On minimal R-complete systems of finite automata

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To the memory of Professor L. Kalmár

From papers by F. Gécseg (see [1], [2]) it is known, that there exist neither finite homomorphically, nor minimal isomorphically R-complete systems of finite automata. In the book by F. Gécseg and I. Peák [3] it is mentioned as an unsolved problem whether or not there exists a minimal homomorphically R-complete system of finite automata.

In this paper we prove that the answer to this problem is in the affirmative. Namely, it is shown that there exists a minimal homomorphically R-complete system of finite automata. Moreover, we prove that there exists a homomorphically R-complete system of finite automata which does not contain any minimal subsystem.

Before proving our statements, we introduce some notions and notations. Take an arbitrary, finite partially ordered set $R=\langle 1, 2, \ldots, n\rangle$ of indices, and for every i $(=1, 2, \ldots, n)$ let an automaton $\mathbf{A}_i=\mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ be given. Suppose that for an automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ with state set $A=A_1\times A_2\times \ldots \times A_n$ the functions $\varphi\colon A_1\times A_2\times \ldots \times A_n\times X\to X_1\times X_2\times \ldots \times X_n$, $\psi\colon A_1\times A_2\times \ldots \times A_n\times X\to Y$ are given. Then $\mathbf{A}=\prod_{i=1}^{n} \mathbf{A}_i[X, Y, \varphi, \psi]$ is called a loop-free or *R-product* of the automata $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, if the conditions $\delta((a_1, a_2, \ldots, a_n), x)=(\delta_1(a_1, x_1), \delta_2(a_2, x_2), \ldots, \delta_n(a_n, x_n))$, $\lambda((a_1, a_2, \ldots, a_n), x)=\psi(a_1, a_2, \ldots, a_n, x)$ hold for arbitrary $(a_1, a_2, \ldots, a_n)\in A$ and $x\in X$, where $(x_1, x_2, \ldots, x_n)=\varphi(a_1, a_2, \ldots, a_n, x)$; moreover $\varphi(a_1, a_2, \ldots, a_n, x)=$ $=(\varphi_1(a_1, a_2, \ldots, a_n, x), \varphi_2(a_1, a_2, \ldots, a_n, x), \ldots, \varphi_n(a_1, a_2, \ldots, a_n, x))$ holds as well, where φ_i $(i=1, 2, \ldots, n)$ is independent of states having indices not less (in the original definition not greater) than i under the partial ordering R. The functions φ and ψ of the R- product are called *feedback function* and *output function,* respectively.

If in the considered R-product $\mathbf{A}$ the set R is completely ordered, then $\mathbf{A}$ is called a *quasi-superposition* of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$.

Let $\mathbf{A}_1=\mathbf{A}_1(X_1, A_1, Y_1, \delta_1, \lambda_1)$ and $\mathbf{A}_2=\mathbf{A}_2(X_2, A_2, Y_2, \delta_2, \lambda_2)$ be arbitrary automata, where $Y_1\subseteq X_2$. Then a quasi-superposition $\mathbf{A}=\prod_{i=1}^{2} \mathbf{A}_i[X_1, Y_2, \varphi, \psi]$ of $\mathbf{A}_1$ and $\mathbf{A}_2$, where $\varphi(a_1, a_2, x)=(x, \lambda_1(a_1, x))$, $\psi(a_1, a_2, x)=\lambda_2(a_2, \lambda_1(a_1, x))$ are for any

$a_1 \in A_1$, $a_2 \in A_2$ and $x \in X_1$, is said to be the *superposition of* $\mathbf{A}_1$ by $\mathbf{A}_2$. The superposition can naturally be generalized for an arbitrary finite system of automata $\mathbf{A}_i = \mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(i=1, 2, \ldots, n)$ with $Y_j = X_{j+1}$ $(j=1, 2, \ldots, n-1)$.

A system $\mathfrak{A}$ of finite automata is called homomorphically (isomorphically) *R-complete,* if for every given finite automaton $\mathbf{A}$ there exists a finite R-product $\mathbf{B}$ of automata from $\mathfrak{A}$, such that an A-subautomaton of $\mathbf{B}$ can be mapped A-homomorphically (A-isomorphically) onto $\mathbf{A}$. $\mathfrak{A}$ is a *minimal* (homomorphically or isomorphically) R-complete system if for arbitrary $\mathbf{C} \in \mathfrak{A}$ the system $\mathfrak{A}/\langle \mathbf{C} \rangle$ is not (homomorphically or isomorphically) R-complete.

Then the following theorem holds(*).

Theorem 1. *There exists a minimal homomorphically R-complete system of finite automata.*

Proof. Denote by Γ a system of finite automata, where the elements of Γ are pair-wise not isomorphic, and simultaneously for every finite automaton $\mathbf{A}$ there exists an element $\mathbf{B}$ of Γ, such that $\mathbf{A}$ is isomorphic to $\mathbf{B}$. It can easily be seen, that Γ is enumerable. Take an arrangement $\Gamma = \langle \mathbf{A}_i(X_i, A_i, Y_i, \delta_i', \lambda_i') | i=1, 2, \ldots \rangle$ of the (enumerable) set Γ.

Let $p_0, p_1, \ldots, p_n, \ldots$ be an infinite sequence of prim numbers, where $p_0 \geqq 2$, $p_1 > p_0$, and for every further p_j $(j=2, 3, \ldots)$, $p_j > p_{j-1} + p_0 \cdot p_1 \cdot \ldots \cdot p_{j-2} \cdot \bar{A}_{j-1}$ holds.

Give the elements of automaton-system $\Delta = \langle \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \ldots \rangle$ as follows: $\mathbf{B}_0 = \mathbf{B}_0(X_0, D_0, Y_0, \delta_0, \lambda_0)$ is an arbitrary automaton, such that $D_0 = \langle 1, 2, \ldots, p_0 \rangle$, furthermore for any pair $u \in D_0$, $x \in X_0$

$$\delta_0(u, x) = \begin{cases} u+1, & \text{if} \quad 1 \leqq u < p, \\ 1, & \text{if} \quad u = p_0. \end{cases}$$

For every further $\mathbf{B}_i (i=1, 2, \ldots)$ let $\mathbf{B}_i = \mathbf{B}_i(C_i \times X_i, D_i \cup C_i \times A_i, Y_i', \delta_i, \lambda_i)$ be, where Y_i' is an arbitrary nonempty and finite set,

$$C_i = \langle 1, 2, \ldots, p_0 \cdot p_1 \cdot \ldots \cdot p_{i-1} \rangle, \tag{1}$$

$$D_i = \langle 1, 2, \ldots, p_i \rangle, \tag{2}$$

and $\lambda_i : (D_i \cup C_i \times A_i) \times C_i \times X_i \to Y_i'$ is arbitrary function, moreover for every triple $s \in D_i$, $(u, a) \in C_i \times A_i$, $(r, x) \in C_i \times X_i$

$$\delta_i(s, (r, x)) = \begin{cases} s+1, & \text{if} \quad 1 \leqq s < p_i, \\ 1, & \text{if} \quad s = p_i, \end{cases} \tag{3}$$

$$\delta_i((u, a), (r, x)) = \begin{cases} (u+1, \delta_i'(a, x)), & \text{if} \quad r = u \quad \text{and} \quad 1 \leqq u < p_0 \cdot p_1 \cdot \ldots \cdot p_{i-1}, \\ (1, \delta_i'(a, x)), & \text{if} \quad r = u \quad \text{and} \quad u = p_0 \cdot p_1 \cdot \ldots \cdot p_{i-1}, \\ 1 (\in D_i), & \text{if} \quad r \neq u. \end{cases} \tag{4}$$

(*) The proof of Theorem 1 is based on an idea of F. Gécseg.

First we prove that Δ is homomorphically R-complete system of finite automata.

Take an arbitrary finite automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$, and let $\langle \Psi_1, \Psi_2, \Psi_3 \rangle$ denote an isomorphism of $\mathbf{A}$ onto a suitable element $\mathbf{A}_i$ in Γ. Let the automata $\mathbf{C}_i = \mathbf{C}_i(X, C_i, C_i \times X_i, \delta_i'', \lambda_i'')$, $\mathbf{B}_i' = \mathbf{B}_i'(C_i \times X_i, D_i \cup C_i \times A_i, Y, \delta_i, \lambda_i^*)$ be constructed in the following way:

For any $r \in C_i$, $x \in X$, $s \in D_i$, $(u, a) \in C_i \times A_i$,

$$\delta_i''(r, x) = \begin{cases} r+1, & \text{if } \quad 1 \leqq r < p_0 \cdot p_1 \cdot \ldots \cdot p_{i-1}, \\ 1, & \text{if } \quad r = p_0 \cdot p_1 \cdot \ldots \cdot p_{i-1}, \end{cases} \tag{5}$$

$$\lambda_i''(r, x) = (r, \Psi_1(x)); \tag{6}$$

let $\lambda_i^*(s, (r, \Psi_1(x)))$ be an arbitrary element in Y given unambigously,

$$\lambda_i^*((u, a), (r, \Psi_1(x))) = \begin{cases} \Psi_3^{-1}(\lambda_i'(a, \Psi_1(x))), & \text{if } \quad r = u \\ \text{arbitrary element in } Y \text{ given} \\ \text{unambigously, otherwise.} \end{cases} \tag{7}$$

From the above constructions it is evident that the superposition $\mathbf{C}_i * \mathbf{B}_i'$ of $\mathbf{C}_i$ by $\mathbf{B}_i'$ exists. On the other hand, using (4), (5) and (6), it can easily be proved that there is an A-subautomaton of $\mathbf{C}_i * \mathbf{B}_i'$ with set of states $B = \langle (u, u, a) | u \in C_i, a \in A_i \rangle$.

Consider the mapping Ψ_2': $B \to A$ given as follows:

For every state $(u, u, a) \in B$ let $\Psi_2'((u, u, a)) = \Psi_2^{-1}(a)$. From constructions (4)—(7) it can be seen that Ψ_2' is an A-homomorphism of the A-subautomaton of $\mathbf{C}_i * \mathbf{B}_i'$ with set of states B onto $\mathbf{A}$. On the other hand, using (2) and (3), it is not difficult to prove that $\mathbf{C}_i$ can be represented as an A-subautomaton of a quasi-superposition of automata $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{i-1}$. So in consequence of construction $\mathbf{B}_i'$, the superposition $\mathbf{C}_i * \mathbf{B}_i'$ is an A-subautomaton of a quasi-superposition of $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_i$. Since $\mathbf{A}$ is arbitrary chosen, Δ is a homomorphically R-complete system of finite automata.

Let us prove that Δ is minimal, i.e. in case of any $\mathbf{B}_i \in \Delta$ the system $\Delta \setminus \langle \mathbf{B}_i \rangle$ is not homomorphically R-complete. To this we shall show, that no R-product of elements in $\Delta \setminus \langle \mathbf{B}_i \rangle$ has any A-subautomaton which can be mapped A-homomorphically onto $\mathbf{B}_i$.

Suppose that contrary to our assumption such R-product there exists. Denote by $\langle \Psi_1, \Psi_2, \Psi_3 \rangle$ a homomorphism of an A-subautomaton of this R-product onto $\mathbf{B}_i$, moreover, let $(e_1, e_2, \ldots, e_m)$ be a state of this A-subautomaton such that $\Psi_2((e_1, e_2, \ldots, e_m)) = s (\in D_i)$.

From (3) it is evident that

$$s \cdot q = s \Leftrightarrow p_i \,\Big|\, |q| \quad (q \in F(C_i \times X_i)). \tag{8}$$

Also from (3) and $\Psi_2((e_1, e_2, \ldots, e_m)) \in D_i$ it can be supposed that for a suitable element x of $C_i \times X_i$ the

$$(e_1, e_2, \ldots, e_m) \cdot x^l = (e_1, e_2, \ldots, e_m) \tag{9}$$

holds, where l is an appropriate natural number. Thus, due to (8), $p_i | l$ also holds. Suppose that the l is minimal among all numbers satisfying (9). For every $i(=1, 2, \ldots, m)$ let l_i be a minimal natural number for which $(e_1, e_2, \ldots, e_i) \cdot x^{l_i} = (e_1, e_2, \ldots, e_i)$ holds, moreover, let φ_i be the ith function-component of the feedback function of the R-product in question. Finally, let $\mathbf{M}_i$ be the i th component-automaton in our R-product.

Suppose that $\mathbf{M}_1 = \mathbf{B}_j (\in \Delta)$. In this case, refering to the equalities $\varphi_1(e_1 \cdot \varphi_1(e_1, e_2, \ldots, e_m, x), x) = \varphi_1((e_1, e_2, \ldots, e_m) \cdot x, x)$, and (4), either $\mathbf{M}_1 = \mathbf{B}_0$, or $e_1 \cdot \varphi_1(e_1, e_2, \ldots, e_m, x) \varphi_1((e_1, e_2, \ldots, e_m) \cdot x, x) \in D_j$ holds. Then, because of (3) and (4), equality (9) holds only in case $e_1 \in D_j$. Hence $l_1 \in \langle p_0, p_1, p_{i-1}, p_{i+1}, p_{i+2}, \ldots \rangle$ that is $p_i \nmid l_1$. If $\mathbf{M}_2$ in the R-product is independent of $\mathbf{M}_1$, $p_i \nmid l_2$ similarly holds. Othervise there are two possible cases.

(a) The number of states in $\mathbf{M}_2$ is less than that in $\mathbf{B}_i$. Hence for arbitrary input word q of $\mathbf{M}_2$ the number of pairly different states from the series $e_2, e_2 \cdot q, e_2 \cdot q^2, \ldots, e_2 \cdot q^s, \ldots$ is less than p_i (see the construction of $\langle p_0, p_1, \ldots \rangle$). Namely, if by the effect of e_1 and x^{l_1} the input word q is given to $\mathbf{M}_2$, then $p_i \nmid l_2$ since $l_2 = l_1 t$, where t is a natural number with $t < p_i$.

(b) The number of states in $\mathbf{M}_2$ is greater than that in $\mathbf{B}_i$. Suppose that by the effect of e_1 and x^{l_1} the input word q is given to $\mathbf{M}_2$. In this case for every natural number k by the effect of e_1 and $x^{k \cdot l_1}$ the automaton $\mathbf{M}_2$ in state e_2 has the input word q^k and $p_i \nmid |q|$. Suppose that $\mathbf{M}_2 = \mathbf{B}_h (\in \Delta, h > i)$ and $e_2 = (s, a) (\in C_h \times A_h)$. Because of (1) and (4), $e_2 \cdot q \notin \langle s \rangle \times A_h$. Therefore, by (4), for any $k (\geqq 1)$ we have $e_2 \cdot q \notin C_h \times A_h$. Thus $e_2 \cdot q^k \in D_h$, which, by (9) and (3), means that $e_2 \in D_h$. Consequently, taking into considerations the minimality of l_2, by (8) we get $l_2 = [l_1, p_j]$, where $[m, n]$ denotes the least common multiple of m and n. Therefore, $p_i \nmid l_2$ holds as well.

Repeating our procedure for the components $e_3, e_4, \ldots, e_m$, finally we get that $p_i \nmid l_m$. Since $l = l_m$ holds *per definitionem,* thus $p_i \nmid l$. Therefore, by (8), $\Psi_2((e_1, e_2, \ldots, e_m)) \notin D_i$. Thus none of the A-subautomaton of the considered R-product can be mapped A-homomorphically onto the A-subautomaton of $\mathbf{B}_i$ with the set of states D_i. Consequently, it also cannot be mapped A-homomorphically onto $\mathbf{B}_i$. Hence the system Δ is minimal, which ends the proof of Theorem 1.

Finally we prove

Theorem 2. *There exists a homomorphically R-complete system of finite automata which does not contain any minimal homomorphically R-complete subsystem.*

Proof. Again let $\Gamma = \langle \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots \rangle$ denote a system of finite automata such that the elments of Γ are pairly not isomorphic and for every finite automaton $\mathbf{A}$ there exists an element $\mathbf{B}$ of Γ which is isomorphic to $\mathbf{A}$. Now let us take the system $\Lambda = \langle \mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n, \ldots \rangle$ where for arbitrary $i(=1, 2, \ldots)$ every automaton $\mathbf{A}_j (j = 1, 2, \ldots, i)$ is a subautomaton of $\mathbf{B}_i$.

It can easily be seen that Λ is homomorphically R-complete system of finite automata. By a result of F. Gécseg [1], no finite subset of Λ is homomorphically R-complete.

Denote by Ω an infinite subset of Λ. It is evident that for every natural number i there is a j with $j \geqq i$ such that $\mathbf{B}_j \in \Lambda \cap \Omega$. Since every $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_j \in \Gamma$ is a subautomaton of $\mathbf{B}_j$, thus Ω is also homomorphically R-complete. It is obvious that Ω is not minimal, which completes the proof of Theorem 2.

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