Characteristically free quasi-automata

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In [2], [3] and [4] we dealt with the cyclic state-independent, well-generated group-type and reversible state-independent quasi-automata, respectively. In this paper we investigate a more general class of quasi-automata: the characteristically free quasi-automata. For the notions and notations which are not defined here we refer the reader to [3] and [7].

1. General preliminaries

The A-sub-quasi-automaton $\mathbf{A}_1=(A_1, F, \delta_1)$ of the quasi-automaton $\mathbf{A}=(A, F, \delta)$ is the *kernel* of $\mathbf{A}$ if

$$A_1 = \langle \delta(a,f) | a\in A, f\in F\rangle. \tag{1}$$

$\mathbf{A}$ is *well-generated* if $A=A_1$. In [3] and [4] the well-generated quasi-automaton is called simply generated quasi-automaton. $\bar{F}^A$ (or simply $\bar{F}$) denotes the characteristic semigroup of $\mathbf{A}$, and $\bar{f}^A$ (or $\bar{f}$) is the element of $\bar{F}^A$ represented by $f(\in F)$.

The well-generated quasi-automaton $\mathbf{A}=(A, F, \delta)$ is said to be *characteristically free* if there exists a generating system G of $\mathbf{A}$ such that

$$\delta(a,f) = \delta(b, g) \Rightarrow a = b, \quad \bar{f} = \bar{g} (a, b\in G; f, g\in F). \tag{2}$$

G is called a *characteristically free generating system* of $\mathbf{A}$, and its elements are called *characteristically free generating elements* of $\mathbf{A}$.

We note that every characteristically free generating system is minimal.

Theorem 1. *The quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free if and only if* $\mathbf{A}$ *is a direct sum of isomorphic characteristically free cyclic quasi-automata.*

Proof. It can easily be seen that the subsets $A_b=\langle\delta(b,f)|f\in F\rangle$ $(b\in G)$ of A form a partition on A, where G is a characteristically free generating system of $\mathbf{A}$. Quasi-automata $\mathbf{A}_b=(A_b, F, \delta_b)$ $(b\in G)$ are characteristically free cyclic quasi-automata. Let b_1, b_2 $(\in G)$ be arbitrary generating elements. The mapping

$$\varphi_{b_1, b_2}: \delta(b_1, f) \to \delta(b_2, f) \quad (f\in F) \tag{3}$$

is an isomorphism of $\mathbf{A}_{b_1}$ onto $\mathbf{A}_{b_2}$.

Conversely, it is clear that the direct sum of isomorphic characteristically free cyclic quasi-automata is characteristically free.

Theorems 1. and 2. are equivalent for A-finite well-generated quasi-automata.

Theorem 2. *The A-finite well-generated quasi-automaton $\mathbf{A}=(A, F, \delta)$ is characteristically free if and only if there exists a generating system G of $\mathbf{A}$ such that*

$$|A| = |G|\cdot O(\bar{F}).$$

(In this case G is a characteristically free generating system.)

Proof. Let G be a generating system of the A-finite well-generated quasi-automaton $\mathbf{A}=(A, F, \delta)$ such that

$$|A| = |G|\cdot O(\bar{F}).$$

Since $|A_b|\leqq O(\bar{F})$ $(b\in G)$ and $A=\bigcup_{b\in G} A_b$ therefore

$$|A| \leqq \sum_{b\in G} |A_b| \leqq |G|\cdot O(\bar{F}) = |A|,$$

thus

$$|A| = \sum_{b\in G} |A_b|.$$

This means that A_b $(b\in G)$ form a partition on A and $|A_b|=O(\bar{F})$. It is evident that the mapping $\bar{f}\to\delta(b,f)$ $(f\in F)$ is one-to-one. Therefore, the quasi-automata $\mathbf{A}_b$ $(b\in G)$ are characteristically free, cyclic, and for every pair b_1, b_2 $(\in G)$, $\mathbf{A}_{b_1}\cong\mathbf{A}_{b_2}$. By Theorem 1, the quasi-automaton $\mathbf{A}=(A, F, \delta)$ is characteristically free, and G is a characteristically free generating system of $\mathbf{A}$.

The necessity of this theorem follows from Theorem 1.

Lemma 1. (I. Babcsányi [3].) *Arbitrary two minimal generating systems of a well-generated quasi-automaton have the same cardinality.*

Corollary 1 and 2 follow immadeately from Theorem 2 and Lemma 1.

Corollary 1. *The A-finite cyclic quasi-automaton $\mathbf{A}=(A, F, \delta)$ is characteristically free if and only if $|A|=O(\bar{F})$.*

The necessity of Corollary 1 is true for infinite quasi-automata; thus we get the following result:

Theorem 3. *If the cyclic quasi-automaton $\mathbf{A}=(A, F, \delta)$ is characteristically free then $|A|=O(\bar{F})$.*

It should be noted that the converse of Theorem 3 does not hold. Indeed, in Example 1 for the quasi-automaton $\mathbf{A}=(A, F_1(X), \delta)$ we show that $|A|=O\big(\overline{F_1(X)}\big)$, but $\mathbf{A}$ is not characteristically free.

Example 1. $A=\langle 1; 2; 3; \ldots\rangle$, $X=\langle x, y\rangle$,

$$\delta(1,x) = 2,\quad \delta(1,y)=1,\quad \delta(i,x)=\delta(i,y)=i+1\quad (i=2,3,\ldots).$$

It can be seen that $\overline{F_1(X)}=\langle\overline{y^i x^j}\mid i,j=0,1,2,\ldots\rangle$. (We note that $x^0=y^0$ is the empty word.)

Corollary 2. *Every minimal generating system of an A-finite characteristically free quasi-automaton is characteristically free.*

In the following example it is shown that Corollary 2 does not hold for infinite quasi-automata.

Example 2. Let N be the set of natural numbers, $A=N\times N$ and $X=\langle x, y\rangle$. The definition of next state function δ is the following:

$$\delta\big((i, 1), x\big) = (i, 2),$$

$$\delta\big((i, 2), x\big) = \delta\big((i, 4), x\big) = (i, 3),$$

$$\delta\big((i, 2j+1), x\big) = \delta\big((i, 2j+4), x\big) = (i, 2j+3),$$

$$\delta\big((i, 1), y\big) = (i+1, 1),$$

$$\delta\big((i, 2), y\big) = \delta\big((i, 4), y\big) = (i, 1),$$

$$\delta\big((i, 2j+1), y\big) = \delta\big((i, 2j+4), y\big) = (i, 2j+2) \quad (i, j = 1, 2, 3, \ldots).$$

The quasi-automaton $\mathbf{A}=(A, F(X), \delta)$ is cyclic. $\langle(1, j)\rangle$ $(j=1, 2, 3, \ldots)$ are minimal generating systems, but only $\langle(1, 1)\rangle$ is characteristically free.

Lemma 2. *The characteristic semigroup of every characteristically free quasi-automaton has a left identity element.*

Proof. Let G be a characteristically free generating system of the quasi-automaton $\mathbf{A}=(A, F, \delta)$ and $b\in G$. There exists an $e\in F$ such that $\delta(b, e)=b$. Thus

$$\forall_{f\in F} f[\delta(b,f) = \delta(\delta(b, e), f) = \delta(b, ef)],$$

that is,

$$\forall_{f\in F} f[\bar{f} = \bar{e}\bar{f}].$$

Theorem 4. *Let a_0 be a characteristically free generating element of the cyclic quasi-automaton $\mathbf{A}=(A, F, \delta)$. $\delta(a_0, h)$ $(h\in F)$ is a characteristically free generating element of $\mathbf{A}$ if and only if there exists a $k\in F$ such that $\delta(a_0, hk)=a_0$ and $\bar{k}\bar{h}$ is a left identity element of $\bar{F}$.*

Proof. Let a_0 be a characteristically free generating element of $\mathbf{A}$, $\delta(a_0, hk)=a_0$ $(h, k\in F)$ and $\bar{k}\bar{h}$ a left identity element of $\bar{F}$. Furthermore, for $f, g(\in F)$, let,

$$\delta(a_0, hf) = \delta(\delta(a_0, h), f) = \delta(\delta(a_0, h), g) = \delta(a_0, hg).$$

Since a_0 is a characteristically free generating element, thus,

$$\bar{h}\bar{f} = \bar{h}\bar{g},$$

that is,

$$\bar{f} = \bar{k}\bar{h}\bar{f} = \bar{k}\bar{h}\bar{g} = \bar{g}.$$

This means that $\delta(a_0, h)$ is a characteristically free generating element of $\mathbf{A}$. Conversely, let $\delta(a_0, h)$ $(h\in F)$ be a characteristically free generating element of $\mathbf{A}$. There exists a $k\in F$ such that $a_0=\delta(a_0, hk)$. Now let $f\in F$ be arbitrary. By Lemma 2,

$\overline{hk}$ is a left identity element of $\bar{F}$. Therefore

$$\delta(\delta(a_0, h), f) = \delta(a_0, hf) = \delta(a_0, hkhf) = \delta(\delta(a_0, h), khf),$$

that is,

$$\bar{f} = \overline{khf}.$$

It is clear that every well-generated state-independent quasi-automaton is characteristically free. The converse of this statement does not hold (see Example 2). However, by Corollary 2, every A-finite strongly connected characteristically free quasi-automaton is state-independent.

Lemma 3. *The characteristic semigroup of a state-independent quasi-automaton is left cancellative.*

Proof. Let the quasi-automaton $\mathbf{A}=(A, F, \delta)$ be state-independent and $\overline{hf}=\overline{hg}$ $(h, f, g \in F)$. Then for an arbitrary state $a(\in A)$.

$$\delta(a, hf) = \delta(\delta(a, h), f) = \delta(\delta(a, h), g) = \delta(a, hg).$$

Since $\mathbf{A}$ is state-independent thus $\bar{f}=\bar{g}$, i.e., the characteristic semigroup $\bar{F}$ of $\mathbf{A}$ is left cancellative.

The converse of Lemma 3 does not hold. Indeed, in Example 3 the characteristic semigroup $\overline{F(X)}$ of the quasi-automaton $\mathbf{A}=(A, F(X), \delta)$ is left cancellative, but $\mathbf{A}$ is obviously not state-independent.

Example 3. $A=\langle 1, 2, 3\rangle$, $X=\langle x, y\rangle$

δ	1	2	3
x	2	1	2
y	2	3	2

$\bar{F}$	$\bar{x}$	$\bar{x}^2$	$\bar{y}$	$\bar{y}^2$
$\bar{x}$	$\bar{x}^2$	$\bar{x}$	$\bar{y}^2$	$\bar{y}$
$\bar{x}^2$	$\bar{x}$	$\bar{x}^2$	$\bar{y}$	$\bar{y}^2$
$\bar{y}$	$\bar{x}^2$	$\bar{x}$	$\bar{y}^2$	$\bar{y}$
$\bar{y}^2$	$\bar{x}$	$\bar{x}^2$	$\bar{y}$	$\bar{y}^2$

$\mathbf{A}$ is not a characteristically free quasi-automaton.

Theorem 5. *A characteristically free quasi-automaton is state-independent if and only if its characteristic semigroup is left cancellative.*

Proof. The necessity obviously follows from Lemma 3. For the proof of sufficiency, let the characteristic semigroup $\bar{F}$ of the characteristically free quasi-automaton $\mathbf{A}=(A, F, \delta)$ be left cancellative. Take the elements $a(\in A)$ and $f, g(\in F)$ such that $\delta(a, f)=\delta(a, g)$. Let G be a characteristically free generating system of $\mathbf{A}$. There are $b(\in G)$ and $h(\in F)$ such that $\delta(b, h)=a$, thus,

$$\delta(b, hf) = \delta(b, hg).$$

Since $\mathbf{A}$ is characteristically free thus $\overline{hf}=\overline{hg}$. But $\bar{F}$ is left cancellative Therefore, $\bar{f}=\bar{g}$. This means that $\mathbf{A}$ is state-independent.

We note that if a characteristically free quasi-automaton is state-independent, then each of its minimal generating systems is characteristically free.

In the following two paragraphs we generalise some results of papers [2] and [4], concerning cyclic state-independent and reversible state-independent quasi-automata for characteristically free quasi-automata.

2. Endomorphism semigroup

Theorem 6. *Let a_0 be a characteristically free generating element of the characteristically free cyclic quasi-automaton $\mathbf{A}=(A, F, \delta)$ and $\delta(a_0, e)=a_0$ $(e\in F)$. Then*

$$E(A) \cong \bar{F}\bar{e}.$$

Proof. Define the following mappings $\alpha_{a_0,h}: A\to A$

$$\alpha_{a_0,h}(\delta(a_0,f)) = \delta(a_0, hf) \quad (f\in F). \tag{4}$$

If $\delta(a_0,f)=\delta(a_0, g)$ $(f, g\in F)$ then, by (2), $\bar{f}=\bar{g}$, thus,

$$\delta(a_0, hf) = \delta(\delta(a_0, h),f) = \delta(\delta(a_0, h), g) = \delta(a_0, hg),$$

i.e., $\alpha_{a_0,h}$ is well-defined. Let $a(\in A)$ and $f(\in F)$ be arbitrary elements. Then there exists a $g(\in F)$ such that $\delta(a_0, g)=a$ Therefore,

$$\alpha_{a_0,h}(\delta(a,f)) = \alpha_{a_0,h}(\delta(a_0, gf)) = \delta(a_0, hgf) =$$
$$= \delta(\delta(a_0, hg),f) = \delta(\alpha_{a_0,h}(\delta(a_0, g)), f) = \delta(\alpha_{a_0,h}(a),f),$$

i.e., $\alpha_{a_0,h}$ is an endomorphism of **A**. Let α be arbitrary endomorphism of **A**. There exists an $h\in F$ such that $\delta(a_0, h)=\alpha(a_0)$. Then for every $a=\delta(a_0, g)\in A$,

$$\alpha(a) = \alpha(\delta(a_0, g)) = \delta(\alpha(a_0), g) = \delta(\delta(a_0, h), g) = \delta(a_0, hg) =$$
$$= \alpha_{a_0,h}(\delta(a_0, g)) = \alpha_{a_0,h}(a),$$

that is, $\alpha=\alpha_{a_0,h}$. Therefore, every endomorphism of **A** is of type (4).

From Lemma 2 it follows that $\bar{e}$ is a left identity element of $\bar{F}$. It can easily be seen that the mapping

$$\alpha_{a_0,h} \to \bar{h}\bar{e} \quad (h\in F)$$

is an isomorphism of $E(A)$ onto $\bar{F}\bar{e}$.

Corollary 3. *The endomorphism semigroup of a characteristically free cyclic quasi-automaton is a homomorphic image of its characteristic semigroup.*

Proof. The mapping $\bar{f}\to\bar{f}\bar{e}$ $(f\in F)$ is an endomorphism of $\bar{F}$.

In Example 2 $\overline{xy}$ is a left identity element of $\overline{F(X)}$.

$$\overline{F(X)} = \langle \overline{x^k}; \overline{y^k}; \overline{x^k y}; \overline{y^l x^k}; \overline{y^l x^{j+1} y} \mid j, k, l = 1, 2, 3, \ldots\rangle,$$

$$\overline{F(X)}\,\overline{xy} = \langle \overline{y^k}; \overline{x^k y}; \overline{y^l x^{j+1} y} \mid j, k, l = 1, 2, 3, \ldots\rangle.$$

Let G be a characteristically free generating system of the characteristically free quasi-automaton $\mathbf{A}=(A, F, \delta)$. Furthermore, $\pi: G\to G$ and $\omega: G\to F$.

Theorem 7. *The mapping $\varphi_{\pi\omega}: A \to A$ for which*

$$\varphi_{\pi\omega}\big(\delta(b,f)\big) = \delta\big(\pi(b), \omega(b)f\big) \quad (b \in G; f \in F) \tag{5}$$

is an endomorphism of **A**. *Furthermore, every endomorphism of* **A** *is of type* (5) *and*

$$\varphi_{\pi\omega} = \bigcup_{b \in G} \varphi_{b,\pi(b)}\, \alpha_{b,\omega(b)},$$

where $\varphi_{b,\pi(b)}$ is a mapping of type (3) *and $\alpha_{b,\omega(b)}$ is a mapping of type* (4).

Proof. Let $\delta(b,f)=\delta(c,g)$ $(b, c \in G; f, g \in F)$. From (2) it follows that $b=c$ and $\bar{f}=\bar{g}$, that is, $\pi(b)=\pi(c)$ and $\overline{\omega(b)f} = \overline{\omega(b)g}$. Therefore, $\varphi_{\pi\omega}$ is well-defined. Let $a=\delta(b, h)$ be an arbitrary state of **A** and $f \in F$. Then

$$\varphi_{\pi\omega}\big(\delta(a,f)\big) = \varphi_{\pi\omega}\big(\delta(b,hf)\big) = \delta\big(\pi(b), \omega(b)hf\big) =$$
$$= \delta\big(\delta(\pi(b), \omega(b)h), f\big) = \delta\big(\varphi_{\pi\omega}(\delta(b, h)), f\big) = \delta\big(\varphi_{\pi\omega}(a), f\big).$$

Therefore, $\varphi_{\pi\omega}$ is an endomorphism of **A**. Let α be an arbitrary endomorphism of **A**, $\alpha(b) \in A_c$ $(b, c \in G)$ and $\alpha(b)=\delta(c, h)$ $(h \in F)$. Since the subsets A_c $(c \in G)$ of A form a partition on A, thus the mapping $\pi: b \to c$ is well-defined. Let $\omega: G \to F$ such that $\delta\big(c, \omega(b)\big)=\alpha(b)$. Then

$$\alpha\big(\delta(b,f)\big) = \delta\big(\alpha(b), f\big) = \delta\big(\delta(c, \omega(b)), f\big) =$$
$$= \delta\big(c, \omega(b)f\big) = \delta\big(\pi(b), \omega(b)f\big) = \varphi_{\pi\omega}\big(\delta(b,f)\big) \quad (b \in G, f \in F),$$

that is, $\alpha=\varphi_{\pi\omega}$. This means that α is a mapping of type (5). Furthermore,

$$\varphi_{\pi\omega}\big(\delta(b,f)\big) = \delta\big(\pi(b), \omega(b)f\big) = \varphi_{b,\pi(b)}\big(\delta(b, \omega(b)f)\big) = \varphi_{b,\pi(b)}\, \alpha_{b,\omega(b)}\big(\delta(b,f)\big),$$

that is,

$$\varphi_{\pi\omega|A_b} = \varphi_{b,\pi(b)}\, \alpha_{b,\omega(b)}.$$

Denote the set of mappings $\varphi_\pi := \bigcup_{b \in G} \varphi_{b,\pi(b)}$ by T and the set of mappings $\alpha_\omega := \bigcup_{b \in G} \alpha_{b,\omega(b)}$ by H. T and H are subsemigroups of $E(A)$ under the usual multiplication of mappings.

Corollary 4. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free then*

$$E(A) = TH \quad \text{and} \quad T \cap H = \{\iota\}.$$

Proof. It is evident that $\varphi_{\pi\omega}=\varphi_\pi \alpha_\omega$ and

$$\varphi_\pi = \alpha_\omega \Leftrightarrow \varphi_\pi = \alpha_\omega = \iota,$$

where ι is the identity element of $E(A)$.

Corollary 5. *If the A-finite quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free and* $\bar{F}$ *is a monoid then*

$$O\big(E(A)\big) = |A|^{|G|}$$

where G is a characteristically free generating system of **A**.

Proof. By Theorem 1, $O(T)$ is equal to the number of the transformations of G, that is, $O(T)=|G|^{|G|}$. Since $\bar{F}$ is a monoid thus, by Theorem 6, $E(A_b)\cong\bar{F}(b\in G)$. By Theorem 2, $O(\bar{F})=\frac{|A|}{|G|}$. Therefore, by Theorem 7, $O(H)=\left(\frac{|A|}{|G|}\right)^{|G|}$. Thus, by Corollary 4,

$$O(E(A)) = O(T)\cdot O(H) = |G|^{|G|}\cdot\left(\frac{|A|}{|G|}\right)^{|G|} = |A|^{|G|}.$$

Theorem 8. *Let the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *be characteristically free. Then*

1) $\varphi_\pi\in G(A)$ *if and only if* π *is a permutation of* G, *where* G *is a characteristically free generating system of* $\mathbf{A}$.

2) $\alpha_\omega\in G(A)$ *if and only if* $G'=\langle\delta(b,\omega(b))|b\in G\rangle$ *is a characteristically free generating system of* $\mathbf{A}$.

Proof. 1) By Theorem 1, $\varphi_{\pi|A_b}$ $(b\in G)$ is an isomorphism. Thus $\varphi_\pi\in G(A)$ if and only if

$$\varphi_{\pi|A_b} = \varphi_{\pi|A_c}(b, c\in G) \Rightarrow A_b = A_c,$$

that is, $b=c$. This means that π is a permutation of G.

2) By (3), $\alpha_\omega\in G(A)$ if and only if for every $b\in G$,

$$\overline{\omega(b)f} = \overline{\omega(b)g} \quad (f, g\in F) \Rightarrow \bar{f}=\bar{g}$$

and $G'=\langle\delta(b,\omega(b))|b\in G\rangle$ is a generating system of $\mathbf{A}$, i.e., G' is a characteristically free generating system of $\mathbf{A}$.

The quasi-automaton $\mathbf{A}=(A, F, \delta)$ is called *reversible* if for every pair $a(\in A)$, $f(\in F)$ there exists a $g\ (\in F)$ such that $\delta(a,fg)=a$. (s. V. M. GLUSKOV [9].)

We note that if $\bar{F}$ is left cancellative (i.e., if the characteristically free quasi-automaton $\mathbf{A}$ is state-independent) then every mapping α_ω is one-to-one. If every $\mathbf{A}_b$ $(b\in G)$ is strongly connected (i.e., $\mathbf{A}$ is reversible) then α_ω is onto. If $\mathbf{A}$ is reversible and state-independent then H is a subgroup of $G(A)$ (see [3] and [4]).

If $\varphi_\pi, \alpha_\omega\in G(A)$ then

$$\varphi_{\pi\omega}(\delta(b,f)) = \delta(\pi(b), \omega(b)f) = \alpha_{\pi(b),\omega(b)}(\delta(\pi(b),f)) =$$
$$= \alpha_{\pi(b),\omega(b)}\,\varphi_{b,\pi(b)}(\delta(b,f)) \quad (f\in F, b\in G),$$

that is,

$$\varphi_\pi\alpha_\omega = \varphi_{\pi\omega} = \bigcup_{b\in G}\alpha_{\pi(b),\omega(b)}\,\varphi_{b,\pi(b)} = \bigcup_{b\in G}\alpha_{b,\omega(\pi^{-1}(b))}\,\varphi_{\pi^{-1}(b),b} = \alpha'_\omega\varphi_\pi$$

where $\alpha'_\omega := \bigcup_{b\in G}\alpha_{b,\omega(\pi^{-1}(b))}$.

We denote the set of mappings $\alpha_\omega\ (\in G(A))$ by H'. H' is a subgroup of H. Let us denote the set of mappings $\varphi_\pi(\in G(A))$ by P. P is a subgroup of T.

Corollary 6. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free then*

$$G(A) = PH' = H'P \quad \textit{and} \quad P\cap H' = \{\iota\}.$$

Proof. It is evident that $PH', H'P\subseteq G(A)$. Let $\alpha\in G(A)$. Then there exist $\varphi_\pi\in T$ and $\alpha_\omega\in H$ such that $\alpha=\varphi_\pi\alpha_\omega$, by Corollary 4. We show that $\varphi_\pi\in P$ and $\alpha_\omega\in H'$. Using the proof of Theorem 7, we get that the mapping $\pi\colon b\to c$ $(b, c\in G)$, where $\alpha(b)\in A_c$, is a transformation of G. Assume that $\alpha(b_1), \alpha(b_2)\in A_c$ $(b_1, b_2, c\in G)$.

Then there exist $h_1, h_2 \in F$ for which $\alpha(b_1)=\delta(c, h_1)$ and $\alpha(b_2)=\delta(c, h_2)$, that is, $b_1=\delta(\alpha^{-1}(c), h_1)$ and $b_2=\delta(\alpha^{-1}(c), h_2)$. By Theorem 1, $A_{b_1}=A_{b_2}$, that is, $b_1=b_2$. Thus π is one-to-one. Since $\alpha(b)\in A_c$ thus $\alpha(A_b)\subseteq A_c$. Thus, for every $c(\in G)$ there exists a $b(\in G)$ such that $\alpha(b)\in A_c$, since α is an automorphism. Therefore, π is a permutation of G, that is, $\varphi_\pi \in G(A)$. This means that $\alpha_\omega=\varphi_\pi^{-1}\alpha\in G(A)$. Since $\varphi_\pi\alpha_\omega=\alpha'_\omega\varphi_\pi$, where $\alpha'_\omega\in H$, thus $\alpha'_\omega=\varphi_\pi\alpha_\omega\varphi_\pi^{-1}\in G(A)$.

Corollary 7. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free, then* P *can be embedded homomorphically into the automorphism group of* H'.

Proof. It is clear that the mapping $\Theta_\pi\colon \alpha_\omega\to\alpha'_\omega$ is an automorphism of H' $(\varphi_\pi\in P, \alpha_\omega, \alpha'_\omega\in H')$. The mapping $\varphi_\pi\to\Theta_\pi$ $(\varphi_\pi\in P)$ is well-defined. Take arbitrary mappings $\varphi_{\pi_1}, \varphi_{\pi_2}$ $(\in P)$ and α_ω $(\in H')$. If

$$\varphi_{\pi_2}\alpha_\omega=\alpha_{\omega_1}\varphi_{\pi_2} \quad\text{and}\quad \varphi_{\pi_1}\alpha_{\omega_1}=\alpha_{\omega_2}\varphi_{\pi_1}\,(\alpha_{\omega_1}, \alpha_{\omega_2}\in H')$$

then

$$\varphi_{\pi_1\pi_2}\alpha_\omega=\varphi_{\pi_1}\varphi_{\pi_2}\alpha_\omega=\varphi_{\pi_1}\alpha_{\omega_1}\varphi_{\pi_2}=\alpha_{\omega_2}\varphi_{\pi_1}\varphi_{\pi_2}=\alpha_{\omega_2}\varphi_{\pi_1\pi_2},$$

thus,

$$\Theta_{\pi_1}\Theta_{\pi_2}(\alpha_\omega)=\Theta_{\pi_1}(\alpha_{\omega_1})=\alpha_{\omega_2}=\Theta_{\pi_1\pi_2}(\alpha_\omega),$$

that is, $\Theta_{\pi_1}\Theta_{\pi_2}=\Theta_{\pi_1\pi_2}$.

We note that if the quasi-automaton $\mathbf{A}$ is reversible and state-independent then $H'=H$ (see I. Babcsányi [4].)

Example 4.

A	1	2	3	4	5	6
x	3	3	3	6	6	6
y	2	1	3	5	4	6

$\bar{F}$	$\bar{x}$	$\bar{y}$	$\overline{y^2}$
$\bar{x}$	$\bar{x}$	$\bar{x}$	$\bar{x}$
$\bar{y}$	$\bar{x}$	$\overline{y^2}$	$\bar{y}$
$\overline{y^2}$	$\bar{x}$	$\bar{y}$	$\overline{y^2}$

$G=\langle 1; 4\rangle$ is a characteristically free generating system of $\mathbf{A}$.

$$\pi_1=\begin{pmatrix}1&4\\1&4\end{pmatrix},\quad \pi_2=\begin{pmatrix}1&4\\1&1\end{pmatrix},\quad \pi_3=\begin{pmatrix}1&4\\4&4\end{pmatrix},\quad \pi_4=\begin{pmatrix}1&4\\4&1\end{pmatrix};$$

$$\omega_1=\begin{pmatrix}1&4\\\bar{x}&\bar{x}\end{pmatrix},\quad \omega_2=\begin{pmatrix}1&4\\\bar{x}&\bar{y}\end{pmatrix},\quad \omega_3=\begin{pmatrix}1&4\\\bar{x}&\overline{y^2}\end{pmatrix},\quad \omega_4=\begin{pmatrix}1&4\\\bar{y}&\bar{x}\end{pmatrix},\quad \omega_5=\begin{pmatrix}1&4\\\bar{y}&\bar{y}\end{pmatrix},$$

$$\omega_6=\begin{pmatrix}1&4\\\bar{y}&\overline{y^2}\end{pmatrix},\quad \omega_7=\begin{pmatrix}1&4\\\overline{y^2}&\bar{x}\end{pmatrix},\quad \omega_8=\begin{pmatrix}1&4\\\overline{y^2}&\bar{y}\end{pmatrix},\quad \omega_9=\begin{pmatrix}1&4\\\overline{y^2}&\overline{y^2}\end{pmatrix}.$$

$$T=\langle \iota=\varphi_{\pi_1}, \varphi_{\pi_2}, \varphi_{\pi_3}, \varphi_{\pi_4}\rangle,\quad H=\langle \alpha_{\omega_i} \mid i=1, 2, \ldots, 9\rangle,$$

$$O(T)=4,\quad O(H)=9,\quad O(E(A))=O(TH)=|A|^{|G|}=6^2=36,$$

$$T\cap H=\{\iota\},\quad P=\langle \iota=\varphi_{\pi_1}, \varphi_{\pi_4}\rangle,\quad H'=\langle \alpha_{\omega_5}, \alpha_{\omega_6}, \alpha_{\omega_8}, \alpha_{\omega_9}=\iota\rangle.$$

$$\varphi_{\pi_4}\alpha_{\omega_5}=\alpha_{\omega_5}\varphi_{\pi_4},\quad \varphi_{\pi_4}\alpha_{\omega_6}=\alpha_{\omega_8}\varphi_{\pi_4}\quad\text{and}\quad \varphi_{\pi_4}\alpha_{\omega_8}=\alpha_{\omega_6}\varphi_{\pi_4},$$

that is, $G(A)=PH'=H'P$, $P\cap H'=\{\iota\}$.

$$|HT|=24.\quad \text{Therefore, } E(A)=TH\neq HT.$$

3. Reduced quasi-automata

In the paper [2] we introduced on the state set A of the quasi-automaton $\mathbf{A}=(A, F, \delta)$ the following congruence relation ϱ:

$$a\varrho b \Leftrightarrow \underset{f\in F}{\forall} f[\delta(a,f)=\delta(b,f)]. \tag{6}$$

The factor quasi-automaton $\bar{\mathbf{A}}:=\mathbf{A}/\varrho$ is said to be the *reduced quasi-automaton belonging to* $\mathbf{A}$. The quasi-automaton $\mathbf{A}=(A, F, \delta)$ is called *reduced* if for arbitrary $a, b\ (\in A)$:

$$a\varrho b \Rightarrow a=b.$$

We note that if $\bar{e}$ is a left identity element of $\bar{F}$ then

$$a\varrho b\,(a, b\in A) \Leftrightarrow \delta(a, e)=\delta(b, e).$$

If the characteristic semigroup $\bar{F}$ of a well-generated quasi-automaton $\mathbf{A}$ is a monoid, then $\mathbf{A}$ is reduced. The proof is obvious; we only note that $\mathbf{A}$ is well-generated if and only if

$$\underset{a\in A}{\forall} a[\delta(a, e)=a],$$

where $\bar{e}$ is a right identity element of $\bar{F}$ (see I. BABCSÁNYI [4]).

Denote the characteristic semigroup of $\bar{\mathbf{A}}=(\bar{A}, F, \bar{\delta})$ by $\bar{\bar{F}}$. Let $\bar{\bar{f}}$ be the element of $\bar{\bar{F}}$ represented by $f\,(\in F)$. Furthermore $\bar{a}$ is the element of $\bar{A}$ represented by $a\,(\in A)$.

Lemma 4. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free then the quasi-automaton* $\bar{\mathbf{A}}=(\bar{A}, F, \bar{\delta})$ *is characteristically free as well.*

Proof. Let G be a characteristically free generating system of $\mathbf{A}$. It is clear that the set $\bar{G}=\langle \bar{a}_0 | a_0\in G\rangle$ is a generating system of $\bar{A}$. Let

$$\bar{\delta}(\bar{a}_0, f)=\bar{\delta}(\bar{b}_0, g) \quad (a_0, b_0\in G;\ f, g\in F),$$

that is,

$$\underset{h\in F}{\forall} h[\delta(a_0, fh)=\delta(b_0, gh)].$$

Since G is characteristically free thus

$$a_0=b_0 \quad \text{and} \quad \underset{h\in F}{\forall} h[\bar{f}\bar{h}=\bar{g}\bar{h}],$$

thus, $\bar{a}_0=\bar{b}_0$ and $\bar{\bar{f}}=\bar{\bar{g}}$. This means that $\bar{G}$ is characteristically free.

Theorem 9. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free then* $E(A)\cong E(\bar{A})$.

Proof. Let G be a characteristically free generating system of $\mathbf{A}$. It is evident that all mappings $\varphi_{\bar{\pi}\bar{\omega}}$ of type (5) are endomorphisms of $\bar{\mathbf{A}}$ ($\bar{\pi}$: $\bar{G}\to\bar{G}$; $\bar{\omega}$: $\bar{G}\to F$).

Take the mapping $\Psi: E(A) \to E(\bar{A})$ for which

$$\Psi(\varphi_{\pi\omega}) = \varphi_{\bar{\pi}\bar{\omega}} \Leftrightarrow \underset{a_0 \in G}{\forall} a_0[\bar{\pi}(\bar{a}_0) = \overline{\pi(a_0)} \quad \text{and} \quad \bar{\omega}(\bar{a}_0) = \omega(a_0)].$$

Since the mapping $a_0 \to \bar{a}_0$ $(a_0 \in G)$ is one-to-one, thus the $\bar{\pi}$ and $\bar{\omega}$ are well-defined.

$$\varphi_{\pi\omega} = \varphi_{\pi'\omega'}(\in E(A)) \Rightarrow \underset{a_0 \in G}{\forall} a_0\big[\underset{f \in F}{\forall} f\big[\delta(\pi(a_0), \omega(a_0)f) = \delta(\pi'(a_0), \omega'(a_0)f)\big]\big] \Rightarrow$$

$$\Rightarrow \underset{a_0 \in G}{\forall} a_0\big[\overline{\delta(\pi(a_0), \omega(a_0))} = \overline{\delta(\pi'(a_0), \omega'(a_0))}\big] \Rightarrow$$

$$\Rightarrow \underset{\bar{a}_0 \in \bar{G}}{\forall} \bar{a}_0\big[\bar{\delta}(\bar{\pi}(\bar{a}_0), \bar{\omega}(\bar{a}_0))\big] = \bar{\delta}(\bar{\pi}'(\bar{a}_0), \bar{\omega}'(\bar{a}_0))\big] \Rightarrow \varphi_{\bar{\pi}\bar{\omega}} = \varphi_{\bar{\pi}'\bar{\omega}'}.$$

Conversely,

$$\varphi_{\bar{\pi}\bar{\omega}} = \varphi_{\bar{\pi}'\bar{\omega}'} \Rightarrow \underset{\bar{a}_0 \in \bar{G}}{\forall} \bar{a}_0\big[\underset{f \in F}{\forall} f\big[\bar{\delta}(\bar{\pi}(\bar{a}_0), \bar{\omega}(\bar{a}_0)f) = \bar{\delta}(\bar{\pi}'(\bar{a}_0), \bar{\omega}'(\bar{a}_0)f)\big]\big] \Rightarrow$$

$$\Rightarrow \underset{\bar{a}_0 \in G}{\forall} \bar{a}_0\big[\underset{f \in F}{\forall} f\big[\bar{\delta}(\overline{\pi(a_0)}, \bar{\omega}(\bar{a}_0)f) = \bar{\delta}(\overline{\pi'(a_0)}, \bar{\omega}'(\bar{a}_0)f)\big]\big].$$

Since $\overline{\pi(a_0)}, \overline{\pi'(a_0)} \in \bar{G}$ and $\bar{G}$ is a characteristically free generating system of $\bar{\mathbf{A}}$ thus

$$\underset{a_0 \in G}{\forall} a_0[\overline{\pi(a_0)} = \overline{\pi'(a_0)}],$$

that is,

$$\underset{a_0 \in G}{\forall} a_0\big[\underset{f \in F}{\forall} f\big[\delta(\pi(a_0), f) = \delta(\pi'(a_0), f)\big]\big].$$

But $\pi(a_0), \pi'(a_0) \in G$ and G is a characteristically free generating system of $\mathbf{A}$. Thus

$$\underset{a_0 \in G}{\forall} a_0[\pi(a_0) = \pi'(a_0)],$$

that is, $\pi = \pi'$. From this, using $\bar{\omega}(\bar{a}_0) = \omega(a_0)$ and $\bar{\omega}'(\bar{a}_0) = \omega'(a_0)$, we get that $\varphi_{\pi\omega} = \varphi_{\pi'\omega'}$. This means that Ψ is one-to-one. It is clear that Ψ is onto.

Let $\varphi_{\pi_1\omega_1}, \varphi_{\pi_2\omega_2} \in E(A)$ and $\delta(a_0, f)$ $(a_0 \in G, f \in F)$ an arbitrary state of $\mathbf{A}$. If $\pi := \pi_1\pi_2$ and $\omega(a_0) := \omega_1(\pi_2(a_0))\omega_2(a_0)$ then

$$\varphi_{\pi_1\omega_1}\varphi_{\pi_2\omega_2}(\delta(a_0, f)) = \varphi_{\pi_1\omega_1}(\delta(\pi_2(a_0), \omega_2(a_0)f)) =$$

$$= \delta(\pi_1\pi_2(a_0), \omega_1(\pi_2(a_0))\,\omega_2(a_0)f) = \varphi_{\pi\omega}(\delta(a_0, f)),$$

that is, $\varphi_{\pi_1\omega_1}\varphi_{\pi_2\omega_2} = \varphi_{\pi\omega}$. But $\bar{\pi}_1\bar{\pi}_2(\bar{a}_0) = \bar{\pi}_1(\overline{\pi_2(a_0)}) = \overline{\pi_1\pi_2(a_0)} = \bar{\pi}(\bar{a}_0)$ and $\bar{\omega}_1(\bar{\pi}_2(\bar{a}_0)) \cdot \bar{\omega}_2(\bar{a}_0) = \bar{\omega}_1(\overline{\pi_2(a_0)})\bar{\omega}_2(\bar{a}_0) = \omega_1(\pi_2(a_0))\omega_2(a_0)$. Therefore,

$$\varphi_{\bar{\pi}_1\bar{\omega}_1}\varphi_{\bar{\pi}_2\bar{\omega}_2}(\bar{\delta}(\bar{a}_0, f)) = \varphi_{\bar{\pi}_1\bar{\omega}_1}(\bar{\delta}(\bar{\pi}_2(\bar{a}_0), \bar{\omega}_2(\bar{a}_0)f)) =$$

$$= \bar{\delta}(\bar{\pi}_1\bar{\pi}_2(\bar{a}_0), \bar{\omega}_1(\bar{\pi}_2(\bar{a}_0))\bar{\omega}_2(\bar{a}_0)f) = \varphi_{\bar{\pi}\bar{\omega}}(\bar{\delta}(\bar{a}_0, f)),$$

that is, $\varphi_{\bar{\pi}_1\bar{\omega}_1}\varphi_{\bar{\pi}_2\bar{\omega}_2} = \varphi_{\bar{\pi}\bar{\omega}}$. Thus Ψ is an isomorphism of $E(A)$ onto $E(\bar{A})$.

We note that if $\pi \neq \pi'$ then $\varphi_{\pi\omega} \neq \varphi_{\pi'\omega'}$. Furthermore,

$$\varphi_{\pi\omega} = \varphi_{\pi\omega'} \Leftrightarrow \underset{a_0 \in G}{\forall} a_0[\overline{\overline{\omega(a_0)}} = \overline{\overline{\omega'(a_0)}}].$$

Corollary 8. *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free, then the characteristic semigroup* $\bar{\bar{F}}$ *of* $\bar{\mathbf{A}}$ *can be embedded isomorphically into the endomorphism semigroup* $E(A)$ *of* $\mathbf{A}$.

Proof. Let G be a characteristically free generating system of $\mathbf{A}$ and π the identity mapping on G. Denote the mapping $\varphi_{\pi\omega}$ by φ_h if

$$\forall_{a_0 \in G} a_0[\omega(a_0) = h].$$

It can clearly be seen that the mapping $\bar{\bar{h}} \to \varphi_h$ $(h \in F)$ is one-to-one. Let $h, k, f \in F$ and $a_0 \in G$. Then

$$\varphi_h \varphi_k(\delta(a_0, f)) = \varphi_h(\delta(a_0, kf)) = \delta(a_0, hkf) = \varphi_{hk}(\delta(a_0, f)),$$

that is, $\varphi_h \varphi_k = \varphi_{hk}$. Thus the mapping $\bar{\bar{h}} \to \varphi_h$ $(h \in F)$ is an isomorphism of $\bar{\bar{F}}$ into $E(A)$.

We note that the characteristic semigroup $\bar{F}$ of the characteristically free quasi-automaton $\mathbf{A}=(A, F, \delta)$ can be embedded homomorphically into $E(A)$. If $O(\bar{F})=1$ then every element of $\bar{F}$ is its left identity element. In this case $H=\{\iota\}$.

Corollary 9. *If the cyclic quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is characteristically free then* $E(A) \cong \bar{\bar{F}}$.

Proof. By Theorem 6, $E(A) \cong \bar{F}\bar{e}$. Since $\bar{e}$ is a left identity element of $\bar{F}$, thus the mapping $\bar{f}\bar{e} \to \bar{\bar{f}}$ $(f \in F)$ is an isomorphism of $\bar{F}\bar{e}$ onto $\bar{\bar{F}}$.

Corollary 10. *The characteristically free quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is reduced if and only if its characteristic semigroup is a monoid.*

Proof. By Lemma 2, there exists a left identity element $\bar{e}$ of $\bar{F}$, that is,

$$\forall_{a \in A} a\big[\forall_{f \in F} f[\delta(a, f) = \delta(a, ef) = \delta(\delta(a, e), f)]\big].$$

If $\mathbf{A}$ is reduced then

$$\forall_{a \in A} a[a = \delta(a, e)],$$

i.e. $\bar{e}$ is the identity element of $\bar{F}$. It is evident that if $\bar{F}$ is a monoid then $\mathbf{A}$ reduced.

The next result follows from Theorem 6 and Corollary 10.

Corollary 11. *The characteristically free cyclic quasi-automaton* $\mathbf{A}$ *is reduced if and only if* $\bar{F} \cong E(A)$.

Lemma 5. *Let the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *be characteristically free and* L *the set of left identity elements of* $\bar{F}$. *Then*

$$\forall_{a_0 \in G} a_0[\bar{a}_0 = \langle \delta(a_0, e) | \bar{e} \in L \rangle],$$

and for arbitrary pair a_0, b_0 $(\in G)$, $|\bar{a}_0| = |\bar{b}_0|$, *where* G *is a characteristically free generating system of* $\mathbf{A}$.

Proof. Let $\bar{a}_0=\bar{b}$ $(a_0\in G, b\in A)$. Then there exist $h\in F$ and $b_0\in G$ for which $\delta(b_0, h)=b$, thus,

$$\forall_{f\in F} f[\delta(a_0,f)=\delta(b,f)=\delta(b_0, hf)],$$

that is, $a_0=b_0$ and $\forall_{f\in F} f[\bar{f}=\bar{h}\bar{f}]$. Therefore, $\bar{h}\in L$. It is evident that if $\bar{e}\in L$ then $\delta(a_0, e)\in\bar{a}_0$. If $\delta(a_0, e_1)=\delta(a_0, e_2)$ $(a_0\in G;\ \bar{e}_1, \bar{e}_2\in L)$ then $\bar{e}_1=\bar{e}_2$, thus the mapping $\delta(a_0, e)\to\bar{e}$ $(\bar{e}\in L)$ is one-to-one; therefore, $|\bar{a}_0|=O(L)$ $(a_0\in G)$.

We note that for every state $a(\in A)$:

$$\bar{a}\supseteqq\langle\delta(a, e)|\bar{e}\in L\rangle$$

and $\bar{a}\subseteqq A_{a_0}$, where $a_0\in G$ and $a=\delta(a_0, h)$ $(h\in F)$.

Corollary 12. (I. Babcsányi [4].) *If the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is reversible and state-independent then* $\bar{a}=\langle\delta(a, e)|\bar{e}\in L\rangle$ $(a\in A)$ *and for every pair* $a, b\ (\in A)$, $|\bar{a}|=|\bar{b}|$.

Corollary 13. (I. Babcsányi [4].) *If the reversible state-independent quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is A-finite and there exists an* $a\ (\in A)$ *such that* $|A_a|$ *is a prime number, then the characteristic semigroup* $\bar{F}$ *of* $\mathbf{A}$ *is a group or every element of* $\bar{F}$ *is its left identity element.*

Proof. By Corollary 12, $|\bar{a}|$ is a divisor of $|A_a|$ $(a\in A)$. If $|A_a|$ is a prime number then $|\bar{a}|=1$ or $|\bar{a}|=|A_a|$. If $|\bar{a}|=1$ then, also by Corollary 12, $|\bar{b}|=1$ for every $b(\in A)$. This implies that $\bar{F}$ is a group. If $|\bar{a}|=|A_a|$ then for every state $b(\in A_a)$,

$$\forall_{f\in F} f[\delta(a,f)=\delta(b,f)].$$

Since for every $h(\in F)$, $\delta(a, h)\in A_a$ thus

$$\forall_{f\in F} f[\delta(a,f)=\delta(a, hf)],$$

that is,

$$\forall_{f\in F} f[\bar{f}=\bar{h}\bar{f}].$$

Therefore, $\bar{h}$ is a left identity element of $\bar{F}$.

Let the characteristically free quasi-automaton $\mathbf{A}=(A, F, \delta)$ be cyclic and a_0 a characteristically free generating element of $\mathbf{A}$. $\delta(a_0, h)$ $(h\in F)$ is a characteristically free generating element of $\mathbf{A}$ if and only if the mapping $\alpha_{a_0,h}$ (see (3)) is an automorphism of $\mathbf{A}$. This means that the cardinal number of the set of characteristically free generating elements equals $O(G(A))$.

In Example 2 $\overline{(i, 1)}=\langle(i, 1)\rangle$; $\overline{(i, 2)}=\langle(i, 2);\ (i, 4)\rangle$; $\overline{(i, 2j+1)}=\langle(i, 2j+1);\ (i, 2j+4)\rangle$ $(i, j=1, 2, 3, \ldots)$. $\bar{F}=\langle\overline{x^k};\ \overline{y^k};\ \overline{xy};\ \overline{y^l x^k}|k, l=1, 2, 3, \ldots\rangle$. $E(A)\cong\bar{F}$ and $G(A)=\{\iota\}$.

Theorem 10. *If the characteristically free quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is cyclic then the quasi-automaton* $\mathbf{E}(\mathbf{A})=(E(A), F, \delta')$ *is well-defined, where*

$$\delta'(\alpha_{a_0,h}, f)=\alpha_{a_0,hf}\quad(f\in F)$$

and $\mathbf{E}(\mathbf{A})\cong\bar{\mathbf{A}}$.

Proof. Since

$$\alpha_{a_0,h} = \alpha_{a_0,k} \Leftrightarrow \underset{f\in F}{\forall} f[\delta(a_0, hf) = \delta(a_0, kf)],$$

thus

$$\alpha_{a_0,h} = \alpha_{a_0,k} \Rightarrow \underset{f\in F}{\forall} f[\alpha_{a_0,hf} = \alpha_{a_0,kf}].$$

Furthermore,

$$\delta'(\alpha_{a_0,h}, fg) = \alpha_{a_0,hfg} = \delta'(\alpha_{a_0,hf}, g) = \delta'(\delta'(\alpha_{a_0,h}, f), g)$$

($h, k, f, g \in F$; a_0 is a characteristically free generating element of **A**.), that is, **E**(**A**) is well-defined. The mapping $\Psi: E(A) \to \bar{A}$ for which

$$\Psi: \alpha_{a_0,h} \to \overline{\delta(a_0, h)} \quad (h \in F)$$

is one-to-one and onto. Finally, we shall show that Ψ is a homomorphism. Take arbitrary elements $\alpha_{a_0,h} \in E(A)$ and $f \in F$. Then

$$\Psi(\delta'(\alpha_{a_0,h}, f)) = \Psi(\alpha_{a_0,hf}) = \overline{\delta(a_0, hf)} =$$
$$= \overline{\delta(\delta(a_0, h), f)} = \bar{\delta}(\overline{\delta(a_0, h)}, f) = \bar{\delta}(\Psi(\alpha_{a_0,h}), f).$$

Theorem 11. *If the characteristically free quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is cyclic, then* $E(E(A))$ *is the semigroup of left translations of* $E(A)$ *and* $E(E(A)) \cong E(A)$.

Proof. Note that $E(E(A))$ denote the endomorphism semigroup of **E**(**A**).
Let $\alpha_{a_0,h}, \alpha_{a_0,k} (\in E(A))$ be arbitrary endomorphisms and $\mu \in E(E(A))$. Then

$$\mu(\alpha_{a_0,h}\alpha_{a_0,k}) = \mu(\alpha_{a_0,hk}) = \mu(\delta'(\alpha_{a_0,h}, k)) = \delta'(\mu(\alpha_{a_0,h}), k) =$$
$$= \delta'(\alpha_{a_0,g}, k) = \alpha_{a_0,gk} = \alpha_{a_0,g}\alpha_{a_0,k} = \mu(\alpha_{a_0,h})\alpha_{a_0,k},$$

where $h, k, g \in F$ and $\mu(\alpha_{a_0,h}) = \alpha_{a_0,g}$. This means that μ is a left translation of $E(A)$. Conversely, if μ is a left translation of $E(A)$, then

$$\mu(\delta'(\alpha_{a_0,h}, f)) = \mu(\alpha_{a_0,hf}) = \mu(\alpha_{a_0,h}\alpha_{a_0,f}) = \mu(\alpha_{a_0,h})\alpha_{a_0,f} =$$
$$= \alpha_{a_0,g}\alpha_{a_0,f} = \alpha_{a_0,gf} = \delta'(\alpha_{a_0,g}, f) = \delta'(\mu(\alpha_{a_0,h}), f),$$

where $f \in F$ and $\mu(\alpha_{a_0,h}) = \alpha_{a_0,g}$, i.e. μ is an endomorphism of **E**(**A**). It is well-known that every monoid is isomorphic to the semigroup of its left translations.

We note that if the quasi-automaton $\mathbf{A}=(A, F, \delta)$ is cyclic and characteristically free, a_0 is a characteristically free generating element of **A**, $\delta(a_0, e) = a_0 (e \in F)$ and $A_e := \langle \delta(a_0, fe) | f \in F \rangle$, then the quasi-automaton $\mathbf{A}_e = (A_e, Fe, \delta_e)$ is well-defined. $\mathbf{A}_e$ is a reduced sub-quasi-automaton of **A** and $\overline{Fe^{A_e}} \cong \bar{\bar{F}}$.

Theorem 12. *If the endomorphism semigroup* $E(A)$ *of the characteristically free cyclic quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *is isomorphic to the direct product of semigroups* E_i $(i=1, 2, \ldots, n)$ *then* $\bar{\mathbf{A}}$ *is isomorphic to the A-direct product of reduced characteristically free cyclic quasi-automata* $\mathbf{A}_i=(A_i, F, \delta_i)$ *and* $E(A_i) \cong E_i$.

Proof. It is sufficient to prove this theorem for $n=2$. Let $E(A) \cong E_1 \otimes E_2$. We can assume that $E(A) = E_1 \otimes E_2$. By Theorem 10, $\mathbf{E}(\mathbf{A}) \cong \bar{\mathbf{A}}$. Let $\alpha_{a_0,h} := (\alpha_{1,h}, \alpha_{2,h})$

($\alpha_{i,h} \in E_i$, $i=1, 2$). Since

$$(\alpha_{1,hf}, \alpha_{2,hf}) = \alpha_{a_0,hf} = \alpha_{a_0,h}\alpha_{a_0,f} = (\alpha_{1,h}, \alpha_{2,h})(\alpha_{1,f}, \alpha_{2,f}) = (\alpha_{1,h}\alpha_{1,f}, \alpha_{2,h}\alpha_{2,f})$$

thus $\alpha_{i,hf}=\alpha_{i,h}\alpha_{i,f}$. This means that the mappings δ_i: $E_i \times F \to E_i$ given by

$$\delta_i(\alpha_{i,h}, f) = \alpha_{i,hf}$$

are well-defined. Furthermore, the quasi-automata $\mathbf{E}_i=(E_i, F, \delta_i)$ are also well-defined.

$$\delta'((\alpha_{1,h}, \alpha_{2,h}), f) = \delta'(\alpha_{a_0,h}, f) = \alpha_{a_0,hf} =$$
$$= (\alpha_{1,hf}, \alpha_{2,hf}) = (\delta_1(\alpha_{1,h}, f), \delta_2(\alpha_{2,h}, f)),$$

that is, $\mathbf{E}(\mathbf{A})=\mathbf{E}_1 \otimes \mathbf{E}_2$. Thus $\overline{\mathbf{A}} \cong \mathbf{E}_1 \otimes \mathbf{E}_2$. It is evident that $\alpha_{a_0,e}$ is a characteristically free generating element of $\mathbf{E}(\mathbf{A})$, where a_0 is a characteristically free generating element of $\mathbf{A}$ and $\delta(a_0, e)=a_0$ ($e \in F$). Prove that $\alpha_{i,e}$ ($i=1, 2$) is a characteristically free generating element of $\mathbf{E}_i$. Let

$$\alpha_{i,f} = \delta_i(\alpha_{i,e}, f) = \delta_i(\alpha_{i,e}, g) = \alpha_{i,g} \quad (f, g \in F).$$

Then for every $h \in F$,

$$\delta_i(\alpha_{i,h}, f) = \alpha_{i,hf} = \alpha_{i,h}\alpha_{i,f} = \alpha_{i,h}\alpha_{i,g} = \alpha_{i,hg} = \delta_i(\alpha_{i,h}, g),$$

that is $\bar{f}^{E_i}=\bar{g}^{E_i}$. Therefore, the quasi-automata $\mathbf{E}_i$ are cyclic and characteristically free. From Theorem 6 it follows that β_h: $\alpha_{i,f} \to \alpha_{i,hf}$ ($f \in F$) is an endomorphism of $\mathbf{E}_i$, and for arbitrary endomorphism β of $\mathbf{E}_i$ there exists an $h \in F$ such that $\beta=\beta_h$.

$$\beta_h = \beta_k (h, k \in F) \Leftrightarrow \underset{f \in F}{\forall} f[\alpha_{i,hf} = \alpha_{i,kf}] \Leftrightarrow \alpha_{i,he} = \alpha_{i,ke}.$$

But $\alpha_{i,h}=\alpha_{i,he}$ and $\alpha_{i,k}=\alpha_{i,ke}$. Therefore, the mapping $\beta_h \to \alpha_{i,h}$ ($h \in F$) is a one-to-one mapping of $E(E_i)$ onto E_i. Since $\beta_f\beta_g=\beta_{fg}$ ($f, g \in F$), thus the mapping $\beta_h \to \alpha_{i,h}$ ($h \in F$) is an isomorphism. Let $\overline{\alpha_{i,h}}=\overline{\alpha_{i,k}}$, that is,

$$\underset{f \in F}{\forall} f[\delta_i(\alpha_{i,h}, f) = \delta_i(\alpha_{i,k}, f)].$$

Thus $\alpha_{i,h}=\alpha_{i,he}=\alpha_{i,ke}=\alpha_{i,k}$. Therefore, the quasi-automata $\mathbf{E}_i$ are reduced.

Corollary 14. *The reduced characteristically free cyclic quasi-automaton* $\mathbf{A}$ *is isomorphic to the A-direct product of reduced characteristically free cyclic quasi-automata* $\mathbf{A}_i$ ($i=1, 2, ..., n$) *if* $E(A) \cong E(A_1) \otimes E(A_2) \otimes \ldots \otimes E(A_n)$.

Example 5.

$\mathbf{A}_1$	1	2
x	1	2
y	2	2

$\mathbf{A}_2$	3	4
x	4	3
y	4	3

$\mathbf{A}_1 \otimes \mathbf{A}_2$	(1,3)	(1,4)	(2,3)	(2,4)
x	(1,4)	(1,3)	(2,4)	(2,3)
y	(2,4)	(2,3)	(2,4)	(2,3).

1 is characteristically free generating element of $\mathbf{A}_1$. 3 and 4 are characteristically free generating element of $\mathbf{A}_2$. $\mathbf{A}_1$ and $\mathbf{A}_2$ are reduced. $E(A_1)=\langle \alpha_1, \beta_1 \rangle$, where $\alpha_1=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $\beta_1=\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ $E(A_2)=\langle \alpha_2, \beta_2 \rangle$, where $\alpha_2=\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$ and $\beta_2=\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$, $E(A_1 \times A_2)=E(A_1) \otimes E(A_2)$.

4. Homomorphism

Let $\mathbf{A}=(A, F, \delta)$ be a quasi-automaton and $I\,(\notin F)$ an arbitrary symbol. Define the semigroup F^I to be $F\cup\{I\}$, multiplication in F is unchanged and I acts as an identity for $F\cup\{I\}$. Furthermore, let φ be a mapping of A into itself and $\delta_\varphi\colon A\times F^I\to A$ such that

$$\delta_\varphi(a,f)=\begin{cases}\delta(a,f) & \text{if } f\in F\\ \varphi(a) & \text{if } f=I\end{cases}\quad (a\in A). \tag{7}$$

Lemma 6. (I. Babcsányi [4].) *The quasi-automaton $\mathbf{A}_\varphi:=(A, F^I, \delta_\varphi)$ is well-defined if and only if φ is an idempotent endomorphism of the quasi-automaton $\mathbf{A}=(A, F, \delta)$ and the restriction of φ to the kernel of $\mathbf{A}$ is the identity mapping. In this case $\mathbf{A}$ is sub-quasi-automaton of $\mathbf{A}_\varphi$.*

Proof. Necessity. Assume that the quasi-automaton $\mathbf{A}_\varphi$ is well-defined. Let $a(\in A)$ be an arbitrary state. Then

$$\varphi(a)=\delta_\varphi(a, I)=\delta_\varphi(a, I^2)=\delta_\varphi\big(\delta_\varphi(a, I), I\big)=\varphi^2(a),$$

that is, $\varphi^2=\varphi$. Furthermore, if $f\in F$ then

$$\delta_\varphi(a, If)=\delta_\varphi(a,fI)=\delta_\varphi(a,f)=\delta(a,f),$$

$$\delta_\varphi\big(\delta_\varphi(a,f), I\big)=\delta_\varphi\big(\delta(a,f), I\big)=\varphi\big(\delta(a,f)\big),$$

$$\delta_\varphi\big(\delta_\varphi(a, I),f\big)=\delta_\varphi\big(\varphi(a),f\big)=\delta\big(\varphi(a),f\big).$$

Since $\mathbf{A}_\varphi$ is well-defined, thus

$$\delta(a,f)=\varphi\big(\delta(a,f)\big)=\delta\big(\varphi(a),f\big).$$

This means that φ is an idempotent endomorphism of $\mathbf{A}$ and $\varphi|A_1=\iota$ (A_1 is the state set of the kernel of $\mathbf{A}$ (see (1))). The proof of sufficiency is similar. Since F is a subsemigroup of F^I and δ coincides with the restriction of δ_φ to $A\times F$, thus $\mathbf{A}$ is sub-quasi-automaton of $\mathbf{A}_\varphi$.

Theorem 13. (I. Babcsányi [4].) *Every homomorphism of the quasi-automaton $\mathbf{A}_\varphi=(A, F^I, \delta_\varphi)$ is a homomorphism of the quasi-automaton $\mathbf{A}=(A, F, \delta)$. Conversely, if Ψ is a homomorphism of $\mathbf{A}$ onto the quasi-automaton $\mathbf{B}=(B, F, \delta')$, then Ψ is a homomorphism of $\mathbf{A}_\varphi$ onto $\mathbf{B}_{\varphi'}$ if and only if $\Psi\varphi=\varphi'\Psi$.*

Proof. Since A is the state set of $\mathbf{A}$ and $\mathbf{A}_\varphi$, furthermore, $\mathbf{A}$ is a sub-quasi-automaton of $\mathbf{A}_\varphi$, thus every homomorphism of $\mathbf{A}_\varphi$ is a homomorphism if $\mathbf{A}$. Conversely, let Ψ be a homomorphism of $\mathbf{A}$ onto $\mathbf{B}$. φ and φ' are mappings of type (7). It is clear that Ψ is a homomorphism of $\mathbf{A}_\varphi$ onto $\mathbf{B}_{\varphi'}$, if and only if

$$\forall_{a\in A}\, a\big[\Psi\varphi(a)=\Psi\big(\delta_\varphi(a, I)\big)=\delta'_{\varphi'}\big(\Psi(a), I\big)=\varphi'\Psi(a)\big],$$

that is, $\Psi\varphi=\varphi'\Psi$.

We note that if φ is the identity mapping of A, then the homomorphisms of $\mathbf{A}$ and $\mathbf{A}_\varphi$ coincide. In this case denote $\mathbf{A}_\varphi$ by $\mathbf{A}_I=(A, F^I, \delta_I)$.

Theorem 14. *Let* $\mathbf{A}=(A, F, \delta)$ *be an arbitrary quasi-automaton. There exists a characteristically free quasi-automaton* $\mathbf{B}=(B, F, \delta')$ *such that* $\mathbf{A}_I$ *is the homomorphic image of* $\mathbf{B}$ *and the characteristic semigroups of* $\mathbf{A}_I$ *and* $\mathbf{B}$ *are equal.*

Proof. Take the quasi-automaton $\mathbf{A}_I=(A, F^I, \delta_I)$. Let G be a generating system of A_I. Define the following relation τ on $G\times F^I$:

$$(b,f)\tau(c,g) \Leftrightarrow b=c \quad \text{and} \quad \bar{f}^{A_I}=\bar{g}^{A_I}(b,c\in G;\ f,g\in F^I).$$

It is clear that τ is an equivalence relation. Let C_τ be the partition on $G\times F^I$ induced by τ. $C_\tau(A)$ is the set of the classes $C_\tau(b,f)$ $(b\in G, f\in F^I)$. Consider the mapping $\delta' : C_\tau(A)\times F^I\to C_\tau(A)$ for which

$$\delta'(C_\tau(b,f),h)=C_\tau(b,fh).$$

Let $g,h\in F^I$. Then

$$\delta'(C_\tau(b,f),gh)=C_\tau(b,fgh)=\delta'(C_\tau(b,fg),h)=\delta'(\delta'(C_\tau(b,f),g),h),$$

that is, the quasi-automaton $\mathbf{C}_\tau(A)=(C_\tau(A), F^I, \delta')$ is well-defined. We prove that $\overline{F^I}$ is the characteristic semigroup of $\mathbf{C}_\tau(A)$:

$$\bar{f}^{A_I}=\bar{g}^{A_I} \Leftrightarrow \underset{h\in F^I}{\forall} h[\bar{h}^{A_I}\bar{f}^{A_I}=\bar{h}^{A_I}\bar{g}^{A_I}] \Leftrightarrow$$

$$\Leftrightarrow \underset{h\in F^I}{\forall} h\big[\underset{b\in G}{\forall} b[C_\tau(b,hf)=C_\tau(b,hg)]\big] \Leftrightarrow$$

$$\Leftrightarrow \underset{C_\tau(b,h)\in C_\tau(A)}{\forall} C_\tau(b,h)[\delta'(C_\tau(b,h),f)=\delta'(C_\tau(b,h),g)] \Leftrightarrow \bar{f}^{C_\tau(A)}=\bar{g}^{C_\tau(A)}.$$

The set $G_I := \langle C_\tau(b,I)\,|\,b\in G\rangle$ is a generating system of $C_\tau(A)$. Let

$$C_\tau(b,f)=\delta'(C_\tau(b,I),f)=\delta'(C_\tau(c,I),g)=C_\tau(c,g)$$

$(b,c\in G; f,g\in F^I)$. Then $b=c$ and $\bar{f}^{A_I}=\bar{g}^{A_I}$. Thus $C_\tau(b,I)=C_\tau(c,I)$ and $\bar{f}^{C_\tau(A)}=\bar{g}^{C_\tau(A)}$, i.e., $\mathbf{C}_\tau(A)$ is characteristically free. The mapping

$$\Psi : C_\tau(b,f)\to\delta_I(b,f)(b\in G, f\in F^I)$$

is a homomorphism of $\mathbf{C}_\tau(A)$ onto $\mathbf{A}_I$.

Example 6. Take again the quasi-automaton $\mathbf{A}$ given in the Example 3.

$\mathbf{A}_I$	1	2	3
I	1	2	3
x	2	1	2
y	2	3	2

$G=\langle 2\rangle$

$\overline{F^I}=\langle \bar{x}, \overline{x^2}, \bar{y}, \overline{y^2}, \bar{I}\rangle$

$\mathbf{C}_\tau(A)$	$C_\tau(2,I)$	$C_\tau(2,x)$	$C_\tau(2,x^2)$	$C_\tau(2,y)$	$C_\tau(2,y^2)$
I	$C_\tau(2,I)$	$C_\tau(2,x)$	$C_\tau(2,x^2)$	$C_\tau(2,y)$	$C_\tau(2,y^2)$
x	$C_\tau(2,x)$	$C_\tau(2,x^2)$	$C_\tau(2,x)$	$C_\tau(2,x^2)$	$C_\tau(2,x)$
y	$C_\tau(2,y)$	$C_\tau(2,y^2)$	$C_\tau(2,y)$	$C_\tau(2,y^2)$	$C_\tau(2,y)$

$$\Psi=\begin{pmatrix} C_\tau(2,I) & C_\tau(2,x) & C_\tau(2,x^2) & C_\tau(2,y) & C_\tau(2,y^2) \\ 2 & 1 & 2 & 3 & 2 \end{pmatrix}$$

Corollary 15. *Let the quasi-automaton* $\mathbf{A}=(A, F, \delta)$ *be well-generated and* $\bar{F}^A$ *a monoid. There exists a characteristically free quasi-automaton* $B=(B, F, \delta')$ *such that* $\mathbf{A}$ *is a homomorphic image of* $\mathbf{B}$ *and* $\bar{F}^B=\bar{F}^A$.

By Theorem 14 the proof is evident. (The identity element of $\bar{F}^A$ acts as I.)

Характеристично свободные квазиавтоматы

А-подквазиавтомат $\mathbf{A}_1=(A_1 F, \delta_1)$ квазиавтомата $\mathbf{A}=(A, F, \delta)$ называется *ягром* автомата $\mathbf{A}$, если $A_1=\langle\delta(a, f)|a\in A, f\in F\rangle$. $\mathbf{A}$ называется *верно-порождённым* если $A=A_1$. Верно-порождённый квазиавтомат $\mathbf{A}$ называется *характеристично свободным* если (2) выполняется. (G есть неприводимая система образующих в квазиавтомате $\mathbf{A}$) $\bar{F}^A$ (или $\bar{F}$) является характеристической подгруппой квазиавтомата $\mathbf{A}$.

Квазиавтомат $\mathbf{A}=(A, F, \delta)$ характеристично свободный тогда и только тогда, когда он прямая сумма изоморфных характеристично свободных циклических квазиавтоматов (Теорема 1.). Если циклический квазиавтомат $\mathbf{A}$ характеристично свободный, тогда $|A|=0(\bar{F})$. (Теорема 3.). Если $\mathbf{A}$ ещё А-конечный, тогда теорема 3. можно повернуть. (Следствие 1.). Характеристично свободный $\mathbf{A}$ от состоянии независимый тогда и только тогда, когда его характеристическая полугруппа авляется с левым сокращением (Теорема 5.).

Во втором цункте получаем все ендоморфизмы характеристично свободных квазиавтоматов (Теорема 6. и 7.)

В третем пункте проводим отношение ϱ (в. ещё [2]) на множестве состояний $\mathbf{A}$ квазиавтомата $\mathbf{A}=(A, F, \delta)$. Отношение ϱ конгруенция. $\mathbf{A}$ называется *ограниченным*, если $a\varrho b$ $(a, b\in A\Rightarrow \Rightarrow a=b$. Если $\mathbf{A}$ характеристично свободный, тогда факторквазиавтомат $\bar{\mathbf{A}}:=\mathbf{A}/\varrho$ квазиавтомата $\mathbf{A}$ тоже характеристично свободный (Лемма 4.) и $E(A)\cong E(\bar{A})$ (Теорема 9). (Через $E(A)$ обозначаем полугруппу всех ендоморфизмых $\mathbf{A}$.)

Если $\mathbf{A}$ характеристично свободный циклический квазиавтомат и $E(A)\cong E_1\otimes E_2\otimes \otimes\ldots\otimes E_n$, тогда $\bar{\mathbf{A}}\cong\mathbf{A}_1\otimes\mathbf{A}_2\otimes\ldots\otimes\mathbf{A}_n$, где $\mathbf{A}_i$ $(i=1, 2, \ldots, n)$ характеристично свободные циклические ограниченные квазиавтоматы и $E(A_i)\cong E_i$ (Теорема 12.).

Если $\mathbf{A}=(A, F, \delta)$ верно — порождённый квазиавтомат и $\bar{F}^A$ обладает двусторонной единицей, тогда существует такой характеристично свободный квазиавтомат $\mathbf{B}=(B, F, \delta')$, что $\mathbf{A}$ есть гомоморфный образ квазиавтомата $\mathbf{B}$ и $\bar{F}^A=\bar{F}^B$ (Следствие 15.).

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