On graphs satisfying some conditions for cycles, II.

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Introduction

In this paper we study another class (containing all cycles) of finite directed graphs, than in Part I. Let a class be introduced as follows: (i) all cycles belong to the class, (ii) whenever a graph G_0 is contained in the class and we replace a simple vertex P of G_0 by a cycle, then the new graph G is again an element of the class, (iii) the class is as narrow as possible with respect to the rules (i), (ii). The members of this class are called the A-constructible graphs. (A more detailed definition will be given in § 1.)

An advantage of this recursive definition is its simplicity; it has, however, the disadvantage that is does not give the A-constructible graphs uniquely (the same graph can be produced in essentially different ways). Therefore another recursive procedure (called Construction B) will be exposed such that it admits a decomposition statement (Theorem 1) and it yields all the A-constructible graphs (Theorem 2). (As it may be foreseen, Construction B is described more elaborately, than Construction A.) Finally, it is shown that the class of B-constructible graphs is wider, than the class of the A-constructible ones. We deal with the question (without solving it completely) how the A-constructible graphs can be characterized in terms of Construction B.

§ 1. The Constructions A, B

1.1.

CONSTRUCTION A. The construction consists of an initial step and a finite number ($\geqq 0$) of ordinary steps.

Initial step. Let us consider a cycle of length n ($\geqq 2$).

Ordinary step. Suppose that the preceding (initial or ordinary) step has produced the graph G_0. Consider G_0 and a cycle z of length m ($\geqq 2$) such that G_0, z are disjoint. Choose a simple vertex P in G_0; denote by e_1, e_2 the edges incoming to P or outgoing from P, resp. Furthermore, choose two different vertices A, B in z. Let us unite G_0 and z such that P is deleted, A becomes the new final vertex of e_1 and B is the new initial vertex of e_2.

A graph G is called A-*constructible* if G can be built up by Construction A[1].

1.2. Let G be a graph. We denote by $K(G)$ the maximum of the numbers $Z(e)$ where e runs through the edges of G. An edge e_0 (of G) is called *extremal* if $Z(e_0)=K(G)$. Denote by G' the subgraph of G consisting of the extremal edges (in G) and the vertices incident to them. G' is not connected in general. The connected components of G' are called the *extremal subgraphs* of G. If an extremal subgraph is a path only (having one or more edges), then we call it an *extremal path.*

1.3.

CONSTRUCTION B. The construction consists of a finite number ($\geqq 1$) of steps any of which is either an inital step or an ordinary one in the following sense.

Initial step. Let us consider a graph G such that

either G is a cycle (of length $\geqq 1$),

or G is I*-constructible[2] and G has no cut vertex (and, of course, G has neither a loop nor a pair of parallel edges with the same orientation).

Ordinary step. Let us consider a graph G_0 and a matrix

$$\begin{pmatrix} A_1 & A_2 \dots A_k \\ B_1 & B_2 \dots B_k \\ G_1 & G_2 \dots G_k \\ P_1 & P_2 \dots P_k \end{pmatrix}$$

(having four rows and k ($\geqq 1$) columns) such that

(α) any of the $k+1$ graphs $G_0, G_1, G_2, \dots, G_k$ is isomorphic to a graph produced in some earlier step of the construction,[3]

(β) $K(G_0)\geqq \max(2, K(G_1), K(G_2), \dots, K(G_k))$,

(γ) $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k$ are pairwise different simple vertices of G_0,

(δ) for any subscript i ($1\leqq i\leqq k$), G_0 has an extremal path[4] a_i with the following properties:

A_i precedes B_i along a_i, and

the set of vertices lying between A_i, B_i on a_i is disjoint to the set $\{A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k\}$,

(ε) for any i ($1\leqq i\leqq k$), P_i is a simple vertex of G_i and $Z(P_i)=1$ holds (in G_i).

Denote by $e_1^{(i)}$, $e_2^{(i)}$ the edges incoming to P_i and outgoing from P_i, resp. (in G_i).

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[1] I.e. if there exists a finite sequence of steps such that the first one is an initial step, the other ones are ordinary steps and the last step produces G.

[2] We call a graph I*-constructible of it can be produced by Construction I exposed in § 3 of [1]. The term "I*-constructible" has been used in the same sense in [2].

[3] It is permitted that both G_{j_1} and G_{j_2} are isomorphic to the result of the *same* previous step, though $j_1\neq j_2$. G_{j_1} and G_{j_2} are considered to be disjoint even in this case.

[4] The paths $a_1, a_2, \dots, a_k$ are not necessarily different.

Let us construct a new graph such that, for every subscript i $(1 \leqq i \leqq k)$, we delete P_i (out of G_i), A_i becomes the new final vertex of $e_1^{(i)}$ and B_i becomes the new initial vertex of $e_2^{(i)}$. (This means that the situation (a) is replaced by the situation (b) on Fig. 1.)

A graph G is called B-*constructible* if G can be built up by Construction B.

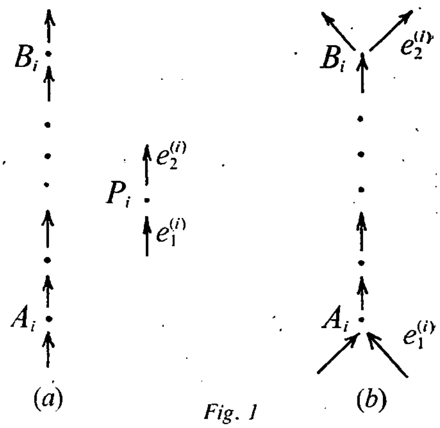

Fig. 1

1.4.

Proposition 1. *Suppose that G is produced by an ordinary step of Construction* B. *Then G has precisely k extremal subgraphs, namely, the part a_i' of a_i from A_i to B_i for each i $(1 \leqq i \leqq k)$.*

Proof. Denote by $Z(e)$, $Z_i(e)$ the number of cycles containing an edge e, meant in G, G_i, respectively. The rules in the ordinary step (chiefly (δ)) imply

$$Z(e) = 1 + Z_0(e) = 1 + K(G_0)$$

whenever e belongs to some a_i'. It is clear that

$$Z(e) = Z_0(e) \leqq K(G_0)$$

is true for the other edges of G_0 and, for any i $(1 \leqq i \leqq k)$,

$$Z(e) = Z_i(e) \leqq K(G_i) \leqq K(G_0)$$

holds (by (β)) if e is an arbitrary edge of G_i.

The above proof and (β) guarantee the following assertion, too:

Proposition 2. *If G can be represented as the result of an ordinary step Construction* B, *then*

$$K(G)(= 1 + K(G_0)) \geqq 3.$$

Proposition 3. *If G is* B-*constructible and $K(G) \geqq 2$, then each extremal subgraph of G is a path and the inner vertices of the extremal paths of G are simple.*

Proof. Case 1. G results by an initial step (of Construction B) only. We assumed $K(G) \geqq 2$, it is hence obvious that $K(G) = 2$ and G is I*-constructible. The conclusion is fulfilled because of Construction I in [1].

Case 2. G is produced by an ordinary step. We use induction: we suppose that G_0 satisfies the conclusion of Proposition 3. Proposition 1 implies that each extremal subgraph of G is a part of an extremal path of G_0, thus Proposition 3 is valid also for G.

The next result is implied immediately by Propositions 1, 2 and the assumptions in Construction B:

Proposition 4. *Let the graph G be represented as the result of an ordinary step of Construction B. Denote the extremal paths of G by $a_1, a_2, \ldots, a_k$; let the initial vertex of a_i be A_i and the final vertex of a_i be B_i (where $1 \leqq i \leqq k$). Then*

the degree of A_i is (2, 1) and we have $Z(e_i^{(1)})=1$, $Z(e_i^{(2)}) \geqq 2$ where $e_i^{(1)}$ and $e_i^{(2)}$ are the edges incoming to A_i with appropriate superscripts,

the degree of B_i is (1, 2) and we have $Z(e_i^{(3)})=1$, $Z(e_i^{(4)}) \geqq 2$ where $e_i^{(3)}$ and $e_i^{(4)}$ are the edges outgoing from B_i with appropriate superscripts.[5]

§ 2. Some notions concerning Construction B

2.1. Let us consider a particular application of Construction B consisting of q steps. We say that the relation $i \prec j$ is true (where $\{i, j\} \subseteqq \{1, 2, \ldots, q\}$) precisely if

$i<j$,

the j-th step is ordinary, and

the graph G resulting in the i-th step is isomorphic to one of the graphs $G_0, G_1, G_2, \ldots, G_k$ used in the j-th step.

We denote by $\lessdot$ the transitive extension of the relation $\prec$ (in the set $\{1, 2, \ldots, q\}$). It is obvious that $\lessdot$ is a partial ordering and $i \lessdot j$ may hold only if $i<j$. The definition of Construction B implies that, to any fixed j, $i \lessdot j$ is satisfiable (by some i) exactly if the j-th step is ordinary.

An application of Construction B, consisting of q steps, is called *connected* when all the $q-1$ relations $1 \lessdot q, 2 \lessdot q, \ldots, q-1 \lessdot q$ are true.

2.2. Two initial steps, occurring in particular performances of Construction B, are called *isomorphic* if the graphs appearing in them are isomorphic.

Let us consider two ordinary steps (again in Construction B) such that the number k is common. Denote the graphs and vertices, occurring in the first of these steps, by $G_0', G_1', A_1', B_1', P_1', \ldots, G_k', A_k', B_k', P_k'$; analogously, let the graphs and vertices of the second step in question be $G_0'', G_1'', A_1'', B_1'', P_1'', \ldots, G_k'', A_k'', B_k'', P_k''$. We call the considered steps to be *isomorphic* if there exist

(i) an isomorphism α of G_0' onto G_0'',

(ii) a permutation π of the set $\{1, 2, \ldots, k\}$, and

(iii) for every choice of i ($1 \leqq i \leqq k$), an isomorphism β_i of G_i' onto $G_{\pi(i)}''$

such that the equalities

$$\alpha(A_i') = A_{\pi(i)}'', \quad \alpha(B_i') = B_{\pi(i)}'', \quad \beta_i(P_i') = P_{\pi(i)}''$$

are fulfilled for each i ($1 \leqq i \leqq k$).

If two ordinary steps are isomorphic, then the originating graphs are again isomorphic.

A performance of Construction B is called *simple* if the i-th and j-th steps in it are not isomorphic unless $i=j$.

[5] It is clear that $e_i^{(1)}$, $e_i^{(3)}$ have been taken from G_i; $e_i^{(2)}$, $e_i^{(4)}$ have been taken from G_0.

2.3. Two applications Q_1, Q_2 of Construction B are said to be *similar* if the number q of their steps is the same and there exists a permutation σ of the set $\{1, 2, \ldots, q\}$ such that

the relation $i \prec_1 j$ holds if and only if $\sigma(i) \prec_2 \sigma(j)$ (where $\prec_l$ means the relation $\prec$ with respect to Q_l, $1 \leqq l \leqq 2$), and

in case of any i ($1 \leqq i \leqq q$), the i-th step of Q_1 is isomorphic to the $\sigma(i)$-th step of Q_2.

§ 3. The inverse construction

3.1. Suppose that a graph G results by an ordinary step of some particular application of Construction B. The main goal of this § is to produce the $k+1$ graphs $G_0, G_1, G_2, \ldots, G_k$ and the $3k$ vertices $A_1, B_1, P_1, A_2, B_2, P_2, \ldots, A_k, B_k, P_k$ (occurring in the ordinary step) by using the properties of G solely. This will lead to the statement that each B-constructible graph can be represented by (one and) only one simple, connected performance of Construction B apart from similarity.

Proposition 5. *If G is a graph mentioned in the initial step of Construction* B, *then there is no Construction* B *which would give G as the result of an ordinary step.*

Proof. Since any graph G occurring in the initial step satisfies $1 \leqq K(G) \leqq 2$ evidently, the statement to be proved follows immediately from Proposition 2.

3.2.

Construction C. Let G be a (finite) graph such that

[α] $$K(G) \geqq 3,$$

[β] every extremal subgraph of G is a path (denote them by $a_1, a_2, \ldots, a_k$; let the initial and final vertex of a_i be A_i, B_i, resp., where $1 \leqq i \leqq k$),

[γ] for any i, each inner vertex of a_i is simple,

[δ] for any i, the degree of A_i is (2, 1) moreover, $Z(e_i^{(1)}) = 1$ and $Z(e_i^{(2)}) \geqq 2$ hold for the edges incoming to A_i if they are denoted appropriately,

[ε] for any i, the degree of B_i is (1, 2), furthermore, $Z(e_i^{(3)}) = 1$ and $Z(e_i^{(4)}) \geqq 2$ are true for the edges outgoing from B_i if they are denoted suitably,

[ζ] for any i, the pair $e_i^{(1)}, e_i^{(3)}$ can be connected by a chain which contains neither A_i nor B_i as an inner vertex; the analogous statement is true for the pair $e_i^{(2)}, e_i^{(4)}$ too,

[η] for any i, each chain connecting $e_i^{(1)}$ and $e_i^{(4)}$ contains either A_i or B_i innerly and the chains connecting $e_i^{(2)}, e_i^{(3)}$ do the same.

Let us form $k+1$ new graphs $G_0, G_1, G_2, \ldots, G_k$ (from G) in the following way:

(1) we take k new vertices $P_1, P_2, \ldots, P_k$,

(2) for any i ($1 \leqq i \leqq k$), let $e_i^{(1)}$ go into P_i (instead of A_i) and let $e_i^{(3)}$ come out of P_i (instead of B_i); denote the resulting (non-connected) graph by G^*,

(3) let $G_0, G_1, G_2, \ldots, G_k$ be the connected components of G^* with such subscripts that[6] whenever $1 \leqq i \leqq k$, then G_i contains $e_i^{(1)}, e_i^{(3)}$, and G_0 contains none of $e_1^{(1)}, e_1^{(3)}, e_2^{(1)}, e_2^{(3)}, \ldots, e_k^{(1)}, e_k^{(3)}$.

[6] [ζ] and [η] guarantee that the number of connected components is $k+1$ and the conditions to be posed are satisfiable.

Thus Construction C is completed.

It is evident that, if [α]—[η] are fulfilled, then G uniquely defines k and the graphs $G_0, G_1, G_2, \ldots, G_k$ resulting by Construction C (apart from the numbering of $G_1, G_2, \ldots, G_k$).

3.3.

Proposition 6. *Assume that the graph G results by an ordinary step of Construction* B *such that the graphs and vertices (occurring in the step) are $G'_0, G'_1, G'_2, \ldots, G'_k$ and $A'_1, B'_1, P'_1, A'_2, B'_2, P'_2, \ldots, A'_k, B'_k, P'_k$, respectively. Then Construction* C *is applicable for G. Let us apply Construction* C *for G; denote the resulting graphs by $G''_0, G''_1, G''_2, \ldots, G''_k$ and the vertices, playing essential roles in the construction, by $A''_1, B''_1, P''_1, A''_2, B''_2, P''_2, \ldots, A''_k, B''_k, P''_k$. In this case $G'_0=G''_0$ and there exists a permutation π of the set $\{1, 2, \ldots, k\}$ which satisfies*

$$G'_i = G''_{\pi(i)}, \quad A'_i = A''_{\pi(i)}, \quad B'_i = B''_{\pi(i)}, \quad P'_i = P''_{\pi(i)}$$

for each i $(1 \leqq i \leqq k)$.

Proof. Let us take into account the obvious fact that the cycles of G'_0 and (essentially) the cycles of $G'_1, G'_2, \ldots, G'_k$ become the cycles of G, moreover, G does not contain any other cycle.

The conditions [α]—[η] of Construction C are true for G; in detail,

[α] is ensured by Proposition 2,

[β], [γ] are by Proposition 3,

[δ], [ε] are by Proposition 4,

[ζ], [η] follow from the suppositions (γ), (δ), (ε) occurring in the ordinary step of Construction B.

The applicability of Construction C has been shown. Using Proposition 1, we can convince ourselves that G''_0 coincides with G'_0 and the system $\{G''_1, G''_2, \ldots, G''_k\}$ equals the system $\{G'_1, G'_2, \ldots, G'_k\}$ (up to labelling). Hence also the coincidence of the vertices A_i, B_i, P_i (as stated in the Proposition) follows.

Theorem 1. *Let two applications Q_1, Q_2 of Construction* B *be considered such that they produce the same graph G. If Q_1 and Q_2 are simple and connected, then they are similar.*

Proof. Denote the number of steps of Q_1, Q_2 by q_1, q_2 respectively. In the sequel, we shall apply Proposition 6 and the last sentence of Section 3.2 without any particular reference.

Let a relation ϱ be defined between the sets $R_1=\{1, 2, \ldots, q_1\}$ and $R_2=\{1,2,\ldots,q_2\}$ followingly: $\varrho(i,j)$ holds precisely when the graph resulting in the i-th step of Q_1 is isomorphic to the graph originating in the j-th step of Q_2 (where $1 \leqq i \leqq q_1$, $1 \leqq j \leqq q_2$). Because Q_1 and Q_2 are simple, ϱ is a one-to-one assignment between some subset R'_1 of R_1 and some subset R'_2 of R_2. We can write $\sigma(i)=j$ instead of $\varrho(i,j)=\dagger$.

Our next purpose is to show that $R'_1=R_1$ and $R'_2=R_2$. Put $i \in R_1$. Since Q_1 is connected, there exists a sequence $i_0, i_1, i_2, \ldots, i_s$ such that

$$i = i_0 \prec_1 i_1 \prec_1 i_2 \prec_1 \ldots \prec_1 i_s = q_1$$

($s \geqq 0$). It is obvious that $\sigma(i_s)=q_2$, thus $i_s \in R_1'$. Whenever i_t belongs to R_1', then i_{t-1} does the same ($1 \leqq t \leqq s$). Consequently, $R_1'=R_1$ and the equality $R_2'=R_2$ follows by an analogous inference (therefore $q_1=q_2$).

We are going to verify that σ establishes a similarity. In order to do this, it remains to show that σ preserves the relation $\lessdot$ (in both directions). If $i \lessdot_1 i^*$, then

$$i = i_0 \prec_1 i_1 \prec_1 i_2 \prec_1 \dots \prec_1 i_w = i^*$$

for suitable numbers $i_0, i_1, \dots, i_w$. For any t ($1 \leqq t \leqq w$), the graph resulting in the $\sigma(t-1)$-th step of Q_2 is utilized in the $\sigma(t)$-th step of Q_2, thus $\sigma(t-1)<\sigma(t)$ (since Q_2 is simple) and $\sigma(t-1) \prec_2 \sigma(t)$. Hence $\sigma(i) \lessdot_2 \sigma(i^*)$. — Conversely, $i \lessdot_2 i^*$ implies $\sigma^{-1}(i) \lessdot_1 \sigma^{-1}(i^*)$ by a symmetrical inference.

Corollary. *Let Q_1, Q_2, G be as in the first sentence of Theorem 1. Denote the number of the steps of these constructions by q_1, q_2, respectively. If Q_1 is simple and connected, then $q_1 \leqq q_2$.*

Proof. We can reduce Q_2 into a simple and connected construction Q_2' followingly:

whenever $1 \leqq i < q_2$ and neither the i-th, q_2-th steps are isomorphic nor the relation $i \lessdot q_2$ holds, then the i-th step is deleted,

whenever $1 \leqq i < j \leqq q_2$ and the i-th, j-th steps are isomorphic, then the j-th step is deleted.

Let us define r as the smallest number with the property that the r-th and q_2-th steps of Q_2 are isomorphic. It is easy to see that

each of the $(r+1)$-th, $(r+2)$-th, ..., q_2-th steps of Q_2 is deleted by virtue of the above rules, and

the r-th step of Q_2 becomes the last step[7] of Q_2'.

We get $q_1=q_2' \leqq q_2$ where q_2' is the number of steps of Q_2'.

§ 4. Interrelations between A-constructibility and B-constructibility

4.1.

Theorem 2. *Each A-constructible graph is B-constructible.*

Proof. For cycles the assertion is trivial. Otherwise, we use induction for the number of edges. Let an A-constructible graph G be considered, suppose that every A-constructible graph, having a fewer number of edges than G, is B-constructible. By the definition of the A-constructibility, there is an A-constructible graph G^* and a simple vertex P of G^* such that G can be produced if we insert a cycle (of length l) for P in G^* (in sense of the ordinary step of Construction A). G^* is B-constructible by the induction hypothesis.

[7] It may happen that some of the first, second, ..., $(r-1)$-th steps of Q_2 are also deleted.

Let us consider a performance Q^* of Construction B which produces G^*. In what follows, our aim is to modify Q^* such that the new construction should give G. For the sake of simplicity, we agree that the construction steps of Q^* will always be mentioned as they are numbered in Q^*.

We define a sequence

$$D_1, D_2, \ldots, D_s \quad (s \geqq 1)$$

of vertices and a sequence

$$j_1, j_2, \ldots, j_s \quad (j_1 > j_2 > \ldots > j_s)$$

of numbers (indicating steps) in the following (recursive) manner:

D_1 is P (a vertex of the graph G^* resulting in the last step of Q^*) and j_1 is the number of the steps of Q^*,

if D_i has already been defined, it belongs to the graph originating in the j_i-th step of Q^* and the step in question is ordinary, then let j_{i+1} ($<j_i$) be such a number that the result of the j_{i+1}-th step occurs among the graphs appearing (as $G_0, G_1, G_2, \ldots, \ldots, G_k$) in the j_i-th step and D_i corresponds to some vertex D_{i+1} of the result of the j_{i+1}-th step (by virtue of an isomorphism mentioned in Construction B, (α)),

if D_i has been defined as a vertex of a graph originating in the j_i-th step of Q^* such that this step is initial, then we put $s=i$ and the process terminates.

We remark that each D_i is a simple vertex of the containing graph.

Next we define s or $s+1$ new construction steps which are called j'_1-th step, j'_2-th step, ..., j'_s-th step and, in some cases, j'_0-th step.

Case 1. $Z(D_s)=1$ in the graph $G^{(1)}$ resulting by the j_s-th step. $G^{(1)}$ is I^*-constructible. The graph $G'^{(1)}$ originating from $G^{(1)}$ by inserting a cycle of length l at D_s (as in the ordinary step of Construction A) is again I^*-constructible. Let the j'_s-th step be initial, let it produce $G'^{(1)}$. — Suppose that the j'_i-th step has been defined ($1 \leqq i < s$), we define a new construction step and call it the j'_{i+1}-th one in the following manner: the new step differs from the j'_{i+1}-th one only in that respect that now the (uniquely determined) graph containing D_{s-i} is replaced by the result or the j'_i-th step. (The graph resulting in the j'_{i+1}-th step will contain a cycle of length l instead of D_s, otherwise it will coincide with the graph originating in the j_{s-i}-th step.)

Let us draw up a new construction Q followingly:

it contains all the steps of Q^* except the last one (in the original ordering),

for every i ($1 \leqq i < s$), let the j'_i-th step be inserted between the j_{s-i+1}-th and $(j_{s-i+1}+1)$-th ones,

the last step of Q is the j'_s-th step.

It is obvious that Q is an application[8] of Construction B and Q produces G.

Case 2. $Z(D_s)=2$ in the result $G^{(1)}$ of the j_s-th step. Let an initial step, called j'_0-th one, be defined in such a manner that it produces a slighthly modified copy of $G^{(1)}$ with the single difference that D_s is replaced by the path a whose length equals the (directed!) distance d of A and B in the last step of the performance of Construction A producing G.

[8] Q is not simple and connected in general even if Q^* has these properties.

Now the j_1'-th step is ordinary such that
$k=1$,
G_0 is the result of the j_0'-th step,
G_1 is the cycle of length $l-d$,
A_1 and B_1 are the beginning and final vertices of a (see how the j_0'-th step is defined), respectively,
P_1 is an arbitrary vertex of G_1.

The further treatment of Case 2 is similar to Case 1. Now both the j_0'-th and i_1'-th steps (in this ordering) are inserted between the j_s-th and (j_s+1)-th ones.

4.2. The collection of A-constructible graphs is properly included in the family of B-constructible ones. An example for a B-constructible graph which is not A-constructible may be the cycle of length 1; a less trivial counter-example can be seen on Fig. 2. (One can check by applying Construction C that this graph is B-constructible. On the other hand, it does not contain any cycle which would be resulted in the last step of Construction A. — The numbers in Fig. 2 indicate the values of $Z(e)$.)

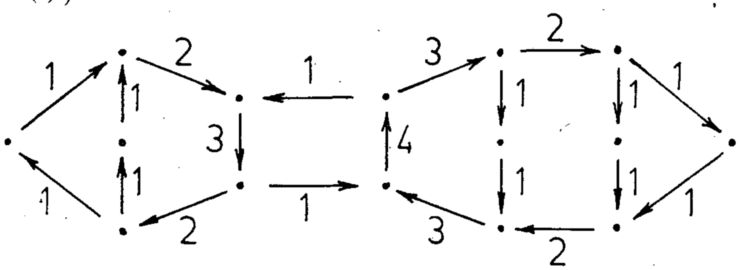

Fig. 2

4.3. The existence of counter-examples (similar to the above one) *disproves* the following statement: whenever each of $G_0, G_1, G_2, \dots, G_k$ in an ordinary step of Construction B is A-constructible, then G is again A-constructible. However, the converse assertion is valid:

Proposition 7. *Let the graph G be the result of an ordinary step of a performance of Construction* B. *If G is* A*-constructible, then each of the graphs $G_0, G_1, G_2, \dots, G_k$ (in the step) are likewise* A*-constructible.*

Proof. It is clear that each step of Construction A augments the number of cycles (of the constructed graph) by one. Moreover, let a performance of Construction A be given and denote the number of steps by r. Let us define a mapping γ of the set $\{1, 2, \dots, r\}$ in the following (recursive) way:
$\gamma(1)$ is the result of the beginning step,
if $(\gamma(1), \gamma(2), \dots, \gamma(j-1)$ are defined and$)$ we execute the j-th step of the construction, then the meaning of $\gamma(1), \gamma(2), \dots, \gamma(j-1)$ remains the same in G as in G_0 (with the small modification that P is now substituted by the path from A to B) and $\gamma(j)$ is defined as the new cycle z (of G)[9]. It is clear that γ is a one-to-one correspondence whose range equals the family of cycles of the constructed graph.

[9] G_0, G are now used as in describing the ordinary step of Construction A.

On the other side, we can convince ourselves by analyzing the ordinary step of Construction B that whenever z is an arbitrary cycle of the constructed graph G, then z has been present in exactly one of $G_0, G_1, G_2, \dots, G_k$ (if this graph is G_i with $i>0$, then apart from the change that P_i is replaced by the chain from A_i to B_i).

Let now G and some G_i $(0 \leqq i \leqq k)$ be as in the Proposition. Denote by Q_2 the application of Construction B in question (yielding G) and let Q_1 be a performance of Construction A which produces again G. Let us define the increasing sequence

$$j_1, j_2, \dots, j_s$$

containing precisely those numbers j for which $\gamma(j)$ is present in G_i (γ is now defined for Q_1). We can compile a performance $Q^{(i)}$ of Construction A from the j_1-th, j_2-th, ..., ..., j_s-th steps of Q_1 (with some modifications which may be left to the reader), it is evident that $Q^{(i)}$ produces G_i. This can be done for every value of i running from 0 to k.

Having Proposition 7, the characterization of A-constructible graphs among the B-constructible ones requires still to clear up the following question:

Problem. Suppose that $G_0, G_1, G_2, \dots, G_k$ are A-constructible graphs ($k \geqq 1$). Let us apply the ordinary step of Construction B for them (with some choices of the vertices having distinguished roles in the step). Let a necessary and sufficient condition be given in order the resulting graph G be again A-constructible.

О графах удовлетворяющих некоторым условиям для циклов, II.

Пусть класс конечных ориентированных графов быть вводим следующим рекурсивным образом: (1) каждый цикл содержается в классе, (2) если G_0 — граф содержаемый в классе и мы заменяем некоторую точку степени (1, 1) графа G_0 циклом, то новый граф находится опять в классе, (3) класс является минимальным ввиду правил (1) и (2). Члены этого класса называются A-конструируемыми графами.

Эта рекурсивная процедура не даёт возможность для однозначного разложения результируемого графа. Вводится другая процедура (называема конструкцией B) так, что она допускает почти единственную декомпозицию и все A-конструируемые графы являются B-конструируемыми.

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(Received Sept. 14, 1976)