# On graphs satisfying some conditions for cycles, I.

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To the memory of my friend Professor Andor Kertész

## Introduction

The aim of the present paper is to give a structural description of the finite directed graphs satisfying the conditions that

to any edge $e$ the number of cycles containing $e$ is 1 or 2, and

there exists a vertex contained in every cycle of the graph.

It is obvious that a graph fulfilling these requirements can have at most one cut vertex.

We rely upon some results of the earlier paper [1]. In §§ 2—3 we give some constructions and prove that they produce the graphs that possess the properties mentioned above and having no cut vertex. The description is extended in § 4 to graphs in which a cut vertex occurs.

## § 1.

By a graph, we mean always a finite directed graph with at least two vertices. We suppose that it is connected and contains neither loops nor parallel edges with the same orientation.

It is assumed that §§ 2—3 of the preceding paper [1] are known to the reader. The terminology introduced in § 2 of [1] is mostly further applied (but the notations $\mathfrak{A}_k(G)$ and $\mathfrak{A}(C)$ do not occur in this paper). We say that e.g. $Z(A)\geqq 1$ is *universally*[1] satisfied in $G$ if it is true for every vertex $A$ of the graph $G$. In accordance with [1], we denote by $C_1$ the class of connected directed finite graphs in which $Z(A)\leqq 2$ and $Z(e)\geqq 1$ are universally valid. Construction I, Theorems 1 and 2 of [1] will be referred to as Construction I*, Theorems 1* and 2*, respectively.

The sum of the indegree and outdegree of a vertex $A$ is called the *total degree* of $A$.

A vertex $A$ of a graph $G$ is called *pancyclic* if $A$ is contained in each cycle of $G$.

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[1] In [1] the word "identically" was applied for expressing the universal quantification.

Let us consider three conditions (imposed upon a graph $G$):

($\alpha$) $1 \leqq Z(e) \leqq 2$ is universally satisfied in $G$,
($\beta$) $G$ has a pancyclic vertex,
($\gamma$) $G$ has no cut vertex.

We define the class $C_5$ as the collection of finite directed graphs fulfilling $(\alpha) \& (\beta) \& (\gamma)$ and we denote by $C_6$ the set of finite directed graphs in which $(\alpha) \& (\beta)$ is satisfied.[2] It is clear that $C_5 \subseteqq C_6$. The condition ($\alpha$) implies the universal validity of $Z(A) > 0$ in $G$.

The vertices of degree (1, 1) are called *simple* vertices. Let $c$ be a path of positive length in the graph $G$, denote the vertices of $c$ by $A_0, A_1, \dots, A_n$ (as they follow in $c$) ($n \geqq 1$); $c$ is called an *arc* (or more precisely, an $(A_0, A_n)$-arc) if its inner vertices $A_1, A_2, \dots, A_{n-1}$ are simple vertices (in $G$).

## § 2.

We describe four constructions. In any construction, the arcs are supposed to have no edge and no inner vertex in common. The lengths of the arcs are arbitrary positive integers.

CONSTRUCTION I. Let $k (\geqq 4)$ be an even number. Take $k+1$ vertices $A, B_1, B_2, \dots, B_k$ and the following $2k$ arcs:

an $(A, B_i)$-arc for each odd number $i$ $(1 \leqq i \leqq k-1)$,
a $(B_i, A)$-arc for each even number $i$ $(2 \leqq i \leqq k)$,
a $(B_i, B_{i-1})$-arc for each odd number $i$ $(3 \leqq i \leqq k-1)$,
a $(B_i, B_{i+1})$-arc for each odd number $i$ $(1 \leqq i \leqq k-1)$,
a $(B_1, B_n)$-arc.

(It is clear that, in a graph $G$ resulted by Construction I, $A, B_1, B_2, \dots, B_k$ and the inner vertices of the arcs are the vertices of $G$, and the edges of the arcs are the edges of $G$.)

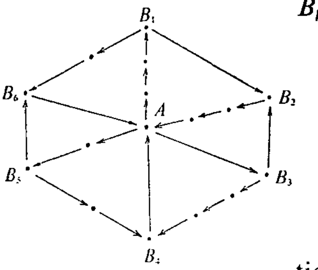

*Fig. 1.* A I-constructible graph ($k=6$)

CONSTRUCTION II/a. Let $k (\geqq 2)$ be an integer. Take the $k+1$ vertices $A, B_1, B_2, \dots, B_k$ and the following $2k+1$ arcs:

an $(A, B_1)$-arc,
a $(B_1, A)$-arc,
an $(A, B_i)$-arc for each odd number $i$ $(3 \leqq i \leqq k-1)$,
a $(B_i, A)$-arc for each even number $i$ $(2 \leqq i \leqq k-1)$,
a $(B_i, B_{i-1})$-arc for each odd number $i$ $(3 \leqq i \leqq k)$,
a $(B_i, B_{i+1})$-arc for each odd number $i$ $(1 \leqq i \leqq k-1)$,
an $(A, B_k)$-arc,
a $(B_k, A)$-arc.

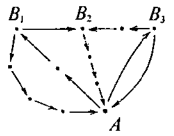

*Fig. 2.* A II/*a*-constructible graph ($k=3$)

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[2] We do not use the notations $C_2$, $C_3$, $C_4$ which occur in [1] but they are not referred to in this paper.

CONSTRUCTION II/b. Let $k(\geqq 2)$ be an integer. Take the $k+1$ vertices $A, B_1, B_2, \ldots, B_k$ and the following $2k+1$ arcs:

an $(A, B_1)$-arc,
a $(B_1, A)$-arc,
a $(B_i, A)$-arc for each odd number $i$ $(3 \leqq i \leqq k-1)$,
an $(A, B_i)$-arc for each even number $i$ $(2 \leqq i \leqq k-1)$,
a $(B_i, B_{i-1})$-arc for each even number $i$ $(2 \leqq i \leqq k)$,
a $(B_i, B_{i+1})$-arc for each even number $i$ $(2 \leqq i \leqq k-1)$,
an $(A, B_k)$-arc,
a $(B_k, A)$-arc.

CONSTRUCTION III. Take the vertices $A, B$, two $(A, B)$-arcs $c_1, c_2$ and two $(B, A)$-arcs $c_3, c_4$ such that $l_1+l_2 \geqq 3$ and $l_3+l_4 \geqq 3$ where $l_j$ is the length of $c_j$ ($j$ can be 1, 2, 3, 4).

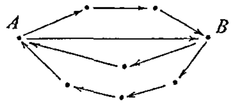

*Fig. 3.*
A III-constructible graph

If a graph $G$ can be built up by Construction I, then it is said that $G$ is I-constructible. The II/a-constructible, II/b-constructible, III-constructible and I*-constructible graphs are meant analogously. $G$ is said to be II-constructible if it is either II/a-constructible of II/b-constructible. A II/a-constructible graph is said to be II/a/o-constructible or II/a/e-constructible if it results with an odd or an even $k$, respectively (by Construction II/a). The II/b/o-constructible and II/b/e-constructible graphs are understood in a similar manner.

**Proposition 1.** *A graph is II/a/e-constructible if and only if it is II/b/e-constructible.*

*Proof.* Let $k$ be even. If the notation of the vertices $B_1, B_2, \ldots, B_k$ is replaced by $B_k, B_{k-1}, \ldots, B_1$ (respectively), then the definitions of II/a/e-constructibility and II/b/e-constructibility are interchanged.

**Proposition 2.** *The sets of*
*I*-constructible graphs,*
*I-constructible graphs,*
*II/a/o-constructible graphs,*
*II/a/e-constructible graphs,*
*II/b/o-constructible graphs and*
*III-constructible graphs*
*are pairwise disjoint.*

*Proof.* It is clear that the total degree of a vertex of a I*-constructible graph is $\leqq 4$ and equality holds precisely in case of cut vertices. On the other hand, the total degree of the vertex $A$ is $\geqq 4$ in case of any of the constructions described above, although $A$ is not a cut vertex. (Indeed, the total degree of $A$ is $k$ for Construction I, $k+2$ for Constructions II/a and II/b, it is 4 for Construction III.) Therefore a I*-constructible graphs cannot belong to any other type mentioned in the proposition.

A III-constructible graph has two vertices (namely $A$ and $B$) whose total degree is 4. If a graph is I-constructible or II-constructible, then all vertices $C(\neq A)$ of it have a total degree $\leqq 3$. Hence a III-constructible graph is neither I-constructible nor II-constructible.

Let $G$ be a II-constructible graph. The $(A, B_1)$-arc and the $(B_1, A)$-arc connect the same vertices $A$ and $B$ (with opposite orientations). The lack of a pair of arcs of this nature in any I-constructible graph implies that $G$ cannot be I-constructible.

To any graph $G$ denote by $\tau(G)$ the pair $(v, w)$ where $v$ is the number of vertices of degree (2, 1) and $w$ is the number of vertices having degree (1, 2). We have

$$\tau(G) = \left(\frac{k-1}{2}, \frac{k+1}{2}\right), \quad \tau(G) = \left(\frac{k}{2}, \frac{k}{2}\right) \quad \text{and} \quad \tau(G) = \left(\frac{k+1}{2}, \frac{k-1}{2}\right)$$

if $G$ is II/a/o-constructible, II/a/e-constructible or II/b/o-constructible, respectively. Consequently, any graph is contained in at most one of these three types.

**Proposition 3.** *If a graph $G$ is I-constructible or II-constructible or III-constructible, then $1 \leqq Z(e) \leqq 2$ holds for any edge $e$ of $G$.*

*Proof.* Let $G$ be I-constructible. Each cycle $c$ of $G$ can be characterized by the sequence of that vertices of $G$ whose degree differs from (1, 1). In this manner, the sequences

$(A, B_i, B_{i-1})$ where $3 \leqq i \leqq k-1$ and $i$ is odd,

$(A, B_i, B_{i+1})$ where $1 \leqq i \leqq k-1$ and $i$ is odd,

$(A, B_1, B_k)$

characterize cycles in $G$, and it is obvious that all the cycles of $G$ have thus been exhausted. This survey of cycles guarantees $1 \leqq Z(e) \leqq 2$.

If $G$ is II/a-constructible, then the inference is similar, namely the cycles are determined by the sequences

$(A, B_1)$

$(A, B_i, B_{i-1})$ where $3 \leqq i \leqq k$ and $i$ is odd,

$(A, B_i, B_{i+1})$ where $1 \leqq i \leqq k-1$ and $i$ is odd,

$(A, B_k)$.

When $G$ is II/b/o-constructible, then the sequences determining the cycles of $G$ are the following ones:

$$\left.\begin{array}{l} (A, B_i, B_{i-1}) \\ (A, B_i, B_{i+1}) \\ (A, B_k) \end{array}\right\} \text{ where } 2 \leqq i \leqq k-1 \text{ and } i \text{ is even.}$$

The II/b/e-constructible graphs do not require a further treatment (by Proposition 1).

It is evident that in any III-constructible graph there are precisely four cycles and $Z(e) = 2$ is universally satisfied.

**Proposition 4.** *If a graph $G$ is I-constructible or II-constructible or III-constructible, then $G \in C_5$.*

*Proof.* The universal validity of $1 \leqq Z(e) \leqq 2$ was stated in Proposition 3. It is clear from the constructions that $G$ has no cut vertex and the vertex $A$ (in any construction) is pancyclic.

## § 3.

**Proposition 5.** *Assume that one of the next five conditions* (a)—(e) *is true for the graph G:*

- (a) *G is a cycle,*
- (b) *G is I*-constructible, it has exactly two cycles and it has no cut vertex,*[3]
- (c) *G is I-constructible,*
- (d) *G is II-constructible,*
- (e) *G is III-constructible.*

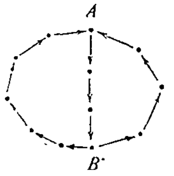

*Fig. 4.* A graph satisfying the condition (b) (occurring in Proposition 5 and Theorem 1)

*Choose two different vertices C, D in G. Take a new (C, D)-arc to G, denote the resulting graph by $G^*$. Suppose that either there is no edge from C to D (in G) or the new arc has at least two edges. Then $G^*$ satisfies one of the following three statements:*

- (1) $G^*$ *fulfils one of* (b), (c), (d), (e),
- (2) $G^*$ *has an edge e such that $Z(e)>2$,*
- (3) $G^*$ *has no pancyclic vertex.*[4]

**Proposition 6.** *Let $G_1$, $G_2$ be two graphs such that each of them fulfils one the requirements* (a)—(e) *exposed in Proposition 5. Let $A_i$ be a pancyclic vertex in $G_i$ (i is 1 or 2). Form the union G of $G_1$ and $G_2$ such that the vertices $A_1$ and $A_2$ are identified with each other (and this vertex is denoted by A). Choose a vertex $C(\neq A_1)$ in $G_1$ and a vertex $D(\neq A_2)$ in $G_2$. Take a new (C, D)-arc to G, denote the resulting graph by $G^*$. Then $G^*$ satisfies one of the statements* (1), (2) *occurring in Proposition 5.*

Since the proofs of Propositions 5 and 6 are lengthy and of technical character, they will be given at the end of the paper as Appendix I and Appendix II, respectively.

*Lemma. Let $G'$ be a subgraph of the graph G such that $G'$ has a cycle. If $G'$ has no pancyclic vertex, then the same holds for G.*

*Proof.* Let $A$ be an arbitrary vertex of $G$. If $A$ belongs to $G'$, then $G'$ has a cycle $a$ which does not contain $A$ (since $A$ is not pancyclic in $G'$). If $A$ is not a vertex of $G'$, then no cycle of $G'$ can contain $A$. We have got that $A$ is not pancyclic in $G$.

**Proposition 7.** *If $G\in C_5$, then one of the requirements* (a)—(e), *occurring in Proposition 5, is true for G.*

*Proof.* Denote by $\varkappa$ the number of cycles of $G$. We use induction on $\varkappa$.

If $\varkappa=1$, then (a) is true; if $\varkappa=2$, then (b) is valid (because of Theorem 2* and ($\gamma$)).

Consider the case when $\varkappa\geqq 3$. Let us select an edge $e_0$ such that $Z(e_0)$ is minimal in $G$. Delete $e_0$ and those vertices $C$ and edges $e$ which satisfy $Z(C)=0$ and $Z(e)=0$ (resp.) in the graph obtained by removing $e_0$. Denote the remaining graph by $G'$. $G'$ exists since $Z(e_0)<3$. It is clear that $1\leqq Z(e)\leqq 2$ holds universally in $G'$. If a vertex $A$ has been pancyclic in $G$, then $A$ is (contained and) pancyclic in $G'$, too.

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[3] In other words: $G$ has been formed by Construction I* from the tree with only one edge, such that $V'\neq 0$ (i.e. Step 3 has really been applied).

[4] The assertions (2) and (3) do not exclude each other.

Our next aim is to show that whenever a vertex $C$ of $G$ does not occur in $G'$, then $C$ is simple. Indeed, any cycle containing $C$ contains also $e_0$, therefore (by $1 \leqq Z(e_0) \leqq 2$ in $G$) the indegree and outdegree of $C$ may be 1 or 2. If e.g. the indegree of $C$ is 2, then $Z(e_0)=2$ and $Z(e')=Z(e'')=1$ (where $e'$ and $e''$ are the edges of $G$ terminating at $C$), contradicting the minimality of $Z(e_0)$. Thus the indegree of $C$ is 1, the outdegree of $C$ is also 1 (by similar reason).

Consequently, $G$ can be represented as an edge-disjoint union of $G'$ and certain arcs $a_1, a_2, \ldots, a_t$ ($t \geqq 1$) such that the inner vertices of any arc $a_i$ ($1 \leqq i \leqq t$) occur neither in $G'$ nor in $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_t$, furthermore, the beginning vertex and end vertex of any $a_i$ belong to $G'$.

Define the graphs

$$G_0, G_1, G_2, \ldots, G_t \qquad (t \geqq 1)$$

successively such that $G_0 = G'$ and $G_i$ proceeds from $G_{i-1}$ (where $1 \leqq i \leqq t$) by adding the edges and inner vertices of $a_i$. We have $G_t = G$. The further proof splits to two cases.

Case 1. $G'$ has no cut vertex. Then, by the induction hypothesis, one of (a)—(e) is valid for $G' = G_0$. We are going to prove that the same holds also for $G_1, G_2, \ldots, G_t$. Suppose that $i$ is the smallest subscript such that each of (a)—(e) is false for $G_i$ ($1 \leqq i \leqq t$). By applying Proposition 5 for $G_{i-1}$ and the arc $a_i$, we get then that either $Z(e) \geqq 3$ is satisfiable in $G_i$ (thus in $G$, too) or $G_i$ (hence, by the Lemma, also $G$) has no pancyclic vertex. Consequently, $G \notin C_5$, this contradicts the assumption.

Case 2. $G'$ has a cut vertex. It is then obvious that the pancyclic vertex $A$ (in $G$) is cut vertex of $G'$, and $G'$ does not possess any other pancyclic or cut vertex. Furthermore, there exists a number $w$ ($0 \leqq w < t$) such that the (single) cut vertex of $G_0, G_1, G_2, \ldots, G_w$ is $A$ but none of $G_{w+1}, G_{w+2}, \ldots, G_t$ has a cut vertex. Moreover, the number of blocks (separated by $A$) of $G_i$ ($1 \leqq i \leqq t$) is either the same as the number of blocks of $G_{i-1}$ or less by one, dependingly on the situation of $a_i$.

Since $G_0 = G'$ satisfies ($\alpha$), the induction hypothesis guarantees the validity of one of (a)—(e) for any block of $G_0$. Similarly to Case 1, we can show that the same holds for the blocks of each $G_i$ (by applying Proposition 5 or Proposition 6 according as the addition of $a_i$ does not or does diminish the number of blocks of $G_{i-1}$).

**Theorem 1.** *Let $G$ be a finite directed connected graph. $G$ belongs to the class $C_5$ if and only if one of the following five conditions is satisfied:*

(a) *$G$ is a cycle,*
(b) *$G$ is $I^*$-constructible, it has exactly two cycles and it has no cut vertex,*
(c) *$G$ is I-constructible,*
(d) *$G$ is II-constructible,*
(e) *$G$ is III-constructible.*

*Moreover,* (a), (b), (c), (d) *and* (e) *pairwise exclude each other.*

*Proof.* It follows from Proposition 2 that $G$ can satisfy at most one of (b)—(e). It is obvious that a graph, obtained by any of the constructions, cannot be a single cycle.

The sufficiency of (a) is trivial, that of (c), (d), (e) has been stated in Proposition 4. It is easy to see that (b) is also sufficient.

The necessity part of the theorem coincides with Proposition 7.

## § 4.

CONSTRUCTION IV. Let

$$G_1, G_2, \ldots, G_t \qquad (t \geqq 2)$$

be (pairwise disjoint) graphs contained in the class $C_5$. Let us choose a pancyclic vertex[5] $A_i$ in any $G_i$. Let us form a graph $G$ such that the vertices $A_1, A_2, \ldots, A_t$ are identified with each other, denote this new vertex by $A$.

Construction IV is completed. The graphs originating by it will be called IV-constructible graphs.

Let us recall the well-known fact that, in any graph, the relation "the edges $e_1$ and $e_2$ are completable to a circuit" is an equivalence relation and the subgraphs determined by the equivalence classes are precisely the blocks separated from each other by the cut vertices of the graph (see e.g. Section 5.4 in [3] or Chapter 3 in [2]).

We have the following immediate consequence of Construction IV:

**Proposition 8.** *Let the graph $G$ result by Construction IV. Then $A$ is a cut vertex of $G$ and $G$ has no other cut vertex. The blocks of $G$, separated by $A$, are the graphs $G_1, G_2, \ldots, G_t$. Whenever $c$ is a circuit (or, particularly, a cycle) of $G$, then all the edges of $c$ belong to the same $G_i$ ($1 \leqq i \leqq t$).*

**Proposition 9.** *If a graph $G$ is IV-constructible, then $G \in C_6$.*

*Proof.* Let $G$ be produced by Construction IV. It is obvious that $G$ is connected. $1 \leqq Z(e) \leqq 2$ holds in $G$ because of the last sentence of Proposition 8 and the validity of these inequalities in every $G_i$. It follows from the construction (more precisely, from the choice of the $A_i$'s) that $A$ is pancyclic.

**Proposition 10.** *If a graph $G$ belongs to the difference set $C_6$—$C_5$, then $G$ is IV-constructible.*

*Proof.* Since $G(\in C_6—C_5)$ satisfies ($\beta$), we can choose a pancyclic vertex $A$ in it. Our next aim is to show that no vertex $C(\neq A)$ of $G$ can be a cut vertex. In the contrary case, some part $G'$ of $G$ (separated by $C$) does not contain $A$, consequently, $A$ does not occur in the cycles consisting of edges of $G'$ what is impossible by ($\beta$).

Since $G$ belongs to $C_6$ but does not belong to $C_5$, it must have a cut vertex. Therefore $A$ is the single cut vertex of $G$. The blocks

$$G_1, G_2, \ldots, G_t \qquad (t \geqq 2)$$

of $G$, separated by $A$, are contained in the class $C_5$. It is evident that $G$ arises from these subgraphs by Construction IV.

By Propositions 9, 10 and Theorem 1, we have reached to a complete description of the graphs belonging to $C_6$. Our results can be summarized in the following assertion:

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[5] This requirement means (by Theorem 1 and the constructions mentioned in it) that
$A_i$ is an arbitrary vertex if $G_1$ satisfies (a),
$A_i$ is a vertex fulfilling $Z(A_i)=2$ if (b) is valid for $G_i$,
$A_i$ is the vertex denoted as $A$ in the corresponding construction if (c) or (d) holds for $G_i$, and
$A_i$ is either $A$ or $B$ (with the notation used in Construction III) if $G_i$ fulfils (e).

**Theorem 2.** *A finite directed graph $G$ is contained in the class $C_6$ if and only if either one of the five conditions* (a), (b), (c), (d), (e) *(occurring in Theorem 1) is true for $G$ or*

(f) *$G$ is IV-constructible.*

*Furthermore, these six conditions pairwise exclude each other.*

## Appendix I.

In this section we verify Proposition 5.

The assumption on the length of the $(C, D)$-arc guarantees the non-existence of parallel edges with coinciding orientation in $G^*$.

We write $Z(e)$ or $Z^*(e)$ according as the number of cycles (containing $e$) is considered in $G$ or in $G^*$.

Instead of (3) we shall sometimes show the assertion

(3′) there are two cycles in $G^*$ having no vertex in common.

It is obvious that (3′) implies (3).

We use the short expression "$(F, H; G)$-path" instead of "a path from $F$ to $H$ in $G$". Let $a$ be an $(F, H; G)$-path and let $b$ be an $(F', H'; G)$-path such that $b$ is a subpath of $a$. If at most one of the equalities $F' = F$ and $H' = H$ holds, then we say that $b$ is a *proper subpath* of $a$. If $F' \neq F$ and $H' \neq H$, then $b$ is called a *strongly proper subpath* of $a$.

If a graph $G$ is I-constructible or II-constructible, then we denote by $\pi(G)$ the value of the numerical parameter $k$ (occurring in Constructions I, II) yielding $G$.

Case 1. $G$ satisfies (a). Then (b) is obviously fulfilled by $G^*$.

Case 2. (b) holds for $G$. Denote by $A$ and $B$ the (uniquely determined) vertices whose degree is (2, 1) and (1, 2) (resp.) in $G$; it is clear that all other vertices of $G$ are simple. Evidently, either the $(C, D; G)$-path or the $(D, C; G)$-path (or both) is uniquely determined by $C$ and $D$.

Case 2/a. There exists only one $(C, D; G)$-path and this is a proper subpath of a $(B, A; G)$-path. Then $Z^*(e) = 3$ for each edge $e$ of the (single) $(A, B; G)$-path.

Case 2/b. There exists only one $(D, C; G)$-path and this is a strongly proper subpath of a $(B, A; G)$-path. Then $G^*$ satisfies the statement (3′).

Case 2/c. There exists only one $(D, C; G)$-path, this is a subpath of a $(B, A; G)$-path and exactly one of the equalities $A = C$ and $B = D$ holds. It is then evident that $G^*$ is II-constructible (with $\pi(G^*) = 2$).

Case 2/d. There exists only one $(C, D; G)$-path and this is a proper subpath of the (single) $(A, B; G)$-path. Then $Z^*(e) = 4$ for each edge $e$ of the $(A, B; G)$-path which is not contained in the $(C, D; G)$-path.

Case 2/e. There exists only one $(D, C; G)$-path and this is a subpath of the $(A, B; G)$-path. Then $Z^*(e) = 3$ for the edges of the $(D, C; G)$-path.

Case 2/f. $A = C$ and $B = D$. Then $G^*$ is III-constructible.

Case 2/g. $C$ is an inner vertex of the $(A, B; G)$-path and $D$ is an inner vertex of the $(B, A; G)$-path. Then the edges of the $(A, C; G)$-path fulfil $Z^*(e)=3$.

Case 2/h. $C$ is an inner vertex of a $(B, A; G)$-path and $D$ is an inner vertex of the $(A, B; G)$-path. Then $Z^*(e)=3$ for the edges of the $(D, B; G)$-path.

Case 2/i. $C$ and $D$ are inner vertices of the two $(B, A; G)$-paths (resp.). Then $Z^*(e)=3$ for the edges of the $(A, B; G)$-path.

It can be checked that every possible subcase of Case 2 has been exhausted.

Case 3. (c) or (d) holds for $G$. It follows from Constructions I, II that the number of the $(A, C; G)$-paths and the number of the $(D, A; G)$-paths is 1 or 2. Denote by $c$ an $(A, C; G)$-path, by $d$ a $(D, A; G)$-path and by $c^*$ the new $(C, D)$-arc (in $G^*$).

Case 3/a. $c$ and $d$ have no vertex in common[6] but $A$. Let $e_1, e_2$ be the edges of $c, d$ (resp.) incident to $A$. One of $e_1$, $e_2$ exists.

Case 3/a/$\alpha$. One of $Z(e_1)$, $Z(e_2)$ equals 2. Then the paths $c^*$, $c$ and $d$ form together a cycle in $G^*$, therefore $Z^*(e_1)$ or $Z^*(e_2)$ is $\geqq 3$.

Case 3/a/$\beta$. $Z(e_1)=Z(e_2)=1$. This is possible only if $G$ is II-constructible with an even $k$, $e_1$ is the first edge of the $(A, B_k)$-arc and $e_2$ is the last edge of the $(B_1, A)$-arc (we have here used the notation of Construction II/a, cf. Proposition 1). It is easy to see that either $Z^*(e)>2$ is satisfiable or $G^*$ is I-constructible (with $\pi(G^*)=\pi(G)+2$).

Case 3/a/$\gamma$. $Z(e_i)=1$ and $e_{3-i}$ does not exist (where $i$ is 1 or 2). Then we can ascertain that either $Z^*(e)>2$ for some edge or $G^*$ is II-constructible (with $\pi(G^*)=\pi(G)+1$).

Case 3/b. $c$ and $d$ have at least two vertices in common. Then $A\neq C$, $A\neq D$ and either $C$ or $D$ is a common vertex of $c$ and $d$. Let $a$ be a cycle (of $G^*$) got by taking the union of $c^*$ and the part $a'$ of $c$ or $d$ from $D$ to $C$. $a$ does not contain $A$. Let $\Gamma$ be the set of cycles $b$ of $G$ such that $a$ and $b$ have a vertex in common. It is clear that $1\leqq|\Gamma|\leqq 3$. Let us recall the survey of cycles of $G$ given in the proof of Proposition 3.

Case 3/b/$\alpha$. $G$ is I-constructible. $G$ has $k(\geqq 4)$ cycles, hence some cycle $b'$ of $G$ is disjoint to $a$, thus (3') is true.

Case 3/b/$\beta$. $G$ is II-constructible with $\pi(G)\geqq 3$. The number of cycles of $G$ is $k+1(\geqq 4)$, this implies again (3').

Case 3/b/$\gamma$. $G$ is II/a-constructible with $\pi(G)=2$ and $C=B_2$, $D=B_1$. (3) is obviously fulfilled.

Case 3/b/$\delta$. $G$ is II/a-constructible with $\pi(G)=2$ and $a'$ is a proper subpath of either the $(B_1, A)$-arc or the $(B_1, B_2)$-arc or the $(A, B_2)$-arc. Then (3') holds.

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[6] It may happen that either $C$ or $D$ equals $A$ (but not both).

Case 3/b/ε. $G$ is II/a-constructible with $\pi(G)=2$ and either $C$ is an inner vertex of the $(B_2, A)$-arc or $D$ is an inner vertex of the $(A, B_1)$-arc. Then $Z^*(e)>2$ holds clearly for the first or last edge of $a'$.

Case 4. $G$ satisfies (e). Since $Z(e)=2$ is universally valid in a III-constructible graph $G$ and $G$ has a path from $D$ to $C$ (however $C$ and $D$ may be chosen) it is evident that $Z^*(e)>2$ is satisfiable in $G^*$.

## Appendix II.

Now we are going to prove Proposition 6.

Similarly to Appendix I (Case 3), let $c$ denote an $(A_1, C; G_1)$-path and let $d$ denote a $(D, A_2; G_2)$-path. Let $e_1$ be the first edge of $c$ and $e_2$ be the last edge of $d$. We use the notations $Z_1, Z_2, Z^*$ according to the function $Z$ is understood in $G_1$, $G_2$, $G^*$ (resp.). $\pi(G)$ has the same meaning as in Appendix I.

Case 1. Either $Z_1(e_1)=2$ or $Z_2(e_2)=2$. Then[7] the conclusion (2) is evidently satisfied.

In the subsequent cases *we shall always assume that* $Z_1(e_1)=Z_2(e_2)=1$. (Therefore $G_1$ may satisfy (b) only if the degree of $A_1$ is (1, 2) in $G_1$; $G_2$ may fulfil (b) only if the degree of $A_2$ is (2, 1) in $G_2$.)

Case 2. $G_1$ and $G_2$ fulfil (a). It is obvious that $G^*$ is II-constructible (and $\pi(G^*)=2$).

Case 3. $G_1$ is a cycle and $G_2$ satisfies (b). Then either $G^*$ is II-constructible (with $\pi(G^*)=3$) or $Z_1(e_1)=3$ (accordingly to that $Z_2(D)$ is 1 or 2).

Case 4. $G_2$ satisfies (b) and $G_1$ is a cycle. The inference is analogous to Case 3 (a distinction is made dependingly on the value of $Z_1(C)$).

Case 5. $G_1$ is a cycle and $G_2$ satisfies (d). This case can be treated by the method of Case 3 (with some improvements); $G^*$ may be II-constructible with $\pi(G^*)=\pi(G_2)+2$.

Case 6. $G_1$ satisfies (d) and $G_2$ is a cycle. The treatment of this case is an improved version of Case 4 (likely to the interrelation of Cases 5 and 3).

Case 7. (b) holds for $G_1$ and (d) holds for $G_2$. Either $G^*$ is II-constructible (with $\pi(G^*)=\pi(G_2)+3$); or one of $Z^*(e_1)$, $Z^*(e_2)$ equals 3.

Case 8. (d) is true for $G_1$ and (b) is true for $G_2$. The treatment is symmetrical to Case 7.

Case 9. $G_1$ and $G_2$ satisfy (d). If $Z_1(C)=Z_2(D)=1$, then $G^*$ is II-constructible (with $\pi(G^*)=\pi(G_1)+\pi(G_2)+2$); otherwise either $Z^*(e_1)$ or $Z^*(e_2)$ equals 3.

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[7] We can perceive that Case 1 comprises a large collection of possible situations; among others, the possibilities when (c) or (e) is valid for $G_1$ or $G_2$ are entirely included.

## О графах удовлетворяющих некоторым условиям для циклов, I.

Цель настоящей работы — дать структурное описание конечных ориентированных графов удовлетворяющих условиям:

для всякого ребра *e*, число циклов содержающих *e* равняется 1 или 2,

существует вершина содержаемая в каждом цикле графа.

Ясно, что граф выполняющий эти требования может иметь не больше чем одну точку сочленения.

Опираемся на результаты предыдушей стати [1]. В §§ 2—3 даём некоторые конструкции и доказываем, что они представляют все графы обладающие вышеупомянутыми свойствами и не имеющими точку сочленения. В § 4 описание распространяется на графы в которых бывает точка сочленения.

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## References

[1] ÁDÁM, A., On some generalizations of cyclic networks, *Acta Cybernet.* v. 1, 1971, pp. 105—119.
[2] HARARY, F., *Graph theory,* Addison—Wesley, Reading, 1969.
[2a] Харари, Ф., *Теория графов,* Мир, Москва, 1973.
[3] ORE, O., *Theory of graphs,* Amer. Math. Soc. Coll. Publ. v. 38, Providence, 1962.

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