A note on optimal performance of page storage

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Introduction

In my earlier papers (see Arató [1], [2]) I have shown that on the basis of the Bayesian approach it is possible to prove that a simple algorithm exists which yields optimal page fault probability rate when the sequence of page references, the so called reference string, forms an independent random sequence with unknown probabilities. With the help of the Bellman equations it was proved that the replacement of the pages depends only on their posterior probabilities. Using this algorithm the expected number of page faults would be minimal.

Benczúr, Krámli and Pergel (see their paper [4] in this volume) pointed out to me that the posterior probabilities depend only on the frequencies of the page references. By the help of their ingenious remark we can prove that under very weak conditions the least frequently used (LFU) algorithm is the optimal one when the reference string is an independent sequence of random variables.

In this paper in a special case I give an elementary proof of the statement of Benczúr, Krámli and Pergel, on the basis of my previous work. The proof will show that the LFU algorithm, which is used in most cases, is the best one. The algorithm means that page should be put on the second level which has the minimal frequency.

The mathematical model I shall use to describe and evaluate the replacement problem of pages is a statistical one on the basis of Bayesian approach. This description seems adequate as it may be used in computer praxis, but we need a proof on the optimality of LFU algorithm without the Bayesian assumption. That this is possible I recall the well known example in the statistical literature, the Wald theorem in the sequential analysis, where the optimality of the likelihood ratio test can be easily proved under the Bayesian conditions (see e.g. Shiryaev [11], De Groot [6]). We have to remark that the "two-armed bandit problem" is meaningless without the Bayesian assumption (see Feldman [8]), but in our case the reference string is independent of the decisions and so the non-Bayesian approach is also allowed and has meaning.

The replacement problems arise in computer system management whenewer the executable memory space available is insufficient to contain all date and programs.

that may be accessed during the execution. An example of this kind of problem is page replacement in virtual memory computers.

In my earlier papers I discussed the problem using the terminology of storage allocation problems. It should be stressed that I consider the problem as being of broader interest (see Easton [7], Casey and Osman [5]). We remember that in virtual memory computer systems a program's address space is divided into equal size blocks called pages. In this paper I consider two-level hierarchies. The first level denotes the faster device, and the backing store represents the larger but slower memory. The first level memory space is divided into page frames each of which may contain a page of some programs. At a given instant of time, not all of a program's pages need reside in the first level memory so that when the program references a page not in the first level, a *page fault* occurs. Supposing that a program's set of pages is $(A_1, A_2, \ldots, A_n)$ and that exactly k of them can be kept in the first level then, if $k<n$, each time a page fault occurs that page is brought into main memory which was demanded and one must be removed from the main memory (demand paging). The purpose of the replacement algorithm is to minimize the average number of page faults. We take as a cost criterion the average number of page faults generated during execution.

In this paper we shall discuss a replacement algorithm in which the main memory is considered full of it contains $k-1$ pages and a new page from the second level is delivered to the k-th place and after delivering the contents of it a page must be removed to the second level.

Any theoretical evaluation of a page replacement algorithm requires a mathematical model of the reference string. Here I shall use for the sequence of requests $\eta_1, \eta_2, \ldots$ the independent identically distributed model

$$P(\eta_t = i) = p_i,$$

where the probability distribution $\{p_i\}$ is unknown. As a first step solving the problem I assume that $p_1 \geqq p_2 \geqq \ldots \geqq p_n$ are known values but the relation is not known between these probabilities and the pages $A_1, \ldots, A_n$. After solving this problem we shall see that the solution, in fact, does not depend on the exact values of the p_i-s.

In all the earlier papers (see Bélády [3], Gelenbe [9], Ingargiola and Korsh [10]) the authors assumed that the probability distribution of the reference string was known and given. The proposed algorithms are depending on the distribution and they are not the exact solutions of the practical problems as the distribution has to be estimated.

Here I do not give propositions for the case when the reference string is a Markov chain with unknown probabilities.

1. A theorem for the case of two pages

In the most elementary case in a program given are two pages A_1, A_2 with request probabilities $p_1 > p_2$ $(=1-p_1)$, but it is not known which probability is related with the first page. The computer is a multiprogrammed one and in the first level memory one page may be kept constantly. In case of a page fault the page of the second level is taken to the main memory and the replacement of one page to the second level occurs after delivering the contents of the demanded page. A sequence of N references is to be made and at each stage either A_1 or A_2 is on the second level, the loss being 1 if a page fault occurs, 0 otherwise. Let ξ denote the a priori probability that A_1 has the less request probability, p_2.

Let η_t $(t=1, 2, \ldots)$ denote the reference string, $\eta_t=i$ $(i=1, 2)$ if the i-th page (A_i) was referenced. Let X_t $(t=1, 2, \ldots)$ denote the random variable which gives that at time moment t which level was referenced

$$X_t = \begin{cases} 1 & \text{page on the first level was referenced,} \\ 0 & \text{page on the second level was referenced.} \end{cases}$$

Let d_t $(t=0, 1, 2, \ldots)$ denote the decision which page has to be removed to level 2

$$d = \begin{cases} 1 & \text{if page } A_1 \text{ goes to level 2,} \\ 2 & \text{if page } A_2 \text{ goes to level 2.} \end{cases}$$

It is obvious that

$$X_t^{(d_{t-1})} = \begin{cases} 1 & \text{if} \quad \eta_t \neq d_{t-1}, \\ 0 & \text{if} \quad \eta_t = d_t. \end{cases}$$

We introduce the non observable random variable w, which gives the relation between the request probabilities (p_1, p_2) and pages A_1, A_2

$$w = \begin{cases} 1 & \text{pages } (A_1, A_2) \text{ having reference probabilities } (p_2, p_1), \\ 2 & \text{pages } (A_1, A_2) \text{ having reference probabilities } (p_1, p_2). \end{cases}$$

The distribution of w is

$$P(w = 1) = \xi, \quad P(w = 2) = 1 - \xi.$$

We seek among all Markov decision rules $\delta=(d_0, d_1, \ldots, d_{N-1})$ (see Shiryaev [11]), where d_t depends only on $\eta_t, \eta_{t-1}, \ldots, \eta_1$, such a $\delta^*=(\delta_0^*, \ldots, d_{N-1}^*)$ for which

$$\max_{\delta} E(X_1^{(d_0)} + \ldots + X_N^{(d_{N-1})}) = E(X_1^{(d_0^*)} + \ldots + X_N^{(d_{N-1}^*)}). \tag{1}$$

Simple calculations give that

$$E(X_1^{(1)}) = p_1\xi + p_2(1-\xi),$$
$$E(X_1^{(2)}) = p_2\xi + p_1(1-\xi),$$

and so

$$E(X_1^{(1)}) - E(X_1^{(2)}) = (p_2 - p_1)(1 - 2\xi). \tag{2}$$

From (2) we get that the difference is greater than 0 if $\xi > 1/2$ (it does not depend on p_1), this means $d_0=1$ if $\xi>1/2$ and $d_0=2$ if $\xi<1/2$. Now we prove the following lemma.

Lemma 1. Let $\xi=1/2$ and $\xi(t)$ denote the a posteriori probabilities, then

$$\xi(t) = P(w = 1 \mid \eta_1, \ldots, \eta_t) = \frac{p_1^k p_2^{t-k}}{p_1^k p_2^{t-k} + p_2^k p_1^{t-k}}, \tag{3}$$

where k denotes the number of occurrences of page A_2 (i.e. $\eta_s=2$, $1 \leqq s \leqq t$).

Proof. On the basis of Bayes' theorem we get

$$\xi(1) = \frac{P(\eta_1|w=1)\,P(w=1)}{P(\eta_1|w=1)\,P(w=1)+P(\eta_1|w=2)\,P(w=2)} =$$

$$= \begin{cases} \dfrac{p_1\xi}{p_1\xi+p_2(1-\xi)} & \text{if} \quad \eta_1 = 2, \\ \dfrac{p_2\xi}{p_2\xi+p_1(1-\xi)} & \text{if} \quad \eta_1 = 1, \end{cases}$$

and from here in case $\xi = 1/2$

$$\xi(1) = \begin{cases} p_1 & \text{if} \quad \eta_1 = 2, \\ p_2 & \text{if} \quad \eta_1 = 1. \end{cases}$$

In the same way we get ($\xi = 1/2$)

$$\xi(2) = \begin{cases} \dfrac{p_1^2}{p_1^2+p_2^2} & \text{if} \quad \eta_1 = 2, \quad \eta_2 = 2, \\ 1/2 & \text{if} \quad \eta_1 = 2, \quad \eta_2 = 1 \quad \text{or} \quad \eta_1 = 1, \quad \eta_2 = 2, \\ \dfrac{p_2^2}{p_1^2+p_2^2} & \text{if} \quad \eta_1 = 1, \quad \eta_2 = 1. \end{cases}$$

By induction the lemma can be easily proved. By the same method as in (2) we can prove that after the first observation η_1 the best decision

$$d_1 = \begin{cases} 1 & \text{if} \quad \xi(1) = p_1 (\text{i.e. } \eta_1 = 2), \\ 2 & \text{if} \quad \xi(1) = p_2 (\text{i.e. } \eta_1 = 1). \end{cases}$$

We prove the following

Theorem. Let $p_1 > 1/2$, $\xi = 1/2$ and N fixed. Let the reference string η_t be an independent, identically distributed sequence of random variables with two states. Then the optimal sequential procedure δ^*, which minimizes the expected number of page faults (see (1)), puts at each stage that page to the second level which has the less request frequency.

Proof. In my paper (Arató [2], Theorem 1) it was proved that the optimal procedure δ^* has the following form (a similar result is known in the "two-armed bandit" problem)

$$d_{t-1}^* = \begin{cases} 1 & \text{if} \quad \xi(t) > 1/2, \\ 2 & \text{if} \quad \xi(t) < 1/2. \end{cases} \tag{4}$$

Comparing (3) and (4) we get (using again the fact $p_1 > 1/2$)

$$d_{t-1}^* = \begin{cases} 1 & \text{if} \quad k > \dfrac{t}{2}, \\ 2 & \text{if} \quad k < \dfrac{t}{2}, \end{cases}$$

and the theorem is proved.

2. Some generalizations

In the general case when the number of pages $n>2$, their number on the first level $1\leqq k<n$ and the a priori distribution $\xi=(\xi_1, \ldots, \xi_{n!})$ of the known probabilities $p_1\geqq p_2\geqq \ldots \geqq p_n$ are given, the optimal sequential procedure has the same construction as in §1 (see Benczúr, Krámli, Pergel [1]). The proof of their theorem, which is a natural generalization of the "many-armed bandit" problem, is not elementary. Here I recall that there may be two types of the replacement algorithms (see Arató [1]). The first type means that the main memory contains $k-1$ pages of the program and there is one place for the content of a page demanded from the second level. After delivering the content of the new page a page must be removed to the second level. The second type algorithm means that the main memory is full when it contains k pages of the program and if a page is requested from the second level then one page from first level has to be sent to the second level (two pages are changed). If we assume that in the Bayesian approach the a priori distribution ξ is uniform

$$\xi_i = P(w = i) = \frac{1}{n!}, \quad (i = 1, 2, \ldots, n!),$$

which is a natural assumption, we get again that the optimal decision rule is the least frequently used algorithm. The proof is based on the following lemma.

Lemma 2. Let $f_1, f_2, \ldots, f_n$ denote the frequencies of the pages $A_1, \ldots, A_n$ in the independent reference string $\eta_1, \ldots, \eta_t$, then

$$P(w = i|\eta_1, \ldots, \eta_t) = \frac{\prod_{j=1}^{n} p_{(w=i)}^{f_j}}{\sum_{l=1}^{n!} \prod_{j=1}^{n} p_{(w=l)}^{f_j}} \tag{5}$$

The proof of (5) is only a slight extension to that of (3) by the same induction method and so will not be repeated here.

Abstract

An example of the replacement problem in computer system management is the page replacement in virtual memory computers. In this note an elementary proof is given that the "least frequently used" algorithm is the optimal one, using the assumption that the references to the pages are indentically distributed independent random variables with unknown distribution. The general case of this problem is discussed in the paper of Benczúr—Krámli—Pergel.

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