looking for a more abstract theory T' (with the interpretation $T' \to T$) for a single theory T, a third theory is employed. Instances of this sort are encountered when a theory is being developed in interaction with the sphere of phenomena. Through study of a certain sphere of phenomena we may come across new facts, and these will be inclued in the theory as new axioms extension of type (2/*b*); or we may discover new objects and new ideas (1/*a*); or we may find the need to formulate new kinds of statements (1/*b*); and these result in the extension of the language, etc. These instances all correspond to the case described above; for the third, auxiliary theory will be furnished by the new observations gained from the sphere of phenomena.

(Note that the definition of inductive extension comprehends such general epitheorems as, for instance, Gödel's incompleteness theorem.)

A distinction can be made between algorithmic and heuristic inductive statements.

Algorithmic extension. The possibility of being able to accomplish a precise extension without studying the phenomenal world is demonstrated by Myhill's theorem [4], according to which there exists an algorithm (constructive procedure) by which for every theory T that incomplete in the Gödel sense** there can be found an extended theory T' such that $T \subset T'$ and $E' = E$ and $T \neq T'$. Extensions of this sort (i.e. those which produce precise theorems) are called algorithmic extensions.

Heuristic extension. If the theory T is incomplete, then the truth or falsity of a statement, in a case where this cannot be decided, can be made more probable by converging to it with a series of heuristic attempts. The probability of the statement in question being true will become the new measure of truth [5]. In this way the theory will be extended, though at the same time the extended theory will necessarily be imprecise.

* Both partition and cover mean the breaking down of some set into subsets. The difference between them is that whereas a partition must resolve into disjunct classes, in the case of cover this stipulation is not made. More specifically, let the partition of the set A be $P = \{P_1, P_2, \ldots, P_n\}$ and the cover of A be $K = \{K_1, K_2, \ldots, K_n\}$. This means $A = P_1 \cup \ldots \cup P_n$ and $A = K_1 \cup \ldots \cup K_n$, but while $i \neq j \Rightarrow P_i \cap P_j = \emptyset$, this is not specified for K. On the other hand, all specifications serving to preserve the structure of the relations, operations, etc. defined on the set hold just as rigorously for K as for P. (See also later under homomorphic interpretation.)

** I.e. In the sense of Gödel's incompleteness theorem.

6.3. Deductive extension. The above process works in reverse in the following case. Consider theory T, the interpretations of which are the theories $T_1, \ldots, T_m$. As our definition of the resulting theory T we take $\hat{T}=T\cup T_1\cup\ldots\cup T_m\cup\{T\to T_1, \ldots, T\to T_m$ are statements describing the interpretations$\}$. In theory $\hat{T}$ the objects of T are variables which display a range of interpretation $T_1\cup\ldots\cup T_m$.

If we are now presented with a new problem, we must first decide whether it is necessary or not to extend the scope of interpretation $T_1\cup\ldots\cup T_m$ in order to be able to solve the problem. If extension is unnecessary, then the theory T can be utilized unchanged to solve the problem. Where this is not possible, then we can make use of theory $\hat{T}$ if to obtain a solution such a theory T_{m+1} is needed which, although differing from all the above theories, can be produced as the interpretation of $\hat{T}$.

T_{m+1} is created by extending theory $\hat{T}$ in such a way that the extended theory will be $\hat{T}\cup T_{m+1}$ (including the statements describing the interpretational relations $T\to T_{m+1}$). For this we need a language $\hat{E}\cup E_{m+1}$. In the creation of this language the generative rules of the original inductive classes must be extended to the new objects and predicates. (For a more detailed treatment of the interrelations of the inductive classes, see the later section on homomorphic interpretation.)

To sum up: In inductive extension the theory T (more precisely the concepts of the theory; that is, the set of objects and the set of predicates) is broken down to equivalence classes, and for the identification of the equivalence classes new, more abstract concepts (objects and predicates) are introduced. Lastly, the rules of substitution are determined (the way in which an element of an equivalence class can be substituted for the abstract concept denoting that class).

By inductive extension we form a new theory T', the interpretation of which is the original theory T (both are of course parts of some theory $\hat{T}=T\cup T'\cup\ldots$) — though it is also possible only to extended the original theory T.* (The latter generally implies a simplification, since in most cases $|T|=\aleph_0$**; in other words, the size of T does not grow with extension, but the axioms and inferences are simplified by virtue of the more general relations.)

In deductive extension we proced in the reverse manner: the cardinality (not the number) of the equivalence classes which correspond to some more abstract concepts is increased. (Here, too, it is possible to produce a new theory or extend the old one.)

7. The effect of extension on the measure of truth

As we have seen, the continuability of the measure of truth is a fundamental point. One way of providing for this continuability is to try to make the new interpretations homomorphic in the following sense:

Definition 10. An interpretation $i\colon T\to T'$ is homomorphic if $i\colon E\to E'$ providing its base is homomorphic. (It is especially important to ensure homomorphism when cover is used instead of partition in the abstraction.)

* In the development of a theory (or theories) the latter procedure corresponds to a continuous, the former to a jump, stage. These occur alternately, the path to the jump stage being prepared for by the continuous.

** $\aleph_0$ (aleph-zero) is the usual symbol for the cardinality of the set of natural numbers.

By the *precision* of a theory T we mean the relation of cardinality between its nucleus and periphery (imprecise part).

Let

$$M = \{t \mid \gamma(t) = 1, t \in T\}$$

$$P = T \setminus M$$

$$P = \frac{|M|}{|M|+|P|}$$

where "p" is the measure of the theory's precision. In accordance with the above, extensions can be ranged in three main classes.

Let the original theory be T and the extended theory T'. Then:

Definition 11. *a)* An extension is proportional if $p=p'$. (We then say that the theories T and T' resemble one other); *b)* An extension achieves an advance in precision if $p<p'$; *c)* An extension loses precision if $p>p'$.

If we want an extension to remain proportional and not to include steps achieving greater precision, then it must not contain steps resulting in greater imprecision either. It is not allowed, for instance, to use hypotheses during the extension. More exactly, theory T is incomplete, in Gödel's sense, if the language $E(E \supset T)$ includes statements such that, though they are not theorems of T, if T were to be (axiomatically) extended with them, a consistent theory would be generated.* If, during the extension, such a statement becomes a theorem of the new theory T', then we say that "hypotheses have been used" during the extension.

With regard to proportional extension it will be remembered that in defining interpretations no stipulation was made that the correspondance i: $T_0 \to T_1$ should be defined everywhere in T. It should therefore be possible to define i with respect to only the nucleus of T and thereby obtain, by deductive extension, a precise theory T_1 from the imprecise theory T. The same can be said of inductive extension, with the difference that here we can utilize the fact that the range of i may also be a subset of T_1, i.e. we do not have to make the condition $i(T)=T_1$.

8. The structure of the Systems Theory (ST)

8.1. The ST is a set of many theories between which connections and interactions of the sort described for an epitheory are possible.

The most abstract part of Systems Theory is General Systems Theory (GST), the interpretations of which are the SST-s (Specific Systems Theories). The SST-s are theories oriented to the individual types of systems (biological, technical, and so on). This two-level classification (GST and SST) is only a rough approximation of the real situation, however. The GST is steadily developing, newer and more abstract levels appear (in the case of inductive extension a whole new theory may be generated), new SST-s are thrown up as new interpretations of the GST; at the same time the articulation of the ST grows more refined, and new levels appear between GST and the SST-s. Nevertheless in the present study all the intermediate

* These are called statements insoluble in T.

levels are oncluded either in the GST, or in the SST category, and only these two are distinguished.

Since all the SST-s are homomorphic interpretations of the GST, it is sufficient to analyse in detail the structure of a single SST; the GST and all the other SST-s will be of similar construction.

Any SST can be partitioned into analytical aspects (AA) and — independently — into analytical levels (AL). These two kinds of partitioning are orthogonal to each other and their joint application is a basic step in using a SST.

An AL is grounded either on a theory or a subtheory. Such a base may be provided by an abstract theory, like information theory, or part of a theory, or even just a statement. An instance is the analysis of a computing centre in terms of some specific parameter (e.g. income or reliability). This theory or subtheory is called the base of the aspect (BA).

As already mentioned in the introduction, the analytical aspects differ from each other in the point of view from which they examine the given type of systems, and consequently in what kind of apparatus they utilize. For instance, there are information theoretical, automata theoretical, control theoretical, and energetical aspects. (A comparative analysis of some of these is provided by Kukhtenko [6].)

The AL-s differ from each other in the detail of the analysis of the given system type; that is, how small are the subsystems that are analysed functionally *only* (like a black box) and how large those that are analysed structurally as well.

Let as look more closely at the development of some interdisciplinary theory.

Consider a variety of theories $T_1, T_2, \ldots, T_n$ dealing with different system types (different phenomenal spheres). (These will correspond later to the SST-s.) If we want to base an interdisciplinary theory on these theories, we must form a theory $\hat{T}=T_1 \cup T_2 \cup \ldots \cup T_n$; then, by extending this theory in an inductive way, we can create a theory T' the interpretations of which are the theories $T_1, T_2, \ldots$*. First of all it is necessary to choose the *analytical aspects.* To do this we shall need to utilize some new subtheories (which are independent of $\hat{T}$), such as information theory, automata theory, etc. If the information theory, for instance, is represented by T_{inf}, the information theoretical aspect will have the form $\hat{T} \cap T_{\text{inf}}$. The analytical aspect obtained are then broken down to the level of the theories $T_1, T_2, \ldots$ and used to form the classification needed for the abstraction.

In contrast to the aspects, the *analytical levels* are obtained as the result of the reverse process. As was mentioned in connection with deductive interpretation, the solving of new problems often requires new interpretations. Such interpretations are the AL-s, the choice of which is determined by the depth of analysis necessary for the solution of the problem. Consequently, the AL-s arise during the search for adequate interpretations of the problems.

Let us investigate in detail the process of forming the analytical aspects (AA-s) and analytical levels (AL-s). This question, it will be noted, belongs more to the application of the ST than to its construction.

* In the case of Systems Theory, $T_1, T_2, \ldots$ correspond to $SST_1, SST_2, \ldots$ and T' corresponds to the GST.

8.2. Formation of the AA. This process will be analysed using the example of the biological ST. Let us take some SST, say Biological Systems Theory (further on BST), and let us search for its information theoretical aspect. (This implies, of course that the information theory (IT) is given too.) The process starts with the deductive extension of the information theory, and we must look for a homomorphic and valid interpretation i: IT→BST.* For this purpose a suitable extension of the BST is needed; this will, in fact, be made in the course of looking for the interpretation.

The interpretation eventually reached will permit us to make conclusions in the scope of the BST by means of information theory, though the manipulation is obviously likely to be extremely difficult. The difficulties can be surmounted by the creation of a congruence relation** $i^{-1} \circ i$ over the BST.

The congruence classes defined in this manner will form a theory, because the mapping i is homomorphic. The new theory is symbolized by $T' = \text{BST}/(i^{-1} \circ i)$ to. Theory T' (the biological information theory) is isomorphic to IT and can be interpreted for the BST in a homomorphic way.

Generalizing to the formation of the aspects of an optional SST in accordance with some BA, the above process can be interpreted as a deductive extension of the memory BA SST in which the rules of deduction (drawing of a valid interpretation i) and the language (provision of a classification $\text{SST}/(i^{-1} \circ i)$) have been extended.***

8.3. Formation of an AL. Let us investigate the application of a theory (T) to solving a problem (p), which will be a statement of some language. (We are not concerned at the moment with selecting an adequate theory T for the given problem; this will be discussed later.)

We introduce the following notation:

$$T \overset{\simeq}{\supset} p \overset{\text{def}}{\Leftrightarrow} (\exists T' \subseteqq T) T' \simeq T'' \,\&\, p \in T''.$$

* It is generally not possible to prescribe that an interpretation be adequate as well, but the adequacy will be prescribed for the minimum subtheory of the BST in which the problem can still be solved.

** By operation $\circ$ is meant a function composition, e.g. $(f \circ g)(x) = g(f(x))$. A congruence relation is an equivalence relation compatible with the operation. In other words, a relation is a congruence relation, if it can be formed as the component of a homomorphism and its inverse; that is, if there exists a homomorphism h to the relation r such that $r = h^{-1} \circ h$.

*** In other words, the factor theory $T' = \text{SST}/(i^{-1} \circ i)$ is a partition on the SST. The set of objects (O') of theory T' is a partition of the objects of the SST. Therefore, to each object of T' can be added an arbitrary number (say n) of fixed variables $(x_1, \ldots, x_n)$ the range of which exactly covers the subclass of the objects of the SST pertaining to the given object. Applying the λ conversion of Church [7] to these variables, an unambiguous correspondance is gained between theory T' and the SST: $e = \lambda(x_1, \ldots, x_n) \circ e'$, where $e \in \text{SST}$ and $e' \in T'$.

To examine this in more detail: Let F' stand for the set of predicates of T'. For every $o \in O'$ we form a word algebra $W_{F'}$ $(o \cup \{x_1, \ldots, x_n\})$. Since every word algebra is free over the generator set, and the union of free algebras is free over the union of generator sets, the mapping $i: o \cup \{x_1, \ldots, x_n\} \to O$ unambiguously defines the homomorphism sought. Here O is the object set of SST.

The theory T' is applicable in the BST because the selected interpretation was homomorphic and thus a congruence partition compatible with the relations was obtained. Ensuring this homomorphism is a basic task in forming the AL's.

A well-know example of an application that has been unsuccessful is biological thermodynamics; a successful example would be biological information theory [8].

If $T \overset{\sim}{\supset} p$, then we say that problem p is *embeddable* in theory T. Let T be a subalgebra of the σ algebra of some universe J, and let $p \in J$. T'' is now a subalgebra of T. To solve p we must look for the minimal *subalgebra* (T'') of T in which p can still be solved ($p \in T''$); this T'' is called the *level* of the theory adequate to problem p. It is easy to see that this level will not be "homogeneous"; that is, the classes of the minimal partition of the selected subalgebra σ will not be of equal cardinality.* If the theory T'' which is adequate to the problem necessitates decisions which cannot be made within the theory T**, then the problem can be solved by the theory T only after this has been extended. In other words, T must be extended to such an algebra σ of the universe J as has a subalgebra adequate to p.

Say the problem is discovering the mechanism of metabolism in the human body. On this universe we can define certain subalgebras σ: e.g. anabolism, catabolism, chemical relations, etc. If the individual organs are described on the anabolic level, the union of the set of algebras we obtain will solve the problem, but it will not be minimal (e.g. we know that anabolic processes taking place in the liver differ from those in the muscles). On the other hand, if the solution of the problem is analysed at the level of chemical reactions, then the number of classes gained in the partition (i.e. the number of concepts) will be a minimum. This means that if such chemical laws are chosen as are common to the metabolism of the liver, muscles, etc., then the concepts that are obtained will be able to describe the whole metabolic process, e.g. the synthesis of starch and dextrose.

References

[1] Curry, H. B., *Foundations of Mathematical Logic*, McGraw-Hill Book Company, Inc. 1963.
[2] Lee, E. T. and L. A. Zadeh, Note on Fuzzy Languages, *Information Sci.* v. 1, 1969, pp. 421—434.
[3] Post, E. L., Finite Combinatory Processes — Formulation 1. *J. Symbolic Logic*, v. 1, 1936, pp. 103—105.
[4] Myhill, J., Notes for a series of lectures on recursive functions, In lecture notes for a summer conference on automata theory, University of Michigan, 1963.
[5] Бердзин, Я. Н., О вычислимости на вероятностных машинах, *Докл. АН СССР* v. 189, №4, 1969, pp. 609—702.
[6] Кухтенко, А. И., Обзор основных направлений развития общей теории систем, Материалы координационного совещания секции технической кибернетики, Киев, 1969, pp. 3—62.
[7] Cohn, P. M., *Universal Algebra*, New York and London, Harper Row. 1965.
[8] Quastler, H., *The Emergence of Biological Organizator*, New Haven and London, Yale University Press, 1964.

(Received October 17, 1973)

* The minimal partition of an algebra σ is taken to mean the infimum (or greatest lower bound) of the partitions generating the algebra.

** Let k be the most detailed partition formed by theory T on the universe (i.e. k is the greatest lower bound of the partitions generating T) and let k'' be one of the partitions generating T''. It follows that the decisions of T'' can be generated in T provided $k'' \geq k$.