Acta Cybernetica, Tom. 2, Fasc. 4, Szeged, 1975.

Endomorphisms of group-type quasi-automata

By I. BABCSÁNYI

In this paper the endomorphisms of group-type quasi-automata are investigated using the concept of the generating system of quasi-automata. For the notions and notations which are not defined here, we refer the reader to [4] or [5].

Let the characteristic semigroup $\bar{F}=F/\varrho_A$ of an arbitrary quasi-automaton $\mathbf{A}=(A, F, \delta)$ be a monoid, and let $\bar{e}$ $(e\in F)$ be the identity element of $\bar{F}$. Take the subset $A'=\langle\delta(a,f)|\, a\in A; f\in F\rangle$ of A and the A-sub-quasi-automaton $\mathbf{A}'=(A', F, \delta')$ of $\mathbf{A}$. It is easy to see that $a\in A'$ if and only if $\delta(a, e)=a$ for an arbitrary state a of $\mathbf{A}$. Furthermore, the characteristic semigroup of $\mathbf{A}'$ is equal to that of $\mathbf{A}$. Assume that the set $A\setminus A'$ is non-empty. Let V be an arbitrary (non-empty) subset of $A\setminus A'$, and let π denote a mapping of V into $A\setminus A'$. Moreover, let α' be an endomorphism of $\mathbf{A}'$. The following holds:

Theorem 1. *The mapping* α: $A\to A$, *defined by*

$$\alpha(a)=\begin{cases}\alpha'(a) & \text{if}\quad a\in A',\\ \alpha'(\delta(a,e)) & \text{if}\quad a\in A\setminus A',\end{cases} \tag{1}$$

is an endomorphism of $\mathbf{A}$. *The mapping* α_π: $A\to A$, *for which*

$$\alpha_\pi(a)=\begin{cases}\alpha'(a) & \text{if}\quad a\in A',\\ \pi(a) & \text{if}\quad a\in V,\\ \alpha'(\delta(a,e)) & \text{if}\quad a\in(A\setminus A')\setminus V\end{cases} \tag{2}$$

holds, is an endomorphism of $\mathbf{A}$ *if and only if*

$$\alpha'(\delta(a,e))=\delta(\pi(a),e) \tag{3}$$

holds for every $a(\in V)$. *Furthermore, if* β *is an endomorphism of* $\mathbf{A}$, *then* β *is a mapping of type* (1) *or* (2).

Proof. α and α_π are well-defined. It can immediately be seen that α is an endomorphism of $\mathbf{A}$. Now let $a(\in V)$, $b(\in(A\setminus A')\setminus V)$ and $f(\in F)$ be arbitrary elements. Assume that the condition (3) holds. Then

$$\alpha_\pi(\delta(a,f))=\alpha'(\delta(a,f))=\alpha'(\delta(a,ef))=\alpha'(\delta(\delta(a,e),f))=$$
$$=\delta(\alpha'(\delta(a,e)),f)=\delta(\delta(\pi(a),e),f)=\delta(\pi(a),ef)=\delta(\alpha_\pi(a),f),$$

and

$$\alpha_\pi(\delta(b,f)) = \alpha'(\delta(b,f)) = \alpha'(\delta(b, ef)) =$$
$$= \alpha'(\delta(\delta(b, e),f)) = \delta(\alpha'(\delta(b, e)),f) = \delta(\alpha_\pi(b),f).$$

These mean that α_π is an endomorphism of **A**. Conversely, if (2) is an endomorphism of **A**, then for every $a(\in V)$ we get

$$\alpha'(\delta(a, e)) = \alpha_\pi(\delta(a, e)) = \delta(\alpha_\pi(a), e) = \delta(\pi(a), e),$$

that is, (3) holds.
Take an arbitrary endomorphism β of **A**. We prove the following implications:

$$a\in A' \Rightarrow \beta(a)\in A',$$
$$a\in A\setminus A' \Rightarrow \beta(a)\in A\setminus A' \quad \text{or} \quad \beta(a) = \beta(\delta(a, e)).$$

If $a\in A'$, there are $b(\in A)$ and $f(\in F)$ such that $\delta(b,f)=a$. Then

$$\beta(a) = \beta(\delta(b,f)) = \delta(\beta(b),f)\in A'.$$

If $a\in A\setminus A'$ and $\beta(a)\in A'$, there are $b(\in A)$ and $f(\in F)$ such that $\beta(a)=\delta(b,f)$. That is,

$$\beta(a) = \delta(b,f) = \delta(b,fe) = \delta(\delta(b,f), e) = \delta(\beta(a), e) = \beta(\delta(a, e)).$$

Let β be an arbitrary endomorphism of **A** and let β' be an endomorphism of **A'** for which $\beta'(a)=\beta(a)$ $(a\in A')$. If $V=\langle a|\beta(a)\in A\setminus A'\rangle$ is a non-empty set, then β is a mapping (2). If V is the empty set, then β is a mapping (1).

Consequently, we can give the endomorphisms of **A**, if we know the endomorphisms of **A'**. In Theorem 3 we give all of the endomorphisms of **A'**, if **A'** is a group-type quasi-automaton.

A non-empty subset B of the state set A of a quasi-automaton $\mathbf{A}=(A, F, \delta)$ is called a *generating system* of **A** if for each state $a(\in A)$ there exists a state $b(\in B)$ and a $f(\in F)$ such that $\delta(b,f)=a$. A generating system B of **A** is *minimal* if none of the proper subset of B is a generating system of **A**. A quasi-automaton is said to be *(finitely) generated* if it has a (finite) generating system. (We note that a quasi-automaton is called *cyclic* if it has an one-element generating system.)

Let the characteristic semigroup $\bar{F}$ of a quasi-automaton $\mathbf{A}=(A, F, \delta)$ be again a monoid and let $\bar{e}$ $(e\in F)$ be the identity element of $\bar{F}$. It can easily be proved that the quasi-automaton **A** has a generating system if and only if

$$\forall_{a\in A}\ a[\delta(a, e) = a]. \tag{4}$$

In the following lemma the theorem of Yu. I. Sorkin [7] concerning finitely generated automata are generalised on generated quasi-automata.

Lemma 1. *If G_1 and G_2 are two minimal generating systems of a generated quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *then* $|G_1|=|G_2|$. [1]

[1] $|A|$ is the cardinal number of the set A.

Proof. Let G_1 and G_2 be two minimal generating systems of $\mathbf{A}$. For every $a_2(\in G_2)$, there exist $a_1(\in G_1)$ and $f(\in F)$ such that $\delta(a_1,f)=a_2$ holds. It can easily be seen that the set

$$G_{12} = \langle a | a \in G_1 \quad \text{and} \quad \underset{f\in F}{\exists} f[\delta(a,f)\in G_2]\rangle$$

is also a generating system of $\mathbf{A}$. Since $G_{12}\subseteqq G_1$ and G_1 is a minimal generating system of $\mathbf{A}$, thus $G_{12}=G_1$. Assume that $\delta(a_1,f)$, $\delta(a_1,h)\in G_2$ $(a_1\in G_1; f, h\in F)$. There exists a $k(\in F)$ such that $\delta(a_1,fk)=\delta(\delta(a_1,f),k)\in G_1$. Since G_1 is a minimal generating system of $\mathbf{A}$, thus $\delta(a_1,fk)=a_1$, that is,

$$\delta(\delta(a_1,f), kh) = \delta(\delta(a_1,fk), h) = \delta(a_1, h)\in G_2.$$

Since G_2 is also a minimal generating system of $\mathbf{A}$, we get that $\delta(a_1,h)= =\delta(\delta(a_1,f), kh)=\delta(a_1,f)$. Furthermore, if $\delta(a_1,f)=\delta(a_1', g)\in G_2(a_1'\in G_1, g\in F)$, then $a_1=\delta(a_1,fk)=\delta(a_1', gk)$, that is, $a_1=a_1'$. Consequently, the mapping $\varphi\colon G_1\to G_2$, for which

$$\varphi(a_1) = a_2 \Leftrightarrow \underset{f\in F}{\exists} f[\delta(a_1,f) = a_2]$$

holds, is an one-to-one mapping of G_1 onto G_2.

We define the following relation ϱ on A:

$$a\varrho b(a, b\in A) \Leftrightarrow \underset{c\in A; f,g\in F}{\exists} (c,f,g)[\delta(c,f) = a, \ \delta(c,g) = b]. \tag{5}$$

If the quasi-automaton $\mathbf{A}=(A, F, \delta)$ is generated then ϱ is a reflexive and symmetric relation. If the quasi-automaton $\mathbf{A}$ is generated and the characteristic semigroup $\bar{F}$ of $\mathbf{A}$ is a group then ϱ is an equivalence relation.

A non-empty subset E of the state set A of a quasi-automaton $\mathbf{A}=(A, F, \delta)$ is called a *strongly connected subset* of A, if for every a, b $(\in E)$ there exists an $f(\in F)$ such that $\delta(a,f)=b$. A partition C of A is called *strongly connected,* if $C(a)$ is a strongly connected subset of A for every $a(\in A)$ ($C(a)$ denotes the class of C containing the element a).

Lemma 2. *If the characteristic semigroup of a generated quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is a group, then* C_ϱ *is a strongly connected partition of* A, *where* C_ϱ *is the partition on* A *induced by* ϱ.

Proof. Let $a, b\in C_\varrho(c)$ $(c\in A)$, then there exist $f\in F$ and $g\in F$ such that $\delta(c,f)=a$ and $\delta(c, g)=b$. Since $\bar{F}$ is a group, there exists an $h(\in F)$ such that $\bar{f}\bar{h}=\bar{g}$, therefore,

$$\delta(a, h) = \delta(\delta(c,f), h) = \delta(c,fh) = \delta(c, g) = b,$$

that is, $C_\varrho(c)$ is a strongly connected subset of A.

Assume that the conditions of this Lemma are satisfied. It can easily be seen that $C_\varrho(a)=\langle\delta(a,f)|f\in F\rangle$ holds for every $a(\in A)$. Thus $\mathbf{C}_\varrho(a)=(C_\varrho(a), F, \delta_a)$ is a strongly connected sub-quasi-automaton of $\mathbf{A}$ for every $a(\in A)$ (cf. CH. A. TRAUTH [6]).

Lemma 3. *If the characteristic semigroup of a generated quasi-automaton* $\mathbf{A}= =(A, F, \delta)$ *is a group, then* $\mathbf{A}$ *has a minimal generating system.*

Proof. By Lemma 2, C_ϱ is a strongly connected partition of A. Let $G(\subseteq A)$ such that $A=\bigcup_{a\in G} C_\varrho(a)$ and if $a\neq b$ $(\in G)$ then $C_\varrho(a)\neq C_\varrho(b)$. We can easily prove that G is a minimal generating system of **A**.

We note that if G is a minimal generating system of **A** then $A=\bigcup_{a\in G} C_\varrho(a)$ and if $a\neq b(\in G)$ then $C_\varrho(a)\neq C_\varrho(b)$.

It is possible that C_ϱ is a strongly connected partition of A if the characteristic semigroup of **A** is not a group. Take the following example:

A	1	2	3	4	5
x	2	1	1	5	4
y	3	2	2	4	5

$C_\varrho(1)=\langle 1, 2, 3\rangle$ and $C_\varrho(4)=\langle 4, 5\rangle$ are strongly connected subsets of A, $C_\varrho(1)\cup C_\varrho(4)=A$ and $C_\varrho(1)\cap C_\varrho(4)=\emptyset$. But $\overline{F(X)}$ is not a group. ($F(X)$ denotes the free semigroup with out identity element generated by $X=\langle x, y\rangle$.) Note that $G=\langle 1, 4\rangle$ is a minimal generating system of **A**.

Theorem 2. *If a quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is finitely generated and* C_ϱ *is a strongly connected partition of* A *then*

$$o(E(A)) \geqq \prod_{i=1}^{k} o(E(C_\varrho(a_i))), \tag{6}$$

where $G=\langle a_1, \dots, a_k\rangle$ *is a minimal generating system of* **A**.

Proof. $E(A)$ and $E(C_\varrho(a_i))$ denote the endomorphism semigroups of the quasi-automaton $\mathbf{A}=(A, F, \delta)$ and $\mathbf{C}_\varrho(a_i)=(C_\varrho(a_i), F, \delta_{a_i})$ $(a_i\in G)$, respectively. Denote by $\alpha=\bigcup_{a_i\in G}\alpha_{a_i}$ the following mapping of A into itself:

$$\alpha(a) = \alpha_{a_i}(a), \quad \text{if} \quad a\in C_\varrho(a_i) \tag{7}$$

where $\alpha_{a_i}\in E(C_\varrho(a_i))$. It can easily be proved that $\alpha\in E(A)$. Furthermore, if

$$\alpha = \bigcup_{a_i\in G}\alpha_{a_i}(\alpha_{a_i}\in E(C_\varrho(a_i))) \quad \text{and} \quad \beta = \bigcup_{a_i\in G}\beta_{a_i}(\beta_{a_i}\in E(C_\varrho(a_i)))$$

such that $\alpha=\beta$, then $\alpha_{a_i}=\beta_{a_i}$ for every a_i $(\in G)$.

Lemma 4. *If a group-type quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is generated, then the sub-quasi-automaton* $\mathbf{C}_\varrho(a)$ *is quasi-perfect and the characteristic group of* $\mathbf{C}_\varrho(a)$ *is equal to the characteristic group of* **A** *for every* $a\,(\in A)$. *Moreover* $\mathbf{C}_\varrho(a)\cong\mathbf{C}_\varrho(b)$ *for every pair* $a, b\,(\in A)$.

Proof. Let $a\,(\in A)$ and $f, g\,(\in F)$ such that

$$\forall_{h\in F}\, h[\delta(a, hf) = \delta(\delta(a, h), f) = \delta(\delta(a, h), g) = \delta(a, hg)].$$

Since **A** is state-independent, thus $\bar{h}\bar{f}=\bar{h}\bar{g}$. Let $\bar{h}=\bar{e}$, where $\bar{e}$ is the identity element of the characteristic group of **A**, then $\bar{f}=\bar{g}$. Consequently, the characteristic group of $\mathbf{C}_\varrho(a)$ is equal to the characteristic group of **A**. A sub-quasi-automaton of a state-independent quasi-automaton is also state-independent, therefore, by Lemma 2, $\mathbf{C}_\varrho(a)$ is quasi-perfect. Let $a, b\ (\in A)$ be arbitrary states. It is clear that the mapping $\delta(a,f)\to\delta(b,f)\ (f\in F)$ is an isomorphic mapping of $\mathbf{C}_\varrho(a)$ onto $\mathbf{C}_\varrho(b)$.

Corollary 1. *If a group-type A-finite quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is generated, then* $O(\bar{F})\,|\,|A|$. [2]

Proof. From Lemma 4 and Theorem 7 of CH. A. TRAUTH [6] we get that $|C_\varrho(a)|=O(\bar{F})$ for every $a\ (\in A)$. $|C_\varrho(a)|=|C_\varrho(b)|$ follows also from Lemma 4 for every pair $a, b\ (\in A)$. Thus $O(\bar{F})=|C_\varrho(a)|\,|\,|A|$.

Corollary 2. *If an A-finite group-type quasi-automaton* $\mathbf{A}=(A,F, \delta)$ *is generated and* $|A|$ *is a prime number, then either* $\bar{F}$ *has only one element or* $\mathbf{A}$ *is quasi-perfect.*

Proof. By Corollary 1, if $|A|$ is a prime number, then either $O(\bar{F})=1$ or $|C_\varrho(a)|= =O(\bar{F})=|A|\ (a\in A)$. If $|A|=|C_\varrho(a)|\ (a\in A)$, then $\mathbf{A}$ is a cyclic quasi-automaton. Cyclic group-type quasi-automaton is quasi-perfect (CH. A. TRAUTH [6]).

Theorem 3. *If a group-type quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is generated, then there exist a subsemigroup* T *and two subgroups* H *and* P *of the endomorphism semigroup* $E(A)$ *of* $\mathbf{A}$ *such that*

$$E(A) = TH, \quad G(A) = PH = HP, \quad T\cap H = \{\iota\}, \quad P \subseteqq T$$

hold, where ι *is the identity element of* $E(A)$. [3]

Proof. Let the group-type quasi-automaton $\mathbf{A}=(A, F, \delta)$ be generated. By Lemma 3, there exists a minimal generating system G of $\mathbf{A}$. Let H denote the set of all endomorphisms (7). By Lemma 4 and Theorem 4 of I. BABCSÁNYI [1], the endomorphisms (7) are automorphisms of $\mathbf{A}$. H is a subgroup of the automorphism group $G(A)$ of $\mathbf{A}$ under the usual multiplication of mappings.

Let π be an arbitrary mapping of G into itself. We define the mapping φ_π: $A\to A$ by

$$\varphi_\pi(\delta(c,f)) = \delta(\pi(c),f) \quad (c\in G,\ f\in F). \tag{8}$$

We show that φ_π is an endomorphism of $\mathbf{A}$. Let a be an arbitrary state of $\mathbf{A}$ and let $c\in G$ and $f, g\in F$ such that $a=\delta(c,f)=\delta(c, g)$. Since $\mathbf{A}$ is state-independent, thus $\delta(\pi(c),f)=\delta(\pi(c), g)$, that is, φ_π is well-defined. If $a=\delta(c, h)\ (c\in G,\ h\in F)$ and $f\in F$ then

$$\varphi_\pi(\delta(a,f)) = \varphi_\pi(\delta(\delta(c, h),f)) = \varphi_\pi(\delta(c, hf)) = \delta(\pi(c), hf) =$$
$$= \delta(\delta(\pi(c), h),f) = \delta(\varphi_\pi(\delta(c, h)),f) = \delta(\varphi_\pi(a),f),$$

that is, $\varphi_\pi\in E(A)$. Let T denote the set of all mappings (8). T is a subsemigroup of $E(A)$. Namely, if $\varphi_\pi, \varphi_{\pi'}\in T$ and $a=\delta(c, h)$, then

$$\varphi_\pi\varphi_{\pi'}(a) = \varphi_\pi\varphi_{\pi'}(\delta(c, h)) = \varphi_\pi(\delta(\pi'(c), h)) =$$
$$= \delta(\pi\pi'(c), h) = \varphi_{\pi\pi'}(\delta(c, h)) = \varphi_{\pi\pi'}(a)$$

that is, $\varphi_\pi\varphi_{\pi'}=\varphi_{\pi\pi'}\in T$.

[2] If n and k are natural numbers then $k|n$ means that n can be divided by k.

[3] $TH=\langle\varphi\alpha|\varphi\in T, \alpha\in H\rangle$.

If π is a permutation of G and $\varphi_\pi(a)=\varphi_\pi(b)$ $(a, b\in A)$ then there exist $c, d\in G$ and $h, k\in F$ such that $\delta(c, h)=a$ and $\delta(d, k)=b$, therefore

$$\delta(\pi(c), h) = \varphi_\pi(\delta(c, h)) = \varphi_\pi(a) = \varphi_\pi(b) = \varphi_\pi(\delta(d, k)) = \delta(\pi(d), k).$$

Let $k'\in\bar{k}^{-1}$, then $\delta(\pi(c), hk')=\delta(\pi(d), kk')=\pi(d)$. Since $\pi(c), \pi(d)\in G$ and G is a minimal generating system of **A**, thus $\pi(c)=\pi(d)$, that is, $c=d$ and $\bar{h}=\bar{k}$. Therefore $a=b$, that is, φ_π is an one-to-one mapping. Now let a be an arbitrary state of **A**, then there exist $d\in G$ and $f\in F$ such that $\delta(d,f)=a$. Furthermore, there exists a $c\in G$ such that $\pi(c)=d$, because π is a permutation of G. Thus

$$\varphi_\pi(\delta(c,f)) = \delta(\pi(c),f) = \delta(d,f) = a,$$

that is, φ_π is onto. Consequently, if π is a permutation of G, then $\varphi_\pi\in G(A)$. Denote by P the set of this automorphisms φ_π. It is obvious, that P is a subgroup of $G(A)$. It can easily be seen that $T\cap H=\{\iota\}$, $P\subseteq T$, $TH\subseteq E(A)$ and PH, $HP\subseteq G(A)$ hold.

Now, we prove that $E(A)\subseteq TH$. Let $\beta\in E(A)$ and $a\in A$. There exist states $c, d(\in G)$ such that $a\in C_\varrho(c)$ and $\beta(a)\in C_\varrho(d)$. Take the mapping π of G into itself such that $\pi(c)=d$. We show that π is well-defined. Let $b\in C_\varrho(c)$ and suppose that $\beta(b)\in C_\varrho(d')$ $(d'\in G)$. There exist $h, h'\in F$ for which $\delta(c, h)=a$ and $\delta(c, h')=b$ hold. Thus $\beta(a)= =\beta(\delta(c, h))=\delta(\beta(c), h)$ and $\beta(b)=\beta(\delta(c, h'))=\delta(\beta(c), h')$, that is, $C_\varrho(d)=C_\varrho(d')$, thus $d=d'$. We define φ_π as in (8). If $\beta(a)=\delta(d, k)$ $(k\in F)$, then let α_c be an automorphism of $\mathbf{C}_\varrho(c)$ such that $\alpha_c(a)=\alpha_c(\delta(c, h))=\delta(c, k)$. (Since $\mathbf{C}_\varrho(c)$ is quasi-perfect, therefore the automorphism group of $\mathbf{C}_\varrho(c)$ is transitive, thus α_c exists (CH. A. TRAUTH [6]).) We prove that α_c depends only on β. Let $b\in C_\varrho(c)$ and $\delta(c, h')=b$ $(h'\in F)$, furthermore $\bar{h}'=\bar{h}l$ $(l\in F)$. Then

$$b = \delta(c, h') = \delta(c, hl) = \delta(\delta(c, h), l) = \delta(a, l).$$

Thus, if $\beta(b)=\delta(d, k')$ $(k'\in F)$, then

$$\delta(d, k') = \beta(b) = \beta(\delta(a, l)) = \delta(\beta(a), l) = \delta(\delta(d, k), l) = \delta(d, kl).$$

Since **A** is state-independent, thus $\bar{k}'=\bar{k}l$, that is,

$$\alpha_c(b) = \alpha_c(\delta(c, h')) = \alpha_c(\delta(c, hl)) = \alpha_c(\delta(\delta(c, h), l)) =$$

$$= \delta(\alpha_c(\delta(c, h)), l) = \delta(\delta(c, k), l) = \delta(c, kl) = \delta(c, k').$$

Thus

$$\varphi_\pi\alpha_c(a) = \varphi_\pi\alpha_c(\delta(c, h)) = \varphi_\pi(\delta(c, k)) = \delta(d, k) = \beta(a).$$

Take this α_c for every c $(\in G)$ and let $\alpha= \bigcup_{c\in G} \alpha_c$. It is clear that $\beta=\varphi_\pi\alpha$, that is, $\beta\in TH$, since $\varphi_\pi\in T$ and $\alpha\in H$. Therefore $E(A)\subseteq TH$, thus $E(A)=TH$.

Suppose that $\beta=\varphi_\pi\alpha\in G(A)$. Since $\alpha\in G(A)$, therefore $\varphi_\pi=\beta\alpha^{-1}\in G(A)$. If $a\in G$ then $\varphi_\pi(a)=\pi(a)$, that is, π is a permutation of G, thus $\varphi_\pi\in P$. We get that $G(A)= =PH$. Finally, we shall show that $PH=HP$. Let $\varphi_\pi(\in P)$ and $\alpha(\in H)$ be arbitrary endomorphisms. Furthermore, let $a=\delta(c, h)$ $(c\in G, h\in F)$ be an arbitrary state of **A** and let $\alpha(a)=\delta(c, k)$ $(k\in F)$. Take the automorphism $\alpha_{\pi(c)}$ of $\mathbf{C}_\varrho(\pi(c))$ such that

$$\alpha_{\pi(c)}(\delta(\pi(c), h)) = \delta(\pi(c), k).$$

It can easily be seen that $\alpha_{\pi(c)}$ depends only on α. Since π is a permutation of G, therefore the mapping $\alpha_{\pi(c)} \to C_\varrho(\pi(c))$ is one-to-one and $\alpha' = \bigcup_{c\in G} \alpha_{\pi(c)} \in H$. Thus

$$\alpha' \varphi_\pi(a) = \alpha' \varphi_\pi(\delta(c, h)) = \alpha'(\delta(\pi(c), h)) = \delta(\pi(c), k) =$$
$$= \varphi_\pi(\delta(c, k)) = \varphi_\pi \alpha(\delta(c, h)) = \varphi_\pi \alpha(a),$$

that is, $\alpha' \varphi_\pi = \varphi_\pi \alpha$. Thus $G(A) = PH \subseteq HP$, therefore $PH = HP$.

Corollary 3. *If a group-type quasi-automaton* $\mathbf{A} = (A, F, \delta)$ *is generated, then*

$$\varphi\alpha = \psi\beta \Rightarrow \varphi = \psi \quad \text{and} \quad \alpha = \beta,$$

where $\varphi, \psi \in T$ *and* $\alpha, \beta \in H$.

Proof. Let $\varphi, \psi \in T$ and $\alpha, \beta \in H$ such that $\varphi\alpha = \psi\beta$, then $\varphi\alpha\beta^{-1} = \psi$. Let G be a minimal generating system of $\mathbf{A}$ and $c \in G$, then $\varphi(\alpha\beta^{-1}(c)) = \psi(c)$. Since $\alpha\beta^{-1}(c) \in C_\varrho(c)$, there exists $f \in F$ such that $\alpha\beta^{-1}(c) = \delta(c, f)$, that is,

$$\psi(c) = \varphi(\alpha\beta^{-1}(c)) = \varphi(\delta(c, f)) = \delta(\varphi(c), f).$$

Since $\varphi(c), \psi(c) \in G$, thus $\varphi(c) = \psi(c)$ $(c \in G)$ and $\bar{f} = \bar{e}$, where $\bar{e}$ is the identity element of $\bar{F}$. We get that $\varphi = \psi$ and $\alpha\beta^{-1}(c) = \delta(c, f) = \delta(c, e) = c$, that is $\alpha = \beta$.

Corollary 4. *Let a group-type quasi-automaton* $\mathbf{A} = (A, F, \delta)$ *be generated. If* $O(\bar{F}) > 1$, *then* P *is isomorphic to a subgroup of the automorphism group of* H. *If* $O(\bar{F}) = 1$ *then* $H = \{\iota\}$.

Proof. Let $\varphi \in P$. We define the following mapping ω_φ of H into itself:

$$\omega_\varphi(\alpha) = \alpha' \Leftrightarrow \varphi\alpha = \alpha'\varphi. \tag{9}$$

ω_φ is one-to-one and onto. Let $\alpha_1, \alpha_2 \in H$ then

$$(\alpha_1\alpha_2)'\varphi = \varphi(\alpha_1\alpha_2) = (\varphi\alpha_1)\alpha_2 = (\alpha_1'\varphi)\alpha_2 = \alpha_1'(\varphi\alpha_2) = \alpha_1'(\alpha_2'\varphi) = (\alpha_1'\alpha_2')\varphi,$$

that is, $(\alpha_1\alpha_2)' = \alpha_1'\alpha_2'$, thus ω_φ is an automorphism of H. Suppose that $\omega_\varphi = \omega_\psi$ $(\varphi, \psi \in P)$, that is,

$$\varphi\alpha = \alpha'\varphi \Leftrightarrow \psi\alpha = \alpha'\psi.$$

Let $\varphi\alpha = \alpha'\varphi$ and $\psi\alpha = \alpha'\psi$, then $\alpha' = \psi\alpha\psi^{-1}$ thus $\varphi\alpha = \psi\alpha\psi^{-1}\varphi$, that is $\psi^{-1}\varphi\alpha = \alpha\psi^{-1}\varphi$ $(\alpha \in H)$. Let $O(\bar{F}) > 1$. Let $\alpha \in H$ such that $\alpha(a) = \delta(a, f)$ and $\alpha(\psi^{-1}\varphi(a)) = \delta(\psi^{-1}\varphi(a), g)$ $(a \in A)$, where $\bar{f} \neq \bar{g}$ $(\in \bar{F})$. α exists if $C_\varrho(a) \neq C_\varrho(\psi^{-1}\varphi(a))$. Then

$$\delta(\psi^{-1}\varphi(a), f) = \psi^{-1}\varphi(\delta(a, f)) = \psi^{-1}\varphi\alpha(a) = \alpha\psi^{-1}\varphi(a) = \delta(\psi^{-1}\varphi(a), g),$$

that is $\bar{f} = \bar{g}$, since $\mathbf{A}$ is state-independent. It is a contradiction. Thus $C_\varrho(a) = C_\varrho(\psi^{-1}\varphi(a))$, that is $\psi^{-1}\varphi = \iota$ and $\varphi = \psi$. Therefore the mapping $\varphi \to \omega_\varphi$ is one-to-one. We prove that this mapping is isomorphism. Let $\varphi, \psi \in P$ and $\alpha \in H$ then $\omega_\varphi\omega_\psi(\alpha) = \omega_\varphi(\alpha_1) = \alpha_2$, where $\psi\alpha = \alpha_1\psi$ and $\varphi\alpha_1 = \alpha_2\varphi$. Then

$$(\varphi\psi)\alpha = \varphi(\psi\alpha) = \varphi(\alpha_1\psi) = (\varphi\alpha_1)\psi = (\alpha_2\varphi)\psi = \alpha_2(\varphi\psi),$$

that is $\omega_{\varphi\psi}(\alpha) = \alpha_2$, thus $\omega_\varphi\omega_\psi = \omega_{\varphi\psi}$.

If $O(\bar{F}) = 1$, then $|C_\varrho(c)| = 1$ $(c \in G)$, that is $H = \{\iota\}$. (In this case $G = A$, $E(A) = T$ and $G(A) = P$.)

Let G and G' be two minimal generating systems of a group-type generated quasi-automaton $\mathbf{A}=(A, F, \delta)$. Let T, P and T', P' be sets which are defined in Theorem 3.

Corollary 5. *$T'=\alpha T\alpha^{-1}$, $P'=\alpha P\alpha^{-1}$ where $\alpha\in H$ and $\alpha(G)=G'$.*[4] *Furthermore $T'\cong T$, $P'\cong P$.*

Proof. Let π be a mapping of G into itself and let π' be a mapping of G' into tself such that

$$\alpha(\pi(c)) = \pi'(\alpha(c)) \quad (c\in G) \tag{10}$$

holds, where $\alpha\in H$ and $\alpha(G)=G'$. The mapping $\pi\to\pi'$ is one-to-one, thus the mapping $\varkappa\colon \varphi_\pi\to\varphi_{\pi'}$ is one-to-one also. Let $a\in A$, then

$$\alpha\varphi_\pi(a) = \alpha\varphi_\pi(\delta(c, h)) = \alpha(\delta(\pi(c), h)) = \delta(\alpha(\pi(c)), h) =$$

$$= \delta(\pi'(\alpha(c)), h) = \varphi_{\pi'}(\delta(\alpha(c), h)) = \varphi_{\pi'}\alpha(\delta(c, h)) = \varphi_{\pi'}\alpha(a)$$

$(c\in G, h\in F)$, that is, $\alpha\varphi_\pi=\varphi_{\pi'}\alpha$ thus $\varphi_{\pi'}=\alpha\varphi_\pi\alpha^{-1}$. It can easily be seen, that the mapping $\varkappa$ is onto, that is $T'=\alpha T\alpha^{-1}$.

$$\varkappa(\varphi_{\pi_1}\varphi_{\pi_2}) = \varkappa(\varphi_{\pi_1\pi_2}) = \alpha\varphi_{\pi_1}\varphi_{\pi_2}\alpha^{-1} = \alpha\varphi_{\pi_1}\alpha^{-1}\alpha\varphi_{\pi_2}\alpha^{-1} = \varkappa(\varphi_{\pi_1})\cdot\varkappa(\varphi_{\pi_2})$$

$(\varphi_{\pi_1}, \varphi_{\pi_2}\in T)$ therefore $T\cong T'$. It is evident that $P'=\alpha P\alpha^{-1}$ and $P\cong P'$.

We note, if G is a minimal generating system of a group-type generated quasi-automaton $\mathbf{A}$ and $\alpha\in H$, then $\alpha(G)$ is also a minimal generating system of $\mathbf{A}$. If $\alpha\neq\beta\in H$ then $\alpha(G)\neq\beta(G)$. Furthermore, if G and G' are two minimal generating systems of $\mathbf{A}$, then there exists $\alpha\in H$ such that $\alpha(G)=G'$ holds. Therefore, the cardinality of the set of all minimal generating systems of $\mathbf{A}$ is equal to $O(H)$.

Theorem 4. *If an A-finite group-type quasi-automaton $\mathbf{A}=(A, F, \delta)$ is generated, $|A|=n$ and $|G|=k$ then*

$$O(G(A)) = k!\cdot\left(\frac{n}{k}\right)^k \quad \textit{and} \quad O(E(A)) = n^k,$$

where G is a minimal generating system of $\mathbf{A}$.

Proof. If $|A|=n$ and $|G|=k$, where G is a minimal generating system of $\mathbf{A}$, then $O(\bar{F})=\frac{n}{k}$. By Lemmas 2 and 4, $|C_\varrho(c)|=\frac{n}{k}$ $(c\in G)$. Since $\mathbf{C}_\varrho(c)$ is quasi-perfect, therefore $O(E(C_\varrho(c)))=|C_\varrho(c)|=\frac{n}{k}$. The number of sets $C_\varrho(c)$ $(c\in G)$ is equal to k, thus $O(H)=\left(\frac{n}{k}\right)^k$. By Theorem 3, $O(P)$ is equal to the number of the permutations of G, that is $O(P)=k!$. By Theorem 3 and Corollary 3, $O(G(A))=O(P)\cdot O(H)=k!\cdot\left(\frac{n}{k}\right)^k$ and $O(E(A))=O(T)\cdot O(H)=k^k\cdot\left(\frac{n}{k}\right)^k=n^k$.

---

[4] $\alpha(G)=\langle\alpha(c)\mid c\in G\rangle$.

Example:

A	1	2	3	4
e	1	2	3	4
f	2	1	4	3

($F=\{e,f\}$ is the Abelian group of degree two, where e is the identity element of F.) Let $abcd$ $(a,b,c,d=1,2,3,4)$ denote the mapping $\varphi\colon A\to A$ such that $\varphi(1)=a$, $\varphi(2)=b$, $\varphi(3)=c$ and $\varphi(4)=d$. It is clear that

$H=\{1234;\ 1243;\ 2134;\ 2143\}$

$T=\{1234;\ 3412;\ 1212;\ 3434\}$

$P=\{1234;\ 3412\}$.

In this example $n=4$ and $k=2$, that is $O(G(A))=2!\cdot 2^2=8$ and $O(E(A))=4^2=16$. But $HT\neq TH=E(A)$, since $|HT|=12$.

We can more easily determine the endomorphisms of a group-type quasi-automaton $\mathbf{A}=(A,F,\delta)$ by means of the following:

Let G be a minimal generating system of $\mathbf{A}'$ (see page 1). Let

$$B_c=\langle b\,|\,b\in A \text{ and } \underset{f\in F}{\exists} f[\delta(b,f)=c]\rangle$$

where $c\in G$. It is evident that this is a partition of A. Furthermore, $C_\varrho(c)\subseteqq B_c$ $(c\in G)$.

Lemma 5. *If α is an arbitrary endomorphism of the group-type quasi-automaton $\mathbf{A}=(A,F,\delta)$, then for every $c(\in G)$, there exists a $d(\in G)$ such that $\alpha(B_c)\subseteqq B_d$.*

Proof. Let $\alpha\in E(A)$ and $a\in B_c$ $(c\in G)$, then there exists an $f(\in F)$ such that $\delta(a,f)=$ $=c$, thus $\delta(\alpha(a),f)=\alpha(c)$. It is obvious, that there exists a $d(\in G)$ such that $\alpha(c)\in B_d$. If $h\in F$ such that $\delta(\alpha(c),h)=d$, then $\delta(\alpha(a),fh)=\delta(\alpha(c),h)=d$, that is, $\alpha(a)\in B_d$.

Эндоморфизмы группа — типных квази-автоматов

В этой работе рассматриваем эндоморфизмы группа-типных квази-автоматов (см. Ch. A. Trauth [6]) при помощи системы образующих квази-автоматов.

Пусть $\mathbf{A}=(A,F,\delta)$ произвольный квази-автомат и $A'=\langle\delta(a,f)|a\in A,f\in F\rangle$. В теореме 1 получаем эндоморфизмы квази-автомата $\mathbf{A}$, если знаем эндоморфизмы A-подквази-автомата $\mathbf{A}'$ квази-автомата $\mathbf{A}$. ($\mathbf{A}'$ можно называться *ядром* квази-автомата $\mathbf{A}$.) Если характеристическая полугруппа $\bar{F}=F/\delta_A$ обладает единицей, тогда $\mathbf{A}'$ является порожденным. Теорема 3 доставляем главный результат этой работы, где даваем эндоморфизмы (автоморфизмы) порожденных группа-типных квази-автоматов и структуру полугруппы эндоморфизмов (группы автоморфизмы): Обозначаем множество отображений (7). H и множество отображений (8) T. H является подгруппой группы автоморфизмов $G(A)$. T является подполугруппой полугруппы эндоморфизмов $E(A)$, $E(A)=TH$ и $T\cap H=\{\iota\}$, где ι есть единица полугруппы $E(A)$. Можно найти такую подгруппу P полугруппы T, что $PH=HP=G(A)$ $(P\cap H=\{\iota\})$. В следствии 4 показаем, что если $O(\bar{F})>1$, тогда P изоморфно вкладывается в группу автоморфизмов группы H, и если $O(\bar{F})=1$, тогда $H=\{\iota\}$. В теореме 4 даваем число эндоморфизмов и автоморфизмов A-конечных порожденных группа-типных квази-автоматов: $O(E(A))=n^k$ и $O(G(A))=$ $=k!\cdot\binom{n}{k}^k$, где $|A|=n$ и $|G|=k$ (G неприводимая система образующих в квази-автомате A).

В следствии 1 показываем, что $O(\bar{F})\,|\,|A|$. Лемма 1 является обобшением теоремы Ю. И. Соркина [7]: Все неприводимые системы образующих квази-автомата являются равномошными. Доказаем, что всякий порожденный группа — типный квази-автомат есть прямая сумма изоморфных полусоверщенных квази-автоматов (лемма 4).

ENTZBRUDER VOCATIONAL SECONDARY SCHOOL
H—9700 SZOMBATHELY, HUNGARY

References

[1] BABCSÁNYI, I., A félperfekt kváziautomatákról (On quasi-perfect quasi-automata), *Mat. Lapok*, v. 21, 1970, pp. 95—102.
[2] BABCSÁNYI, I., Ciklikus állapot-független kváziautomaták (Cyclic state-independent quasi-automata), *Mat. Lapok*, v. 22, 1971, pp. 289—301.
[3] FLECK, A. C., Preservation of structure by certain classes of functions on automata and related group theoretic properties, Computer Laboratory, Michigan State University, 1961, (preprint).
[4] GÉCSEG, F. & I. PEÁK, Az automaták algebrai elmélete (Algebraic theory of automata), *Mat. Lapok*, v. 17, 1966, pp. 77—134.
[5] GÉCSEG, F. & I. PEÁK, *Algebraic theory of automata*, Budapest, 1972.
[6] TRAUTH, CH. A., Group-type automata, *J. Assoc. Comput. Mach.*, v. 13, 1966, pp. 170—175.
[7] SORKIN, I. YU. (Ю. U. Соркин), Теория определяющих соотношений для автоматов, *Problemy Kibernet.*, v. 9, 1963, pp. 45—69.

(Received June 15, 1974)