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On two problems of A. Salomaa

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In this paper we solve two problems raised by A. Salomaa in his book [1]. Namely, we show that all right derivatives of a stochastic language are stochastic. Conversely, if there exists an integer k such that all right derivatives of a language L with respect to all words of length k are stochastic languages then L is stochastic language, too. Furthermore, it is proved that the family of stochastic languages remains unaltered if the components of the output vectors and the cut points are allowed to be arbitrary real numbers. Proving these statements, we give affirmative answers to Problems 3.1 and 5.1 of A. Salomaa.

Before studying these problems we recall some definitions from [1].

By an alphabet I we mean a finite non-empty set. The elements of I are called letters, sometimes input signs. A word over I is a finite string consisting of zero or more letters. The empty word λ is a string consisting of zero letters. If a word P consists of $k(\geqq 0)$ letters then the length of this word is $\lg(P)=k$. The set of all words over I is denoted by $W(I)$. If $P, Q\in W(I)$ then PQ denotes their catenation.

A language L is a subset of $W(I)$. The void language is the language consisting of no words. The union (or sum) of two languages L_1 and L_2 is denoted by $L_1\vee L_2$ and their catenation is defined by $L_1L_2=\{P\,|\,P=P_1P_2,\ P_1\in L_1,\ P_2\in L_2\}$. If L_2 consists of one word Q only then L_1L_2 is denoted by L_1Q.

If given a word P over I and a language $L\subseteq W(I)$ then the right (left) derivative of L with respect to the word P is defined by $L//P=\{Q\,|\,QP\in L\}$ ($L\backslash\backslash P=\{Q\,|\,PQ\in L\}$).

A vector is called stochastic if its each component is nonnegative real number and the sum of its components equals to 1. Moreover, a stochastic matrix is a square matrix whose each row is a stochastic vector.

By a finite probabilistic automaton — or, shortly, probabilistic automaton — over an alphabet I we mean an ordered triple $PA=(S, s_0, M)$, where $S=\{s_1, s_2, \ldots, s_n\}$ is a finite non-empty set, the set of all internal states of PA, s_0 is an n-dimensional stochastic row vector, the initial distribution, whose ith component equals to the probability of PA to be in the state s_i at the beginning of its working; finally, M is a mapping of I into the set of all stochastic matrices of type $n\times n$. For every $x\in I$, $p_{i,j}(x)$ denotes the (i,j)th entry of the matrix $M(x)$. This is the transition probability of PA to go from the state s_i into the state s_j under the input sign x.

We may extend the domain of the function M from I to $W(I)$ by defining: $M(\lambda)=E_n$, $M(Px)=M(P)M(x)$ for every $Px\in W(I)$. (Here E_n is the n-dimensional identity matrix.) The stochastic row vector $s_0 M(P)$ is called the distribution of states caused by the word P. Further on this row vector is often denoted by $PA(P)$.

If V_i is the n-dimensional coordinate column vector whose ith component equals to 1 then for every word $P \in W(I)$, $p_i(P) = PA(P)V_i$ is the probability of PA to go into the state s_i under the word P.

Let PA be the probabilistic automaton defined above and $\bar{S}_1$ an n-dimensional column vector whose each component is either 0 or 1; $\bar{S}_1$ is called output vector. To each such vector $\bar{S}_1$ there corresponds a subset S_1 of S and conversely, where S_1 is given by: $s_i \in S_1$ if and only if the i th component of $\bar{S}_1$ equals to 1. Moreover, let η be a real number such that $0 \leqq \eta < 1$. The language represented in PA by S_1 and the cut point η is defined by $L(PA, \bar{S}_1, \eta) = \{P \mid PA(P)\bar{S}_1 > \eta\}$. A language L is η-stochastic if and only if for some PA and $\bar{S}_1$, $L = L(PA, \bar{S}_1, \eta)$. Furthermore, a language L is stochastic if and only if for some $\eta\,(0 \leqq \eta < 1)$, L is η-stochastic. Now we are ready to state

Theorem 1. All right derivatives of a stochastic language with respect to any word are stochastic languages. Conversely, if there is an integer k such that all right derivatives of a language L with respect to all words of length k are stochastic then L is a stochastic language.

Proof. In order to prove the first part of the theorem take an arbitrary stochastic language $L = L(PA, \bar{S}_1, \eta)$ represented in the probabilistic automaton $PA = (S(= \{s_1, s_2, \ldots, s_n\}), s_0, M)$ over the alphabet $I = \{x_1, x_2, \ldots, x_r\}$. For any $i = 1, 2, \ldots, n$ and $x \in I$ let $q_i(x) = V_i^* M(x)\bar{S}_1$, where V_i^* is the transpose of the vector V_i. Thus $q_i(x)$ is the probability of PA to go from the state s_i into one of the states of S_1 under the input sign x.

Since our statement is obviously valid for the empty word thus, in the sequel, we may confine ourself to derivatives with respect to words of length exceeding 0. By $L/\!/(x_i x_j) = (L/\!/x_j)/\!/x_i$ $(x_i, x_j \in I)$, it is enough to prove the first statement of Theorem 1 for letters. To make our discussions simplier, further on we shall deal with $L/\!/x_1$ only. Thus $q_i(x_1)$ will simply be denoted by q_i.

If for every $i = 1, 2, \ldots, n$, $q_i = 0$ then $L \subseteqq W(I)\{x_2, x_3, \ldots, x_r\}$, therefore, $L/\!/x_1$ is the void language, which is clearly a stochastic one. Hence we may assume that there is at least one index i with $q_i \neq 0$. Let $i_1, i_2, \ldots, i_l$ be all different indices such that the product $q_{i_1} q_{i_2} \ldots q_{i_l} \neq 0$ and let $q = q_{i_1} + q_{i_2} + \ldots + q_{i_l}$.

We may assume that $\eta < q$. Indeed, by a theorem of R. Bukharev and P. Turakainen in [1], every stochastic language is η'-stochastic for any η' with $0 < \eta' < 1$. Furthermore, it can easily be seen that if given a finite probabilistic automaton $PA' = (S', s_0', M')$ then for any language $L' = L(PA', \bar{S}_1', \eta'')$ and η' with $0 < \eta' < \eta''$ one can construct a probabilistic automaton PA'' by adding a new state s to the set of the internal states of PA' such that there is no transition from s to S' and from any state of S' to s, moreover, L' can be represented in PA'' with the cut point η' and the same set S_1'.

Now let $S^* = \{s_1, s_2, \ldots, s_{ln}\}$ and

$$PA^* = \left(S^*, \frac{1}{q}(q_{i_1}s_0, q_{i_2}s_0, \ldots, q_{i_l}s_0), M^*\right),$$

where

$$M^*(x) = \begin{Vmatrix} M(x) & & 0 \\ & M(x) \ddots & \\ 0 & & M(x) \end{Vmatrix}$$

for any $x \in I$. Define

$$\bar{S}_1^* = \begin{Vmatrix} V_{i_1} \\ V_{i_2} \\ \vdots \\ V_{i_l} \end{Vmatrix}, \quad L^* = L(PA^*, \bar{S}_1^*, \eta/q).$$

Obviously PA^* is a probabilistic automaton and L^* is a stochastic language. We clame that $L//x_1 = L^*$. To prove this statement it is enough to verify that for every word P,

$$qPA^*(P)\bar{S}_1^* = PA(Px_1)\bar{S}_1.$$

Indeed, if $P \in W(I)$ is an arbitrary word then

$$qPA^*(P)\bar{S}_1^* = (q_{i_1}s_0, q_{i_2}s_0, \ldots, q_{i_l}s_0) \begin{Vmatrix} M(P) & & 0 \\ & M(P) \ddots & \\ 0 & & M(P) \end{Vmatrix} \begin{Vmatrix} V_{i_1} \\ V_{i_2} \\ \vdots \\ V_{i_l} \end{Vmatrix} =$$

$$= \sum_{j=1}^{l} q_{i_j} s_0 M(P) V_{i_j} = \sum_{i=1}^{n} q_i s_0 M(P) V_i = \sum_{i=1}^{n} p_i(P) q_i = PA(Px_1)\bar{S}_1.$$

The second part of the theorem is also trivial in the case $k=0$. Thus let $k=1$. First we prove that if L is a stochastic language then for every letter x the catenation Lx is stochastic too.

Let again $L = L(PA, \bar{S}_1, \eta)$ be a stochastic language, where $PA = (S(= \{s_1, s_2, \ldots, s_n\}), s_0, M)$ is a finite probabilistic automaton over the alphabet $I = \{x_1, x_2, \ldots, x_r\}$. Without loss of generality we may assume that $S_1 = \{s_1, s_2, \ldots, s_l\}$ for a certain integer $l \leq n$. For arbitrary letter $x \in I$ let $M_i(x)$ denote the i th row of the matrix $M(x)$. For every $i \in \{1, 2, \ldots, l\}$ there exists a $j(i) \in \{1, 2, \ldots, n\}$ such that $p_{i,j(i)}(x_1) \neq 0$. To every such pair $(i, j(i))$ let us correspond the following probabilistic automaton:

$$PA^i = (S^i(= \{s_1^i, s_2^i, \ldots, s_n^i, s_{n+1}^i\}), (s_0, 0), M^i),$$

where

$$M^i(x_1) = \begin{Vmatrix} & M_1(x_1) & & 0 \\ & \vdots & & \vdots \\ & M_{i-1}(x_1) & & 0 \\ p_{i,1}(x_1) \ldots p_{i,j(i)-1}(x_1) & 0 & p_{i,j(i)+1}(x_1) \ldots p_{i,n}(x_1) & p_{i,j(i)}(x_1) \\ & M_{i+1}(x_1) & & 0 \\ & \vdots & & \vdots \\ & M_n(x_1) & & 0 \\ & M_{j(i)}(x_1) & & 0 \end{Vmatrix}$$

if $i \neq j(i)$,

$$M^i(x_1) = \left\| \begin{matrix} & M_1(x_1) & & 0 \\ & \vdots & & \vdots \\ & M_{i-1}(x_1) & & 0 \\ p_{i,1}(x_1) \dots p_{i,j(i)-1}(x_1) & 0 & p_{i,j(i)+1}(x_1) \dots p_{i,n}(x_1) & p_{i,j(i)}(x_1) \\ & M_{i+1}(x_1) & & 0 \\ & \vdots & & \vdots \\ & M_n(x_1) & & 0 \\ p_{j(i),1}(x_1) \dots p_{j(i),j(i)-1}(x_1) & 0 & p_{j(i),j(i)+1}(x_1) \dots p_{j(i),n}(x_1) & p_{j(i),j(i)}(x_1) \end{matrix} \right\|$$

if $i=j(i)$. Moreover, in both cases

$$M^i(x) = \left\| \begin{matrix} M(x) & 0 \\ M_{j(i)}(x) & 0 \end{matrix} \right\|$$

if $x \neq x_1$.

It is clear that $L(PA^i, V_{n+1}, 0) \subseteq W(I)x_1$, where V_{n+1} is the $n+1$-dimensional column vector whose $n+1$th component is 1 and all others are zero. We shall now prove that for every word P,

$$s_0 M(P) V_i = \frac{1}{p_{i,j(i)}(x_1)} (s_0, 0) M^i(Px_1) V_{n+1}.$$

Further on we often use the following notation. If given an arbitrary finite probabilistic automaton $PA'=(S', s_0', M')$ over the alphabet I' and $s'_{i_0} s'_{i_1} \dots s'_{i_{\lg(P)}} \in W(S')$ then

$$p(s'_{i_0} s'_{i_1} \dots s'_{i_{\lg(P)}} | P)$$

denotes the transition probability of PA' to go from the state s'_{i_0} into $s'_{i_{\lg(P)}}$ through the states $s'_{i_1}, \dots, s'_{i_{\lg(P)-1}}$ under the input word P.

Let now $P \in W(I)$ be an arbitrary word. For every $i=1, 2, \dots, l$ define

$$A_i = \{Qs_i | Q \in W(S), p(Qs_i|P) > 0\},$$
$$B_i = \{Q^i s^i_{n+1} | Q^i \in W(S^i), p(Q^i s^i_{n+1} | Px_1) > 0\}.$$

We say that a $Q \in A_i$ $(i=1, 2, \dots, l)$ has the property Φ^i_t for some $t \in \{0, 1, \dots, \lg(P)-1\}$ — in notation $Q \in \Phi^i_t$ — if $Q = s_{i_0} s_{i_1} \dots s_{i_{\lg(P)}}$ such that $i_t = i$, $i_{t+1} = j(i)$, $P = P' x_1 P''$, $\lg(P') = t$. (The fact that Q does not have the property Φ^i_t will be denoted by $Q \notin \Phi^i_t$.)

Let $\varphi_i : A_i \to B_i$ $(i=1, 2, \dots, l)$ be a mapping given by

$$\varphi_i(s_{i_0} s_{i_1} \dots s_{i_{\lg(P)}}) = s^i_{j_0} s^i_{j_1} \dots s^i_{j_{\lg(P)}} s^i_{n+1},$$

where $j_0 = i_0$ and if $s_{i_0} s_{i_1} \dots s_{i_{\lg(P)}} \in \Phi^i_t$ for certain $t \in \{0, 1, \dots, \lg(P)-1\}$ then $j_{t+1} = n+1$ otherwise $j_{t+1} = i_{t+1}$. We shall now prove some properties of the mappings φ_i $(i=1, 2, \dots, l)$.

Assume that $Q = s_{i_0} s_{i_1} \dots s_{i_{\lg(P)}} \in A_i$, $Q' = s_{i'_0} s_{i'_1} \dots s_{i'_{\lg(P)}} \in A_i$ and $Q \neq Q'$. Then there exists an integer t, $-1 \leqq t \leqq \lg(P)-1$ such that $i_{t+1} \neq i'_{t+1}$. Let $\varphi_i(Q) = s^i_{j_0} s^i_{j_1} \dots s^i_{j_{\lg(P)}} s^i_{n+1}$ and $\varphi_i(Q') = s^i_{j'_0} s^i_{j'_1} \dots s^i_{j'_{\lg(P)}} s^i_{n+1}$. Now we distinguish three cases.

1. $t=-1$. Then $j_0=i_0\neq i'_0=j'_0$. Thus $\varphi_i(Q)\neq\varphi_i(Q')$.
2. $t\geqq 0$, $Q\notin\Phi^i_t$, $Q'\notin\Phi^i_t$. Then $j_{t+1}=i_{t+1}\neq i'_{t+1}=j'_{t+1}$ and again $\varphi_i(Q)\neq\varphi_i(Q')$.
3. $t\geqq 0$, $Q\in\Phi^i_t$, $Q'\notin\Phi^i_t$. Now $j_{t+1}=n+1, j'_{t+1}\neq n+1$. Thus $\varphi_i(Q)\neq\varphi_i(Q')$.

Since these are all possible cases, we get that φ_i is a one to one mapping for every $i=1, 2, \ldots, l$.

Let $s^i_{j_0}s^i_{j_1}\ldots s^i_{j_{\lg(P)}}s^i_{n+1}\in B_i$ be an arbitrary word. Since this is clearly the image of the word $s_{i_0}s_{i_1}\ldots s_{i_{\lg(P)}}\in A_i$, where $i_0=j_0$ and for every $t\in\{0, 1, \ldots, \lg(P)-1\}$ if $j_{t+1}=n+1$ then $i_{t+1}=f(i)$ otherwise $i_{t+1}=j_{t+1}$ thus we have that φ_i is one to one mapping of A_i onto B_i for every $i=1, 2, \ldots, l$.

Finally, since φ_i is a one to one mapping of A_i onto B_i and

$$p(Q|P)=\frac{1}{p_{i,j(i)}(x_1)}p(\varphi_i(Q)|Px_1)$$

for any $i\in\{1, 2, \ldots, l\}$ and $Q\in A_i$ thus we get:

$$s_0M(P)V_i=\sum_{Q\in A_i}p(Q|P)=\sum_{\varphi_i(Q)\in B_i}p(\varphi_i(Q)|Px_1)=\frac{1}{p_{i,j(i)}(x_1)}(s_0, 0)M^i(Px_1)V_{n+1}.$$

Define

$$S^*=\{s_1, s_2, \ldots, s_{(n+1)l}\},\quad p=\sum_{i=1}^{l}\frac{1}{p_{i,j(i)}(x_1)},$$

$$s^*_0=\frac{1}{p}\left(\frac{1}{p_{1,j(1)}(x_1)}(s_0, 0), \frac{1}{p_{2,j(2)}(x_1)}(s_0, 0), \ldots, \frac{1}{p_{l,j(l)}(x_1)}(s_0, 0)\right)$$

and for every $x\in I$ take

$$M^*(x)=\begin{Vmatrix}M^1(x) & & & 0\\ & M^2(x) & & \\ & & \ddots & \\ 0 & & & M^l(x)\end{Vmatrix},\qquad S^*_1=\begin{Vmatrix}V_{n+1}\\ V_{n+1}\\ \vdots\\ V_{n+1}\end{Vmatrix}.$$

Moreover, consider the stochastic language $L^*=L(PA^*, S^*_1, \eta/p)$, where $PA^*=(S^*, s^*_0, M^*)$ is obviously a probabilistic automaton over the alphabet I. In order to prove that $L^*=Lx_1$ it is enough to show, by $L^*\subseteq W(I)x_1$, that $Px_1\in Lx_1$ if and only if $Px_1\in L^*$ for arbitrary $P\in W(I)$. But this can be seen immediately because

$$pPA^*(Px_1)\bar{S}^*_1=$$

$$=\left(\frac{(s_0, 0)}{p_{1,j(1)}(x_1)}, \frac{(s_0, 0)}{p_{2,j(2)}(x_1)}, \ldots, \frac{(s_0, 0)}{p_{l,j(l)}(x_1)}\right)\begin{Vmatrix}M^1(Px_1) & & & 0\\ & M^2(Px_1) & & \\ & & & \\ 0 & & & M^l(Px_1)\end{Vmatrix}\begin{Vmatrix}V_{n+1}\\ V_{n+1}\\ \vdots\\ V_{n+1}\end{Vmatrix}=$$

$$=\sum_{i=1}^{l}\frac{(s_0, 0)}{p_{i,j(i)}(x_1)}M^i(Px_1)V_{n+1}=\sum_{i=1}^{l}s_0M(P)V_i=PA(P)\bar{S}_1.$$

Since x_1 is an arbitrary letter thus the language Lx is stochastic for any $x\in I$.

Now let L be a language over I such that all the languages $L//x_1, L//x_2, \ldots$ $\ldots, L//x_r$ are stochastic. Thus the languages $(L//x_1)x_1, (L//x_2)x_2, \ldots, (L//x_r)x_r$ are also stochastic. They can be represented, respectively, in the probabilistic automata $PA_{x_1}=(S_{x_1}, (s_0)_{x_1}, M_{x_1}), PA_{x_2}=(S_{x_2}, (s_0)_{x_2}, M_{x_2}), \ldots, PA_{x_r}=(S_{x_r}, (s_0)_{x_r}, M_{x_r})$ by the sets $S_{x_{1_1}}, S_{x_{2_1}}, \ldots, S_{x_{r_1}}$ and the cut points $\eta_1, \eta_2, \ldots, \eta_r$, where every automaton PA_{x_i} is constructed in a way analogous to the construction of PA^*. It follows from our discussions above that $L(PA_{x_i}, \bar{S}_{x_{i_1}}, 0) \subseteqq W(I)x_i$ for every $i=1, 2, \ldots, r$.

First we deal with the case $\eta_1\eta_2\ldots\eta_r \neq 0$. Then, as it was noted in the proof of the first part of Theorem 1, we may assume that $\eta_1=\eta_2=\ldots=\eta_r=\eta$. Define

$$n = \sum_{i=1}^{r} \operatorname{card}(S_{x_i}), \quad PA = (S(=\{s_1, s_2, \ldots, s_n\}), s_0, M),$$

where

$$s_0 = \frac{1}{r}((s_0)_{x_1}, (s_0)_{x_2}, \ldots, (s_0)_{x_r}), \quad M(x) = \begin{Vmatrix} M_{x_1}(x) & & & 0 \\ & M_{x_2}(x) & & \\ & & \ddots & \\ 0 & & & M_{x_r}(x) \end{Vmatrix}$$

for arbitrary $x \in I$. Let

$$\bar{S}_1 = \begin{Vmatrix} \bar{S}_{x_{1_1}} \\ \bar{S}_{x_{2_1}} \\ \vdots \\ \bar{S}_{x_{r_1}} \end{Vmatrix}, \quad L^* = L(PA, \bar{S}_1, \eta/r).$$

It follows immediately that $L^* = \bigvee_{i=1}^{r} (L//x_i)x_i$ because for every word $P \in W(I)$ and $x_i \in I$,

$$rPA(Px_i)\bar{S}_1 = \sum_{j=1}^{r} PAx_j(Px_i)\bar{S}_{x_{j_1}} = PA_{x_i}(Px_i)\bar{S}_{x_{i_1}}.$$

If there is at least one index i such that $\eta_i=0$ we may assume, without loss of generality, that $\eta_1=\eta_2=\ldots=\eta_j=0$ but the product $\eta_{j+1}\eta_{j+2}\ldots\eta_r \neq 0$ for an integer $j \leqq r$. Since by a theorem in [1] every 0-stochastic language is regular, the language $\bigvee_{i=1}^{j} (L//x_i)x_i$ is regular. Moreover, in the same way as it was done in the previous case, it can be proved that the language $\bigvee_{i=j+1}^{r} (L//x_i)x_i$ is stochastic. Thus, using a theorem of P. Turakainen (see [1]) by which the sum of a stochastic and a regular language is stochastic, we get that $L^* = \bigvee_{i=1}^{r} (L//x_i)x_i$ is a stochastic language.

Finally, since $L=L^*$ or $L=L^* \vee \{\lambda\}$ we have that L is stochastic.

We continue our proof by induction. Assume that the second part of the theorem holds true for a certain integer $k \geqq 1$, and assume that for every word xP of length $k+1$ the language $L//xP$ is stochastic. Since $L//xP=(L//P)//x$ thus by our result for the case $k=1$ and the inductive hypothesis we get that L is stochastic.

We now prove

Theorem 2. The family of stochastic languages remains unaltered if the components of $\bar{S}_1$ as well as η are allowed to be arbitrary real numbers.

Proof. We distinguish two cases.

1. The components of $\bar{S}_1$ are arbitrary nonnegative reals.

Assume that $PA=(S(=\{s_1, s_2, \dots, s_n\}), s_0, M)$ is a probabilistic automaton over the alphabet $I=\{x_1, x_2, \dots, x_r\}$ and consider the language $L=L(PA, \bar{S}_1, \eta)= =\{P\in W(I)|PA(P)\bar{S}_1>\eta\}$, where the components of $\bar{S}_1$ are arbitrary nonnegative numbers and η is an arbitrary real number. Let

$$\bar{S}_1 = \begin{Vmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{Vmatrix}$$

and $v=\max\{v_1, v_2, \dots, v_n\}$. Since Theorem 2 is trivial if $v=0$, therefore, we shall deal with the case $v>0$ only. Moreover, we may assume that $0\leqq\eta<v$ because if $\eta\geqq v$ then L is void and if $\eta<0$ then clearly $L=L(PA, \bar{S}_1', 0)$, where a component of $\bar{S}_1'$ equals to 0 or 1 depending on whether the same component of $\bar{S}_1$ is 0 or positive. Thus in both cases L is stochastic.

Define $S^*=\{s_1, s_2, \dots, s_{n+2}\}$, $s_0^*=(s_0, 0, 0)$ and let $PA^*=(S^*, s_0^*, M^*)$ be a probabilistic automaton over the alphabet $I^*=\{x_1, x_2, \dots, x_{r+1}\}$, where

$$M^*(x) = \begin{Vmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & M(x) & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & 0 & \dots\ 0 & 0 & 1 \\ 0 & 0 & \dots\ 0 & 0 & 1 \end{Vmatrix}$$

for every $x\in I$ and

$$M^*(x_{r+1}) = \begin{Vmatrix} & & & v_1/v & 1-v_1/v \\ & & & v_2/v & 1-v_2/v \\ & 0 & & \vdots & \vdots \\ & & & v_n/v & 1-v_n/v \\ 0 & 0 & \dots\ 0 & 0 & 1 \\ 0 & 0 & \dots\ 0 & 0 & 1 \end{Vmatrix}.$$

Let $\bar{S}_1^*$ denote the $n+2$-dimensional column vector whose $(n+1)$th component is 1 and the others are zero. Define $L^*=L(PA^*, \bar{S}_1^*, \eta/v)$. L^* is stochastic because $0\leqq\eta/v<1$. Our purpose is to show that $L^*=Lx_{r+1}$. Thus, by Theorem 1, it follows that $L=(Lx_{r+1})//x_{r+1}$ is a stochastic language. Since $L^*\subseteqq W(I)x_{r+1}$, therefore in order to prove this equation it is enough to verify that for every word $P\in W(I)$,

$$vPA^*(Px_{r+1})\bar{S}_1^* = PA(P)\bar{S}_1.$$

Indeed,

$$vPA^*(Px_{r+1})\bar{S}_1^* = v(s_0, 0, 0)\, M^*(P) \left\| \begin{array}{cccccc} & & & & v_1/v & 1-v_1/v \\ & & & & v_2/v & 1-v_2/v \\ & & 0 & & \vdots & \vdots \\ & & & & v_n/v & 1-v_n/v \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{array} \right\| \left\| \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{array} \right\| =$$

$$= (s_0, 0, 0) \left\| \begin{array}{cccccc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & M(P) & & \vdots & \vdots \\ & & & & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{array} \right\| \left\| \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \\ 0 \\ 0 \end{array} \right\| = s_0 M(P)\bar{S}_1 = PA(P)\bar{S}_1.$$

2. There exists at least one negative number among the components of $\bar{S}_1$. This case is traceable to the previous one by adding to η and to each component of $\bar{S}_1$ a number which is not smaller than the absolut value of the minimum of the components of $\bar{S}_1$.

After having written the article the author obtained knowledge of the fact, that among others the same problems had been solved in a different way by P. Turakainen in [2].

О двух проблемах А. Саломаа

В этой статьи мы решили две проблемы, поставленные А. Саломаа в [1]. Именно покажем, что правосторонные частные стохастические языки, образованные с любыми цепочками, являются стохастическими, наоборот, если имеется такой целое число k, что у одного языка все правосторонные частные, образованные всеми цепочками длиной k, стохастические, тогда он сам является стохастическим. Далее покажем, что семейство стохастических языков не расширяется, если компоненты выходного вектора любые действительные числа.

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