On some open problems of applied automaton theory and graph theory (suggested by the mathematical modelling of certain neuronal networks)

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Introduction

The branch of investigations to which this paper is devoted was initiated by U. Kling and Gy. Székely [14]. They studied some kind of electronical networks simulating certain nervous activities, and facilitating the quantitative treatment of such phenomena. The description of structure and function of such networks was continued in the articles [2], [7], [3] etc., using mathematical tools.

This paper contains a (more or less detailed) survey of the mathematical considerations mentioned above and — primarily — a list of numerous open problems. The article consists of four chapters. In Chapter I certain finite directed graphs are considered. The questions raised here may mostly be viewed as "variations on the theme" of describing graph classes each of which is a natural extension of the class of single cycles (in one or another sense). Chapter II starts with a systematization of the behaviour of autonomous continuous automata and, as a par-

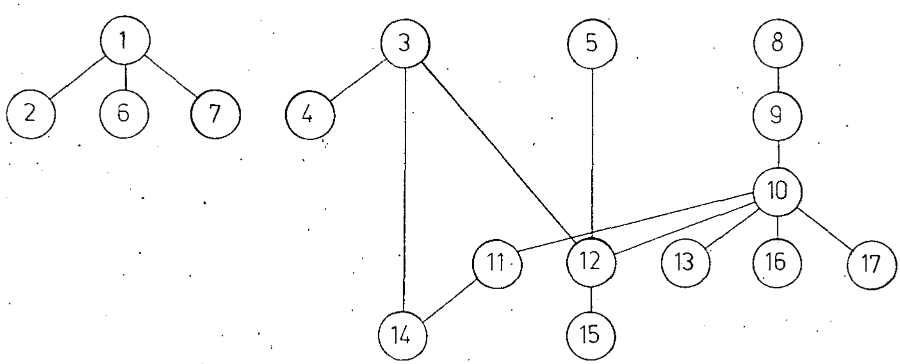


Fig. 1
Subordination of the sections of the paper

ticular case, deals chiefly with the network notions without supposing any special graph structure. In Chapter III the behaviour of networks with special structure is treated and the problem of speed of propagating actions is posed. In Chapter IV some questions of stochastic behaviour of networks are touched.

The first half of the paper contains a number of assertions, too. Some of them (e.g. Proposition 8) are easy consequences of the concepts defined or are "folkloristically" known to the specialists of the topics (as Propositions 12—14). Only the statements of § 4 and § 7 could be regarded as (more or less) original results.

The subject of Chapter I has its own importance in the theory of graphs. The considerations referred to in Chapters II—IV may be estimated rather as an attempt how a certain type of questions admits an exact mathematical treatment, than settled, definitive scientific advances.

The exposed problems are partly strictly determined ones (as Problem 14), partly proposals for making researches in some intuitively encircled field (e.g. Problem 7), or of transitional character between these extremities.

In the assembling of the material of this paper I was not free from some subjectiveness. The variety of ideas in the first chapter has followed from my affection for structural descriptions; on the other hand, the fact that I am no probability theorist has implied that the (very important) questions of stochastic behaviour appear in a smaller extent than they would deserve.

I. Structural problems (Problems concerning graphs)[1]

§ 1.

In this § we deal always with strongly connected directed finite graphs (see Chapter 16 of [12]). The graph with one vertex and without edges is excluded. We do not allow loops and parallel edges with the same orientation. By a *cycle* (of a graph) we mean a circuit (without repeated vertices) along which each edge e is passed through in sense of the orientation of e. For any edge e the number of cycles containing e is denoted by $Z(e)$; similarly, $Z(A)$ is the number of cycles in which a vertex A occurs. The strong connectedness implies $Z(e) \geqq 1$ for every edge and $Z(A) \geqq 1$ for every vertex of the graph. (Conversely, if a connected graph is not strongly connected, then $Z(e)=0$ for some edge of it.)

If a graph G can be represented as the union of two subgraphs G_1, G_2 (each having at least one edge) such that G_1 and G_2 have only one vertex A in common, then A is called a *cut vertex*[2]. If a vertex A of a graph G is contained in every cycle of G, then A is called a *pancyclic vertex*.

Now let four properties (α), (β), (γ), (δ) of graphs be defined. (For the sake of brevity we shall say e.g. "($\alpha\beta$)-graph" instead of "graph satisfying (α) and (β)".)

(α) $Z(e) \leqq 2$ for every edge e of the graph,

(β) $Z(A) \leqq 2$ for every vertex A of the graph,

(γ) the graph has no cut vertex,

[1] Chapter I has already been propagated in preprint form under the title "Some open questions concerning finite directed graphs".

[2] The term "articulation vertex" is also used.

(δ) the graph has a pancyclic vertex.

The main question to be proposed in this § is:

PROBLEM 1 (see [3]). *Describe the structure of all* (α)*-graphs.*

In some particular cases, Problem 1 has been solved. § 3 of [3] deals with the structural description of (β)-graphs, and any (β)-graph is clearly an (α)-graph. On the other hand, [4] is devoted to the description of (αγδ)-graphs and the extension of this to (αδ)-graphs.

It seems that the difficulty of solving Problem 1 lies in getting an overview of the (αγ)-graphs, therefore we formulate separately

PROBLEM 2. *Describe the structure of all* (αγ)*-graphs.*

In § 7 we shall give a relative solution of Problem 1 presupposing that Problem 2 is settled.

§ 2.

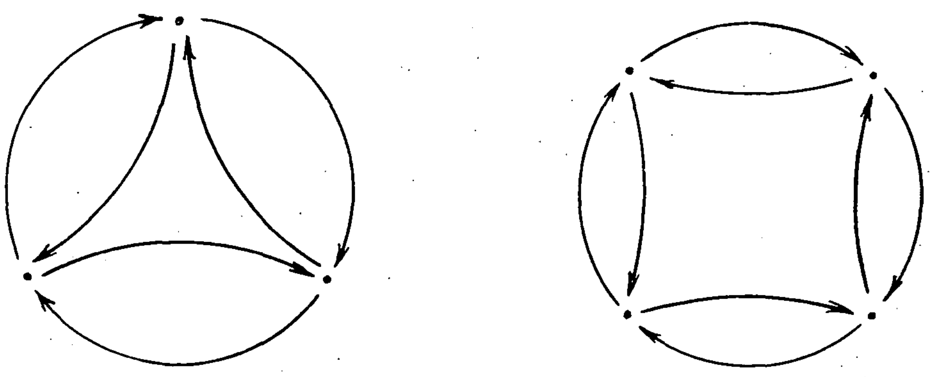
The terminology of § 1 is continued, especially, the condition of strong connectedness is maintained. Let us recall Property (δ) and define two related properties:

(ε) The graph has a vertex A such that
for any choice of the vertex B there is a cycle containing both A and B, and
for any choice of the edge e there is a cycle containing both A and e.

(ζ) Each pair of cycles has at least one vertex in common.

It is trivial that each (δ)-graph is an (εζ)-graph and each (εζ)-graph is an (ε)-graph. The examples on Fig. 2 show that these inclusions (concerning the classes of (δ)-graphs, (εζ)-graphs and (ε)-graphs) are proper[3]. The connection of (ε) and (ζ) is questionable as follows:

PROBLEM 3. *Does there exist a* (ζ)*-graph which does not satisfy* (ε)*?*



(a) (b)

Fig. 2

[3] The graph on Fig. 2/b was called to my attention by Dr. B. Zelinka.

§ 3.

Let H be a set with $n(\geqq 2)$ elements. A permutation α of H is called *cyclic* if α fixes q elements and permutes the remaining $n-q$ ones cyclically (where $0 \leqq q \leqq n-2$ or $q=n$). α is *fully cyclic* if it is cyclic with $q=0$. By a *fully cyclic automorphism* of a graph such an automorphism is understood which acts as a fully cyclic permutation on the set of vertices.

In contrasting with § 1, now we do not restrict ourselves to connected graphs[4]. Let us choose some integers $n, k, m_1, m_2, \ldots, m_k$ such that

(3.1) $$k < n, \quad 1 \leqq m_1 < m_2 < \ldots < m_k < n.$$

We define the (labelled) graph $G = G(n; m_1, m_2, \ldots, m_k)$ as follows:

the vertices of G are denoted by $P_1, P_2, \ldots, P_n$;

the edge from P_i to P_j (where $1 \leqq i \leqq n$, $1 \leqq j \leqq n$) exists in G if and only if there is a number h $(1 \leqq h \leqq k)$ for which the congruence

$$i - j \equiv m_h \pmod{n}$$

holds.

Proposition 1. *A graph G can be expressed as $G(n; m_1, m_2, \ldots, m_k)$ if and only if G has a fully cyclic automorphism.*

Proof. Let $G(n; m_1, m_2, \ldots, m_k)$ be considered. The following (fully cyclic) vertex permutation α is obviously an automorphism:

$$\alpha(P_i) = \begin{cases} P_{i+1} & \text{if} \quad 1 \leqq i < n, \\ P_1 & \text{if} \quad i = n. \end{cases}$$

Conversely, suppose that a graph G (having n vertices) possesses a fully cyclic automorphism α. For an arbitrary vertex A, let us introduce the notation

$$A = P_1, \quad \alpha(A) = P_2, \quad \alpha^2(A) = P_3, \quad \alpha^3(A) = P_4, \quad \alpha^{n-1}(A) = P_n$$

and let $m_1, m_2, \ldots, m_k$ be defined by the conditions

$m_1 < m_2 < \ldots < m_k$,

the edges $\overrightarrow{P_n P_{m_1}}, \overrightarrow{P_n P_{m_2}}, \ldots, \overrightarrow{P_n P_{m_k}}$ exist in G, and

there are no other edges from P_n.

It is easy to see that G equals $G(n; m_1, m_2, \ldots, m_k)$. □

For given n and k, two sequences $(m_1, m_2, \ldots, m_k)$ and $(m_1', m_2', \ldots, m_k')$ (fulfilling (3.1)) are called *equivalent* (for n) if there exists a number r $(1 \leqq r < n)$ and a permutation π of the set $\{1, 2, \ldots, k\}$ such that r is relatively prime to n and the congruences

(3.2) $$rm_1 \equiv m'_{\pi(1)}, \quad rm_2 \equiv m'_{\pi(2)}, \ldots, \quad rm_k \equiv m'_{(\pi)k}$$

are valid modulo n.

[4] The meaning of "graph" is unchanged in any other respect.

Proposition 2. *The equivalence defined above is a reflexive, symmetric and transitive relation (in the set of all sequences* $(m_1, m_2, \dots, m_k)$ *when* n *and* k *are fixed).*

In the *proof* we shall use the fact that the residue classes, consisting of numbers relatively prime to n, form a multiplicative group.

The reflexivity holds since 1 may be chosen for r. — If some r, π establish a connection of type (3.2), then the solution r' $(1 \leqq r' < n)$ of $rr' \equiv 1 \pmod n$ and π^{-1} establish a connection between the two sequences having interchanged roles. — If $rm_h \equiv r'_{\pi(h)}$ and $r'm'_h \equiv m''_{\pi'(h)} \pmod n$, then let r'' $(1 \leqq r'' < n)$ be defined by $r'' \equiv rr' \pmod n$; it follows $r'' m_h \equiv r' r m_h \equiv r' m'_{\pi(h)} \equiv m''_{\pi' \pi(h)} \pmod n$ (where h may be $1, 2, \dots, k$). □

Next we state an evident assertion:

Proposition 3. *If two graphs are described in terms of the formalism* $G(n; m_1, m_2, \dots, m_k)$ *and they are isomorphic, then* n *and* k *are common.* □

Proposition 4. *If (with the same* n *and* k*) the sequences* $(m_1, m_2, \dots, m_k)$ *and* $(m'_1, m'_2, \dots, m'_k)$ *are equivalent, then the graphs* $G = G(n; m_1, m_2, \dots, m_k)$ *and* $G' = G(n; m'_1, m'_2, \dots, m'_k)$ *are isomorphic.*

Proof. To an arbitrary vertex P_i of G, let $\beta(P_i)$ be the vertex $P'_{i'}$, of G' whose subscript is defined by $ri \equiv i' \pmod n$ where the equivalence is established by r. We may check that β is an isomorphism. □

Since Propositions 3 and 4 do not determine fully when two graphs in question are isomorphic, we raise

PROBLEM 4. *Let a condition for two sequences* $(m_1, m_2, \dots, m_k)$, $(m'_1, m'_2, \dots, m'_k)$ *be stated which is necessary and sufficient in order* $G(n; m_1, m_2, \dots, m_k)$ *and* $G(n; m'_1, m'_2, \dots, m'_k)$ *be isomorphic.*

I have conjectured [1] that the converse of Proposition 4 is also valid, i.e. that the equivalence is necessary for isomorphism, too*. The counter-examples due to Elspas and Turner [10] show that the conjecture is false in the class of *all* graphs having a fully cyclic automorphism; namely, $G(8;\ 1,\ 2,\ 5)$ is isomorphic to $G(8; 1, 5, 6)$ and $G(16; 1, 2, 7, 9, 14, 15)$ is isomorphic to $G(16; 2, 3, 5, 11, 13, 14)$ although neither $(1, 2, 5)$ and $(1, 5, 6)$ (for $n=8$) nor $(1, 2, 7, 9, 14, 15)$ and $(2, 3, 5, 11, 13, 14)$ (for $n=16$) are equivalent.

It was proved by Đoković, Elspas, Toida and Turner (see [9], [10], [16]) that my conjecture is valid within each of the following four subclasses of the mentioned class:

(i) the class of graphs the number of whose vertices is a prime,

(ii) the class of graphs whose adjacency matrices have non-repeated eigenvalues only,

(iii) the class of graphs with $k=3$, $m_1 + m_3 = n$ and $m_2 = n/2$,

(iv) the class of graphs with $k=4$, $m_1 + m_4 = m_2 + m_3 = n$ and $(m_1, n) = (m_2, n) = 1$ (the parentheses denote here largest common divisor).

They use — somewhat surprisingly — mostly tools lying outside graph theory (e.g. techniques of matrix theory).

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* Remark added in proof (November 21, 1977). For category-theoretical generalizations see [18].

§ 4.

This § is[5] a continuation of the preceding one. Our present aim is to re-formulate Problem 4 in terms of automorphisms. For the sake of simplicity, we do not make a distinction between a permutation π of the set $\{1, 2, \dots, n\}$ and the permutation of vertices defined by $P_i \to P_{\pi(i)}$.

Let a vertex permutation π of the graph $G(n; m_1, m_2, \dots, m_k)$ be called *special* when $\pi\alpha\pi^{-1}$ is an automorphism where

$$\alpha(i) = \begin{cases} i+1 & \text{if} \quad 1 \leqq i < n \\ 1 & \text{if} \quad i = n \end{cases}$$

Proposition 5. *Consider a vertex permutation π of $G = G(n; m_1, m_2, \dots, m_k)$. Let us introduce another labelling $P_1', P_2', \dots, P_n'$ of the vertices of G by the equalities $P_i' = P_{\pi(i)}$. If π is special, then G (provided with the new notation) equals*

$$G(n; \pi^{-1}(m_1 + \pi(n)), \pi^{-1}(m_2 + \pi(n)), \dots, \pi^{-1}(m_k + \pi(n)))$$

where the k sums (after the semicolon) are thought to be reduced modulo n and ordered increasingly. If π is not special, then the new notation of vertices does not allow to write G as $G(n; m_1, m_2, \dots, m_k)$ (for any choice of the sequence $(m_1, m_2, \dots, m_k)$).

Proof. Let π be special. This means that $\pi\alpha\pi^{-1}$ is an automorphism. The deductions $\pi(P_i) = P_{\pi(i)} = P_i'$ and

$$\pi\alpha\pi^{-1}(P_i') = \pi\alpha(P_i) = \pi(P_{i+1}) = P_{i+1}'$$

show[6] that $\pi\alpha\pi^{-1}$ acts in the same manner as α in the sufficiency proof of Proposition 1. There are evidently k edges incoming to $P_n' (= P_{\pi(n)})$, these edges are outgoing from the vertices

$$P_{\pi(n)+m_1} = P'_{\pi^{-1}(\pi(n)+m_1)}, \quad P_{\pi(n)+m_2} = P'_{\pi^{-1}(\pi(n)+m_2)}, \dots, P_{\pi(n)+m_k} = P'_{\pi^{-1}(\pi(n)+m_k)}$$

where the sums are meant mod n.

Conversely, suppose that π is not special. There is a pair (P_i, P_j) such that exactly one of the edges $\overrightarrow{P_i P_j}$ and $\overrightarrow{(\pi\alpha\pi^{-1}(P_i))(\pi\alpha\pi^{-1}(P_j))}$ exists. We have the equalities

$$P_i = P'_{\pi^{-1}(i)}, P_j = P'_{\pi^{-1}(j)},$$

$$\pi\alpha\pi^{-1}(P_i) = \pi\alpha(P_{\pi^{-1}(i)}) = \pi(P_{1+\pi^{-1}(i)}) = P'_{1+\pi^{-1}(i)},$$

$$\pi\alpha\pi^{-1}(P_j) = P'_{1+\pi^{-1}(j)}$$

(the subscripts have sometimes to be reduced mod n), thus the fully cyclic permutation $P_i' \to P_{i+1}'$ is not an automorphism. □

Proposition 6. *The graphs $G = G(n; m_1, m_2, \dots, m_k)$ and $G' = G(n; m_1', m_2', \dots, m_k')$ are isomorphic if and only if there exists a pair (π, ϱ) satisfying the following three properties:*

[5] The reader may neglect this § unless he is particularly interested in Problem 4.

[6] We omit the separate treatment of the case $i = n$.

π is a permutation of the set $\{1, 2, \ldots, n\}$ and — as a permutation of the vertex set of G — π is special,

ϱ is a permutation of the set $\{1, 2, \ldots, k\}$,

the congruence

$$\pi(m'_{\varrho(h)}) \equiv m_h + \pi(n) \pmod n$$

holds for each h $(1 \leqq h \leqq k)$.

Proof. Let β be an isomorphism of G onto G', denote by γ the mapping of the vertex set of G onto the vertex set of G' satisfying $\gamma(P_i) = P'_i$ $(1 \leqq i \leqq n)$. We introduce new notations $P''_1, P''_2, \ldots, P''_n$ by the formula $\beta(P''_i) = P'_i$, we can now write also G in the form $G(n; m'_1, m'_2, \ldots, m'_k)$. Proposition 5 is applicable (with $\beta^{-1}\gamma$ in the role of π).

Assume the existence of π and ϱ that satisfy the conditions. Proposition 5 assures that G can be made isomorphic to $G(n; m'_1, m'_2, \ldots, m'_k)$ by introducing the notations $P''_i = P_{\pi(i)}$ $(1 \leqq i \leqq n)$. □

PROBLEM 5. *Let a method be given which, for an arbitrary graph $G = G(n; m_1, m_2, \ldots, m_k)$ gives a survey of all (different) systems*

$$\{\pi^{-1}(m_1 + \pi(n)), \pi^{-1}(m_2 + \pi(n)), \ldots, \pi^{-1}(m_k + \pi(n))\}$$

where π runs through the special permutations of the vertex set of G and the sums are meant modulo n. □

By virtue of Proposition 6, a solution of Problem 5 would imply the solution of Problem 4.

§ 5.

We study finite directed graphs without loops and pairs of parallel edges (with coinciding or opposite orientation). We denote by $M(G)$ the lenght of a shortest cycle of the graph G. If the number k satisfies[7] $2 \leqq k < M(G)$, then we assign a new graph $\mathfrak{A}_k(G)$ to the graph G in the following way:

the vertex set of $\mathfrak{A}_k(G)$ equals the vertex set of G,

the edge $\overrightarrow{AB}$ exists in $\mathfrak{A}_k(G)$ if and only if $(A \neq B$ and) there is a path in G from A to B the length of which is smaller than k.

$\mathfrak{A}_2(G)$ is G itself. If G is a cycle of length n, then $\mathfrak{A}_k(G) = G(n; 1, 2, \ldots, k-1)$ where the right-hand side is to be understood as in § 3.

Proposition 7. *If G is a subgraph of H (with the same vertex set as H), then there exists at most one number k fulfilling $\mathfrak{A}_k(G) = H$.*

Proof. Let H equal $\mathfrak{A}_k(G)$ for some k. Denote by $A_1, A_2, \ldots, A_{M(G)}$ the vertices of a shortest cycle of G (in the natural ordering). Suppose $2 \leqq k' < M(G)$ and $k' \neq k$, denote by k'' the larger of k and and k'. The edge $\overrightarrow{A_{M(G)}A_{k''}}$ exists in precisely one of $\mathfrak{A}_k(G)$ and $\mathfrak{A}_{k'}(G)$, thus $\mathfrak{A}_k(G) \neq \mathfrak{A}_{k'}(G)$. □

[7] The letter k is now used in another sense, than in the previous sections.

Let C be a class consisting of directed graphs. Then we denote by $\mathfrak{A}(C)$ the class of all graphs $\mathfrak{A}_k(G)$ where G runs through the members of C and, for any G, k runs through the numbers satisfying $2 \leq k < M(G)$.

By virtue of this definition, every member H of $\mathfrak{A}(C)$ has at least one subgraph $G(\in C)$ such that $\mathfrak{A}_k(G)=H$ for some k. C is called a *decomposition class* if every member H of $\mathfrak{A}(C)$ can be represented with exactly one $G(\in C)$ in the form $\mathfrak{A}_k(G)$.

Proposition 8. *The class C consisting of all cycles is a decomposition class.*

Proof. Let A be a vertex of a member H of $\mathfrak{A}(C)$. Denote the outdegree of A by $\varrho(A)$ and the set of end vertices of the edges outgoing from A by $\sigma(A)$. Then $\varrho(A)$ is common for the vertices of H and — denoting it by ϱ — $|\sigma(A)|=\varrho$.

Let G be an arbitrary cycle such that $\mathfrak{A}_k(G)=H$. It may be seen easily that $\varrho=k-1$, furthermore,

$$|\sigma(A) \cap \sigma(B)| = k-2$$

if A and B are adjacent vertices in G but[8]

$$|\sigma(A) \cap \sigma(B)| \leq k-3$$

for any other choice of A and B $(A \neq B)$.

We have reconstructed the pair (G, k) in terms of H only. □

Let the notion of (β)-graphs be recalled (see § 1).

PROBLEM 6. *Prove or disprove that the family of (β)-graphs is a decomposition class.*

This problem was raised in § 4 of [3] as Conjecture 3 together with some related conjectures.

Problem 6 is a particular case of the subsequent question (of rather heuristic than exact nature):

PROBLEM 7. *Let us determine decomposition classes, comprehensive as far as possible, among the finite directed graphs.*

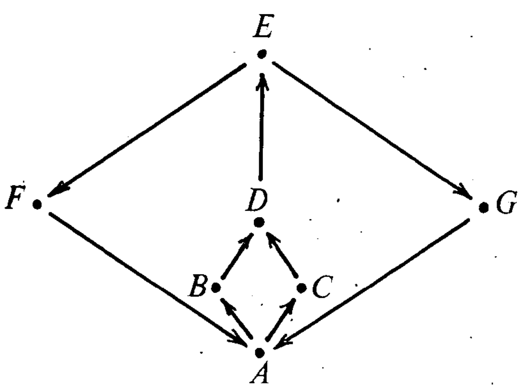

Fig. 3

§ 6.

A graph G is called *cyclically simple* if the intersection of any pair of different cycles of G is (empty or) a path. The graph on Fig. 3 is not cyclically simple because the intersection of the cycles $(ABDEF)$ and

[8] We utilize here the fact $k<n$ where n is the length of G.

$(ACDEG)$ consists of two paths (of lenghts zero and one, resp.) being not connected with each other.

The future development of the methods of analyzing directed graphs structurally, with a particular emphasis to their cycles (see [3], [4], [5]), will perhaps enable us to make remarkable attacks towards the following general research direction:

PROBLEM 8. *Let the structure of all cyclically simple graphs be described.*

I note that [5] terminates with an open question. This problem is somewhat particular, this fact and the lengthiness of the previous definitions (before all, the B-constructibility) do not permit to recapitulate it here within reasonable size.

§ 7.[9]

Let V be a finite set and

$$\mathfrak{H} = \{H_1, H_2, \ldots, H_t\} \quad (t \geqq 1)$$

be a family of subsets of V. The pair $(V, \mathfrak{H})$ is called a *hyper-tree* if the following four conditions are fulfilled:

(A) $H_1 \cup H_2 \cup \ldots \cup H_t = V$,

(B) $|H_i| \geqq 2$ for each i $(1 \leqq i \leqq t)$,

(C) $|H_i \cap H_j| \leqq 1$ whenever $1 \leqq i < j \leqq t$,

(D) to each pair (i, j) (where $1 \leqq i < j \leqq t$) there exists precisely one sequence

$$i = i_0, i_1, i_2, \ldots, i_m = j \quad (m \geqq 1)$$

satisfying the properties (i), (ii):

(i) $i_0, i_1, \ldots, i_m$ are pairwise different numbers chosen from the set $\{1, 2, \ldots, t\}$,

(ii) $H_{i_{p-1}} \cap H_{i_p} \neq \emptyset$ whenever $1 \leqq p \leqq m$.

Let some evident consequences of the above definition of hyper-tree be stated. $H_i \nsubseteq H_j$ if $i \neq j$ (by (B) and (C)). The intersection in (ii) has exactly one element (by (C)). If $2 \leqq p+1 < q \leqq m$, then $H_{i_p} \cap H_{i_q}$ is empty in (D) (by the unicity of the sequence).

The elements of V are called the *vertices* of the hyper-tree, the nembers of $\mathfrak{H}$ are the *hyper-edges* of it.

CONSTRUCTION. The construction consists of three steps.

Step 1. Let a hyper-tree $(V, \mathfrak{H})$ be considered. We assign to any vertex $A (\in V)$ an $(\alpha\gamma)$-graph $v(A)$. At the beginning of the procedure, $v(A)$ and $v(B)$ are viewed to be disjoint if A, B are different vertices.

Step 2. Let $\mu(A, H_i)$ be a mapping such that

(a) μ is defined on the set of pairs $A (\in V)$, $H_i (\in \mathfrak{H})$ such that $A \in H_i$,

(b) the value of $\mu(A, H_i)$ is a vertex of $v(A)$, and

(c) $i \neq j$ implies $\mu(A, H_i) \neq \mu(A, H_j)$ (where, of course, $A \in H_i \cap H_j$).

[9] This § is addressed only to readers interested in Problem 1 or how graphs are built up from their blocks.

Step 3. For any $H_i (\in \mathfrak{H})$, let us identify all the vertices $\mu(A, H_i)$ with each other (A runs through the elements of H_i). Denote the resulting graph by G.

The description of the Construction is completed.

In Step 3, any $v(A)$ is embedded into G. We denote by $v^*(A)$ the result of this embedding. The graphs $v^*(A)$ are not necessarily disjoint, unlike the $v(A)$'s.

Proposition 9. *Let P be a vertex of the graph G (constructed above). P is a cut vertex of G if and only if there is a $H_i (\in \mathfrak{H})$ such that P is the result of the identification of the vertices $\mu(A, H_i)$ (in sense of Step 3).*

Proof. We use the following characteristic property of cut vertices: P is a cut vertex if and only if there exist two vertices Q, R such that Q and R are adjacent to P and every chain between Q and R contains P.

Necessity. Suppose that P has not been produced by identification. Then P is a vertex of some (well-determined) $v^*(A)$ and any vertex adjacent to P (in G) is in the same $v^*(A)$. $v^*(A)$ is an $(\alpha\gamma)$-graph, hence it has no cut vertex. For every choice of Q and R, these vertices may be connected by a chain within $v^*(A)$ which does not pass through P; the same holds obviously in G, too.

Sufficiency. Assume that P originates from identification. Let us choose two elements A and B of H_i. There is a vertex Q in $v^*(A)$ and a vertex R in $v^*(B)$ such that they are adjacent to P, i.e. they can be connected by a chain a_1 whose vertices are Q, P, R (in this ordering). Suppose that there exists a chain a_2 (in G) which connects Q and R and does not contain P. The union of a_1 and a_2 is a circuit a. Passing along a and considering the common vertices of the subgraphs of type $v^*(A)$, we can form a sequence

$$H_{i_1}, H_{i_2}, \ldots, H_{i_w} \quad (w \geqq 2)$$

of some members of $\mathfrak{H}$ such that every pair of neighbouring members of the sequence is a pair of distinct and non-disjoint elements, moreover also H_{i_1} and H_{i_w} have an element in common. If $w>2$, then this contradicts the condition (D) in the definition of hyper-trees. The case $w=2$ is impossible by (C). □

Proposition 9 implies immediately

Proposition 10. *The blocks of G coincide with the subgraphs $v^*(A)$ where A runs through the elements of V.* □

Proposition 11. *Every graph produced by our Construction is an (α)-graph and all (α)-graphs can be represented in this manner.*

Proof. Any circuit (and, particularly, any cycle) of G is entirely included in some $v^*(A)$, thus the Construction leads to an (α)-graph if all the $v(A)$'s were (α)-graphs.

Conversely, consider the blocks of an (α)-graph G. Every block is an $(\alpha\gamma)$-graph. If

we assign an element A to any block,

V denotes the set of the A's ($|V|$ is the number of blocks), we assign a H_i to the situation (whenever it occurs) that the same vertex of G occurs in two or more blocks, and

$\mathfrak{H}$ denotes the family of H_i's ($|\mathfrak{H}|$ is the number of cut vertices of G), then the conditions of the Construction are obviously satisfied. □

Remark. In this § we have utilized only very few properties of the (α)-graphs. Let us consider an arbitrary class of connected graphs (directed or non-directed) — say, the (τ)-graphs — such that a graph G is a (τ)-graph precisely if each block of G is again a (τ)-graph.[10] Then the connection of (τ)-graphs and ($\tau\gamma$)-graphs can be described analogously to the previous treatment.

II. Problems on the functioning of networks

§ 8.

By an *autonomous continuous automaton* we understand in the sequel a partial function $\varphi(x, t)$ fulfilling the seven conditions as follows:

(i) x runs through the elements of a set X,

(ii) t runs through the non-negative real numbers,

(iii) each value of φ is an element of X,

(iv) for any fixed $x(\in X)$, either $\varphi(x, t)$ is defined for every t or there exists a bound $b_x(\geqq 0)$ (depending on x) such that $\varphi(x, t)$ is defined precisely when $0 \leqq t < b_x$,

(v) $\varphi(x, 0) = x$ if the left-hand side is defined (i.e. if $b_x > 0$),

(vi) whenever $\varphi(x, 0)$ is defined, then there exists an $r_x(>0)$ (depending on x) such that either

$0 < t \leqq r_x$ implies $\varphi(x, t) = x$ or

$0 < t \leqq r_x$ implies $\varphi(x, t) \neq x$,

(vii) the equality

$$\varphi(x, t_1 + t_2) \doteq \varphi(\varphi(x, t_1), t_2) \tag{8.1}$$

is required for every triple $x(\in X)$, $t_1(\geqq 0)$, $t_2(\geqq 0)$ (in such a sense that either both sides of (8.1) are defined or none of them).

Now we turn to a heuristical explanation of what the definition of autonomous continuous automata expresses. The second variable t of φ is interpreted as time. The first variable x corresponds to the possible states of a system (working in time). The function φ itself has the following meaning: whenever the system is in a state x at the initial instant 0 and the function value $\varphi(x, t)$ is defined, then the system will take the state $\varphi(x, t)$ at the instant t. (The case when $\varphi(x, t)$ in undefined means that our mathematical model is unable to say what will be the state of the system at t.)

In (iv) three possibilities are allowed, namely, either b_x is 0 or b_x is a positive real number or b_x does not occur. The situation $b_x = 0$ means that, at least in sense of the mathematical treatment of the system's behaviour, the system is not capable to take the state x. (In the following §§ this will arise when the edge $\overrightarrow{PQ}$ exists in a graph, and the state x attributes the value 1 to P and a value lying in the interval[11] (0, 1] to Q.) If a positive b_x exists, then our mathematical apparatus

[10] Of course, the blocks are ($\tau\gamma$)-graphs.

[11] By (0, 1] the set of instants t fulfilling $0 < t \leqq 1$ is denoted.

describes the functioning of the system not longer than within a time interval having a finite (positive) length. When b_x is not defined, then our model is able to prognosticate the system's states for every future instant.

The aim of the condition (vi) is to exclude the automata whose functioning is very irregular. It does not permit the occurrence of the following (abnormal enough) situation: for any positive r (however small it be!) there exist two instants t, t' in the interval $(0, r]$ such that the state of the system at t is x and its state at t' differs from x (i.e., roughly speaking, the state x is densely mixed with other state(s) in the neighbourhood of the initial instant). — Such situations can scarcely arise in the work of real systems, but they are logically imaginable.

Condition (vi) is rather of technical character; in contrast with this, (vii) expresses an important characteristic feature of the considered automata. The notation $\varphi(x, t)$ (containing no symbol denoting effects which come from outside!) is interpreted that the automaton works autonomously; in addition to this, (vii) postulates that the laws of its functioning do not change with time, in other words, the distinguished role of the initial instant is abolished and t may be viewed as the length of a time interval (situated anywhere in the non-negative semiaxis). (We can also say, on the basis of (vii), that the system neither oldens nor learns.)

In accordance with the above intuitive considerations, we introduce the following terminologies. The elements of X are also called *states* (of the automaton $\mathbf{A}$, to be defined later). The variable t is interpreted as time, thus its values are called *instants*. If $\varphi(x; 0)$ is defined, then x is called a *permitted* state. The condition (vi) is said the *quasi-continuity* of φ, (vii) is said the *homogeneity* of φ *in time*.

A pair $\mathbf{A}=(\varphi, x_0)$ is called an *initial autonomous continuous automaton* (or, for the sake of brevity, an *initial automaton*) if φ is an autonomous continuous automaton and x_0 is a (fixed) permitted element of X. We call x_0 the *initial state* of $\mathbf{A}$. $\varphi(x_0, t)$ is said the state of $\mathbf{A}$ at the instant t.

If x is a state of φ, then we denote by H_x the set of positive numbers t_0 satisfying $\varphi(x, t_0)=x$. — We write also shortly H (instead of H_x) when only one state is considered. If some states $x_1, x_2, \dots$ are viewed, then the simple notations $H_1, H_2, \dots$ may be used (instead of $H_{x_1}, H_{x_2}, \dots$, resp). Similar simplifications will be used for other quantities depending on states too.

Proposition 12. *If x is a state of the autonomous continuous automaton φ such that H_x is neither empty nor the set of all positive numbers, then H_x contains a smallest element p_x.*

Proof. Suppose that H is not empty and (the existence of smallest element does not hold, i.e.) for every $t_0(\in H)$ there is a t_1 satisfying both $0<t_1<t_0$ and $t_1\in H$. Our aim is to show that every positive number belongs to H. By iterating the step of determining t_1, we get an infinite decreasing sequence

$$t_0 > t_1 > t_2 > t_3 > \dots$$

consisting of elements of H. The difference $d_i=t_i-t_{i+1}$ converges to 0 if i tends to the infinity. We have

$$\varphi(x, d_i) = \varphi(\varphi(x, t_{i+1}), d_i) = \varphi(x, t_{i+1}+d_i) = \varphi(x, t_i) = x$$

(since φ is homogeneous in time) for each i, thus there exists an r such that $\varphi(x, t)=x$ whenever $0<t\leqq r$ (by the quasi-continuity of φ).

Let now t^* be an arbitrary positive number. There is an integer u and a (real) number v such that $t^*=ur+v$ and $0 \leqq v < r$; hence

$$\varphi(x, r) = x,$$

$$\varphi(x, 2r) = \varphi(\varphi(x, r), r) = \varphi(x, r) = x,$$

$$\dots$$

$$\varphi(x, ur) = \varphi(\varphi(x, (u-1)r), r) = \varphi(x, r) = x,$$

$$\varphi(x, t^*) = \varphi(x, ur+v) = \varphi(\varphi(x, ur), v) = \varphi(x, v) = x,$$

this means $t^* \in H$. □

In the set of permitted states of an autonomous continuous automaton, we introduce the (binary) relation σ as follows: $\sigma(x, y)$ exactly if there exist two non-negative numbers t_1, t_2 such that $\varphi(x, t_1)=y$ and $\varphi(y, t_2)=x$. It is obvious that σ is reflexive, symmetric and transitive, consequently the set of permitted states splits into equivalence classes modulo σ (called σ-classes). A σ-class is said *trivial* if it consists of one element only.

If $\varphi(x, t)$ (is meaningful and) equals x for every non-negative t, then x is called a *steady* state. Any steady state forms a trivial σ-class (but the trivial σ-classes are not exhausted in this manner). If the σ-class containing x is non-trivial, then we say that x is a *properly periodic* state[12] and its *period* is p_x. x is *periodic* if it is either steady or properly periodic.

The behaviour of an initial automaton whose initial state x_0 is steady is trivial. If x_0 is properly periodic, then the behaviour may be derived from our next statement:

Proposition 13. *Let K be a non-trivial σ-class of an autonomous continuous automaton. Then the following five assertions hold:*

(I) *(the period p_x is common in K, i.e.) there exists a positive number p_K such that $p_x=p_K$ for every $x(\in K)$,*

(II) *whenever $x \in K$ and $t^* \geqq 0$, then $\varphi(x, t^*)$ (exists and) belongs to K,*

(III) *whenever x and y are arbitrary elements of K, then there exists one and only one instant t fulfilling both $0 \leqq t < p_K$ and $\varphi(x, t)=y$,*

(IV) *whenever $x \in K$ and $t^* \geqq p_K$, then $\varphi(x, t^*)=\varphi(x, v)$ where v is the single number determined by the inequalities $0 \leqq v < p_K$ and the condition that $(t^*-v)/p_K$ is an integer,*

(V) *the cardinality of K is continuum.*

Proof. (I) Choose two arbitrary elements x, y of K. Since $\varphi(x, t_1)=y$ is satisfiable with some t_1, we have

$$\varphi(y, p_x) = \varphi(\varphi(x, t_1), p_x) = \varphi(x, t_1+p_x) = \varphi(\varphi(x, p_x), t_1) = \varphi(x, t_1) = y$$

hence $p_y \leqq p_x$. A symmetrical inference shows that $p_x \leqq p_y$. Hence p_x is the same for every choice of x in K.

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[12] Proposition 13 will justify this terminology.

(II) Let u, v be two numbers such that $t^*=up_x+v$, $0\leqq v<p_x$ and u is an integer. An easy induction shows that $\varphi(x, up_x)$ (exists and) equals x for every u (similarly to the final part of the proof of Proposition 12). Moreover,

$$\varphi(x, t^*) = \varphi(x, up_x+v) = \varphi(\varphi(x, up_x), v) = \varphi(x, v)$$

(where $\varphi(x, v)$ is necessarily meaningful!) and

$$\varphi(\varphi(x, t^*), p_x-v) = \varphi(\varphi(x, v), p_x-v) = \varphi(x, v+p_x-v) = \varphi(x, p_x) = x,$$

consequently, x and $\varphi(x, t^*)$ are in the same σ-class.

(III) We have $\varphi(x, t^*)=y$ with some $t^*(\geqq 0)$. The number v, seen in the preceding section of the proof, is a convenient v. We are going to show the unicity of t. Suppose (contrarily to this) that $\varphi(x, t)=\varphi(x, t')=y$ where $0\leqq t<t'<p_K$; hence

$$\varphi(x, p_K+t-t') = \varphi(\varphi(x, t), p_K-t') =$$

$$= \varphi(\varphi(x, t'), p_K-t') = \varphi(x, t'+p_K-t') = \varphi(x, p_K) = x,$$

this equality and the obvious inequalities

$$0 < (p_K-t' <) p_K+t-t' < p_K$$

contradict the definition of p_x.

(IV) This assertion was already verified in the proof of (II).

(V) If x is fixed, then (III) establishes a one-to-one mapping between the elements (denoted by y) of K and the numbers being in the interval $[0, p_K)$. □

Now we turn to the behaviour of an automaton when it starts with a non-periodic state.

Proposition 14. *Let an autonomous continuous automaton be considered. Suppose that a permitted state x of it is not periodic. Then exactly one of the subsequent two assertions is valid:*

(A) *The states $\varphi(x, t)$ are pairwise different as far as they are defined (the bound b_x may or may not exist).*

(B) *There is a number $c_x(\geqq 0)$ such that*

(i) *for all choices of t such that $t>c_x$, the states $\varphi(x, t)$ are (defined and) periodic and moreover, they belong to a common σ-class K,*

(ii) *for the choices of t such that $0\leqq t<c_x$, the states $\varphi(x, t)$ are non-periodic and pairwise different, moreover, any of them differs from $\varphi(x, c_x)$, and*

(iii) *if $(c_x>0$ and) $\varphi(x, c_x)$ is a periodic state, then it belongs to the class K mentioned in* (i).

Proof. Assume that (A) does not hold, i.e. there are two instants t_1, t_2 such that $(0<)t_1<t_2$ and $\varphi(x, t_1)=\varphi(x, t_2)$. We want to show that all statements of (B) are true.

Case 1: there is a t' such that $t_1<t'<t_2$ and $\varphi(x, t')\neq\varphi(x, t_1)$. In sense of the formulae $\varphi(\varphi(x, t_1), t'-t_1)=\varphi(x, t')$, $\varphi(\varphi(x, t'), t_2-t')=\varphi(x, t_2)=\varphi(x, t_1)$, $\varphi(x, t_1)$ and $\varphi(x, t')$ are in the same σ-class K, hence they are properly periodic. Denote by $J(\neq\emptyset)$ the set of all instants t^* fulfilling $\varphi(x, t^*)\in K$. Proposition 13,

(II) implies that J is an interval of form (c_x, ∞) or $[c_x, \infty)$ where c_x is the infimum of J. The statements (i), (iii) are obvious, (ii) can be verified easily by indirect method (namely, supposing that t_1, t_2 may be chosen so that $t_1 < t_2 < c_x$, we get a contradiction with the definition of c_x).

Case 2: $\varphi(x, t') = \varphi(x, t_1)$ whenever t' is between t_1 and t_2. It is clear that $\varphi(x, t_1)$ is a steady state. If we denote now by J the set of instants t^* fulfilling $\varphi(x, t^*) = \varphi(x, t_1)$, then J and its infimum c_x will again satisfy (i), (ii), (iii). □

A (non-periodic, permitted) state is called *aperiodic* or *pre-periodic* if it satisfies assertion (A) or assertion (B) of Proposition 14, respectively. The number c_x is called the *length* of the pre-period. An aperiodic state x is called *bounded aperiodic* or *boundless aperiodic* depending on the existence of the bound b_x. Table 1 shows the hierarchy of the notions introduced for states.

PROBLEM 9. *Analyze how the periodicity properties are modified if some of the conditions* (i)—(vii) *is replaced by a weaker one.*

(E.g. we can suppose, instead of (ii), that t varies on a totally ordered set T, the cardinality of T may differ from the continuum.)

properly periodic	periodic		
steady			
pre-periodic			permitted
boundless aperiodic		non-periodic	
bounded aperiodic	aperiodic		

Table 1

Remarks. In this § we have considered automata whose functioning starts with an *instantaneous* initial state. The somewhat modified notion when the behaviour in an interval of *positive* duration is regarded to be the starting condition has been studied by Konikowska [15].

It is worthy of mention that the behaviour of autonomous systems of differential equations has certain similarities to the above results (see [17], § 15).

§ 9.

By a *network* we understand a triple (G, Σ, φ) such that

(1) G is a finite directed graph whose vertices are denoted by $P_1, P_2, \ldots, P_n$,

(2) Σ is a subset of the set of real numbers,

(3) the set of all mappings x of the set $\{P_1, P_2, \ldots, P_n\}$ into Σ is denoted by X, and

(4) φ is an autonomous continuous automaton, the set of states of φ is X, the function φ is given in terms of the graph structure of G.

By virtue of the above definition a state of the network in question is denoted as a vector

$$(x(P_1), x(P_2), \ldots, x(P_n)). \tag{9.1}$$

For the sake of convenience, we write the simpler notation

$$(x_1, x_2, \ldots, x_n) \tag{9.2}$$

instead of the vector (9.1) and we identify the mapping x to (9.2).

Thus $\varphi(x, t)$ may be decomposed into n functions

$$\alpha_1(x, t), \alpha_2(x, t), \ldots, \alpha_n(x, t) \qquad (9.2)$$

where $\alpha_i(x, t)$ is the i-th component of the vectorial form of the state[13] $\varphi(x, t)$.

This means that if x_i is the state of the vertex P_i of the network at some instant t_0 (for any i, $1 \leqq i \leqq n$) and we use the symbol x for the sequence $(x_1, x_2, \ldots, x_n)$, then $\alpha_i(x, t)$ denotes the state of some P_i at $t_0 + t$.

§ 10.

Now we define a particular type of networks which will be called *simple* networks in the sequel[14]. This notion arises (by abstraction) from the networks studied in Chapter 2 of [14]. Let G be a graph (without loops and multiple edges). Let Σ be the closed interval [0, 1] (i.e. the set of real numbers y fulfilling $0 \leqq y \leqq 1$). Let the positive number τ, called *recovery time* (of the vertices), be characteristic for the network (τ is the same for each vertex). The function φ is determined by the subsequent four rules (in which $\{y\}$ is defined by

$$\{y\} = \begin{cases} y & \text{if } \; 0 \leqq y < 1 \\ 0 & \text{if } \; y = 1 \end{cases}$$

and ϱ_x denotes the maximum of the values $\{x_1\}, \{x_2\}, \ldots, \{x_n\}$):

(1) a state x is permitted if and only if the equality $x_i = 1$ and the existence of the edge $\overrightarrow{P_i P_j}$ imply $x_j = 0$ (for any i, j where $1 \leqq i \leqq n$, $1 \leqq j \leqq n$),

(2) if x is a permitted state and $0 < t < (1 - \varrho_x)\tau$, the we define

$$\varphi(x, t) = (z_1, z_2, \ldots, z_n)$$

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[13] The functions $\alpha_i(x, t)$ are *not* autonomous continuous automata. (Indeed, let us recall the definition of autonomous continuous automata in § 8. If the set of states of the entire network is denoted by X, then (iii) is not satisfied; if X denotes the state of a single vertex, then (i) is not fulfilled.) — We avoid the notation of type $\alpha_i(x_i, t)$ since it lacks to be a function (because there are interactions among the vertices; hence $\alpha_i(x, t)$ may depend on each of $x_1, x_2, \ldots, x_n$, not only on x_i).

[14] The reader who is interested in this subject may find a more detailed explanation in [2] (mainly in Section 3). A short summary of the definition is contained also at the beginning of § 6 of [3].

in the following manner:

$$z_i = \begin{cases} 1 & \text{if } \; x_i = 1 \\ 0 & \text{if } \; \text{there is a } j \text{ such that } \overrightarrow{P_j P_i} \text{ exists and } x_j = 1, \\ x_i + \dfrac{t}{\tau} & \text{otherwise,} \end{cases}$$

(3) if x is a permitted state, $\varrho_x > 0$ and there are two subscripts i, j such that $\overrightarrow{P_i P_j}$ exists and $x_i = x_j = \varrho_x$, then $\varphi(x, (1-\varrho_x)\tau)$ is undefined,

(4) if x is a permitted state, $\varrho_x > 0$ and the assumption in (3) does not hold, then we define

$$\varphi(x, (1-\varrho_x)\tau) = (z_1, z_2, \ldots, z_n)$$

in the following manner

$$z_i = \begin{cases} 0 & \text{if there is a } j \text{ such that } \overrightarrow{P_j P_i} \text{ exists and} \\ & \text{one of the equalities } x_j = \varrho_x, \, x_j = 1 \text{ is true,} \\ \min(1, x_i + 1 - \varrho_x) & \text{otherwise}^{15}. \end{cases}$$

It can be verified that we have defined an autonomous continuous automaton φ (among others, either the value $\varphi(x, t)$ can be determined or the fact that $\varphi(x, t)$ is not defined can be stated consistently by successive application of (1)—(4) for an arbitrary choice of the state x and the non-negative instant t).

Our next aim is to clear up the intuitive basis of the above definition. The vertices of the network represent (idealized) neurons, the edges of the network are inhibitory connections. Each neuron is (at an instant) either in inhibited state or in firing state (denoted by 0, 1, resp.) or in the so-named recovery state[16] (being between these extremities). The inhibition is understood in such a manner that if a neuron P is in firing state and the edge $\overrightarrow{PQ}$ exists, then Q is in inhibited state. There is a permanent "background effect" manifesting itself in such a way that a neuron, being in recovery state, strives to be firing (unless, of course, it will be inhibited by another state in the meantime), and a firing neuron remains in firing state (till it gets an inhibition). The inhibition is produced instantaneously (and, more precisely, right-continuously), the duration of a full recovery phase (beginning with an inhibited state and leading to a firing state) is τ, this duration does not depend on which of the neurons is considered. If two neurons, connected with each other, reach the firing state at the same instant, then the behaviour of the network is studied no more. Let the state of a neuron, being in recovery state at an instant t, be considered; if the neuron is in recovery phase since a duration of length t_0, then its state is denoted by t_0/τ (clearly $0 < t_0/\tau < 1$), hence the firing state 1 will be reached continuously at the instant $t + \tau - t_0$ (except when the recovery phase is interrupted by a new inhibition) (see Fig. 4).

[15] The definition of ϱ_x implies that $x_i + 1 - \varrho_x > 1$ if and only if $x_i = 1$.

[16] The terminology "recovery" occurs in another sense in the literature, too, than its meaning in the sequel.

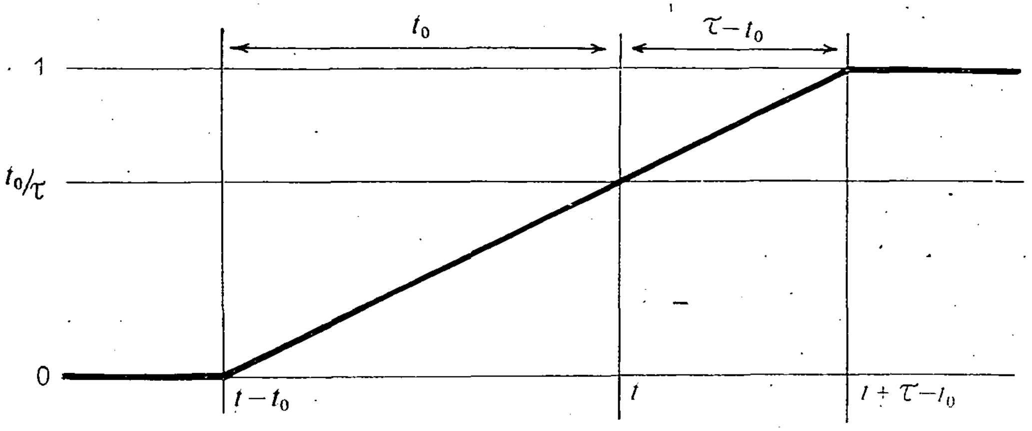

Fig. 4

It is shown in Sections 4—5 of [2] that the functioning of such a network happens in *discrete* steps substantially, and the essential instants can be calculated from the initial state. It follows from the treatment that no boundless aperiodic state occurs in this type of networks. A matrix

$$\begin{pmatrix} \Psi_0^0 & \Psi_1^0 & \dots & \Psi_{q+1}^0 \\ \Psi_0^1 & \Psi_1^1 & \dots & \Psi_{q+1}^1 \\ & \dots & & \\ \Psi_0^m & \Psi_1^m & \dots & \Psi_{q+1}^m \\ & \dots & & \end{pmatrix} \tag{10.1}$$

(having an infinity of rows) is there considered whose entries are sets of vertices (corresponding to the smaller or larger values of $\alpha_i(x, T_m)$ where x is the initial state and T_m is the m-th essential instant), and rules are stated how the entries of the m-th row may be expressed in terms of the entries of the $(m-1)$-th row. The following three problems are mentioned in the last section of [2]:

PROBLEM 10. *Express the entries* $\Psi_0^m, \Psi_1^m, \dots, \Psi_{q+1}^m$ *of* (10.1) *in terms of the entries* $\Psi_0^0, \Psi_1^0, \dots, \Psi_{q+1}^0$ *by formulae which are closed as far as possible.*

PROBLEM 11. *Describe the periodicity properties of a network (starting with an arbitrary initial state) by use of the matrix* (10.1).

PROBLEM 12. *Characterize the graphs G possessing the stability property that whenever x and y are periodic or pre-periodic states, then there exist t and t′ such that* $\varphi(x, t) = \varphi(y, t')$.

(Obviously, Problems 10 and 11 are closely related to each other.)

§ 11.

It seems that from various considerations (belonging not to mathematics but to more or less experimental sciences) it is imaginable to derive various network types. In [2] it was elaborated only in one special case how some concrete type of networks can be introduced and analyzed.

Among the diversity of possibilities, we turn now to the network type suggested by Chapter 3 of [14] where they are called "*complex networks*". We are going to give a definition of these networks that uses mathematical tools (analogously to how the network notion appearing in Chapter 2 of [14] was abstractly defined in Section 3 of [2]). (The motivation will be given at the end of the §.) The definition consists of four parts.

(I) The graph G contains $2n$ vertices which are presented in a matrix form:

$$\begin{pmatrix} P_1 & P_2 & \dots & P_n \\ Q_1 & Q_2 & \dots & Q_n \end{pmatrix}. \tag{11.1}$$

The n edges $\overrightarrow{P_1Q_1}, \overrightarrow{P_2Q_2}, \dots, \overrightarrow{P_nQ_n}$ are always present in G, every other edge of G starts from some Q_i.

(II) Σ consists of the real numbers a satisfying either $0 \leqq a \leqq 1$ or $a=2$.

(III) A state

$$x = \begin{pmatrix} x(P_1) & x(P_2) & \dots & x(P_n) \\ x(Q_1) & x(Q_2) & \dots & x(Q_n) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ z_1 & z_1 & \dots & z_n \end{pmatrix} \tag{11.2}$$

is permitted precisely when

(a) $x_1 \neq 2, x_2 \neq 2, \dots, x_n \neq 2$,

(b) $x_i = 1$ implies $z_i \neq 1$,

(c) $z_i = 2$ implies $x_i = 1$,

(d) $z_i = 2$ and the existence of $\overrightarrow{Q_iP_j}$ imply $x_j = 0$, and

(e) $z_i = 2$ and the existence of $\overrightarrow{Q_iQ_j}$ imply $z_j = 0$.

Before exposing the final part of the definition, we agree that two (fixed) positive real numbers τ_x, τ_z are characteristic for the activity of the network, and we introduce some notations (concerning (11.2)). Let σ be the minimum of the positive numbers among the $2n$ quantities

$$\tau_x(1-x_1), \tau_x(1-x_2), \dots, \tau_x(1-x_n), \tau_z(1-z_1), \tau_z(1-z_2), \dots, \tau_z(1-z_n).$$

Γ^p is the set of subscripts i fulfilling $x_i = 1$. $\Gamma^{q,2}$ and $\Gamma^{q,1}$ are the sets of i's satisfying $z_i = 2$ or $z_i = 1$, respectively. Δ^p and Δ^q are the sets of i's for which $x_i = 1 - \frac{\sigma}{\tau_x}$ or $z_i = 1 - \frac{\sigma}{\tau_z}$ hold, respectively.[17] If Ψ is an arbitrary subset of $\{1, 2, \dots, n\}$, then let $\chi_p(\Psi)$ be the set of those P_i's for which $\overrightarrow{Q_jP_i}$ exists in G with a suitable $j (\in \Psi)$. The set $\chi_q(\Psi)$ (consisting of some Q_i's) is similarly defined (with $\overrightarrow{Q_jQ_i}$).

[17] If τ_x or τ_z equals σ, then the definitions of Δ^p or Δ^q must be somewhat modified. Thismay be left to the reader (for an analogy, see the definition of Δ_k and Footnote 4 in [2]).

(IV) Now define the values of the entries of the matrix

$$\varphi(x, t) = \begin{pmatrix} \alpha_1(x_1, t) & \alpha_2(x_2, t) & \dots & \alpha_n(x_n, t) \\ \gamma_1(z_1, t) & \gamma_2(z_2, t) & \dots & \gamma_n(z_n, t) \end{pmatrix}$$

where x is the same as in (11.2) and t satisfies (separately) $0<t<\sigma$ or $t=\sigma$ in the following way:

[a] if $t<\sigma$, then

$$\alpha_i(x_i, t) = \begin{cases} 0 & \text{if } P_i \in \chi_p(\Gamma^{q,2}) \\ 1 & \text{if } i \in \Gamma^p \\ x_i + \dfrac{t}{\tau_x} & \text{otherwise,} \end{cases}$$

$$\gamma_i(z_i, t) = \begin{cases} 0 & \text{if } Q_i \in \chi_q(\Gamma^{q,2}) \\ 1 & \text{if } i \in \Gamma^{q,1} \\ 2 & \text{if } i \in \Gamma^{q,2} \\ z_i + \dfrac{t}{\tau_z} & \text{otherwise,} \end{cases}$$

[b] for $t=\sigma$:

$$\alpha_i(x_i, \sigma) = \begin{cases} 0 & \text{if } P_i \in \chi_p(\Delta^p \cap \Delta^q) \\ 1 & \text{if } i \in \Gamma^p \text{ and } P_i \notin \chi_p(\Delta^p \cap \Delta^q) \\ 1 & \text{if } i \in \Delta^p \\ x_i + \dfrac{\sigma}{\tau_x} & \text{otherwise} \end{cases}$$

$$\gamma_i(z_i, \sigma) = \begin{cases} 0 & \text{if } Q_i \in \chi_q(\Delta^p \cap \Delta^q) \\ 1 & \text{if } i \in \Gamma^{q,1} - (\Gamma^p \cup \Delta^p) \text{ and} \\ & \quad Q_i \notin \chi_q(\Delta^p \cap \Delta^q) \\ 2 & \text{if } i \in \Delta^q \cap (\Gamma^p \cup \Delta^p) \\ 2 & \text{if } i \in (\Gamma^{q,2} \cup \Gamma^{q,1}) \cap (\Gamma^p \cup \Delta^p) \text{ and} \\ & \quad Q_i \notin \chi_q(\Delta^p \cap \Delta^q) \\ z_i + \dfrac{\sigma}{\tau_z} & \text{otherwise.} \end{cases}$$

It may happen that this definition of $\alpha_i(x_i, \sigma)$ or $\gamma_i(z_i, \sigma)$ is not consistent (because the first and third conditions in both definitions do not exclude each other in general). If such a contradiction arises, then we do not define $\varphi(x, t)$ for the instants that are $\geqq \sigma$.

If the values $\alpha_i(x_i, \sigma)$ and $\gamma_i(z_i, \sigma)$ are meaningful, then the definition of φ can be continued such that 0 is replaced by σ and some instant $\sigma'(>\sigma)$ will play the role of σ. This may be continued piece-wise till the infinity unless a contradiction is sometimes produced (in the manner seen above).

PROBLEM 13. *Let the network type introduced in this § be studied, let the analogies and dissimilarities to* [2] *be discovered.*

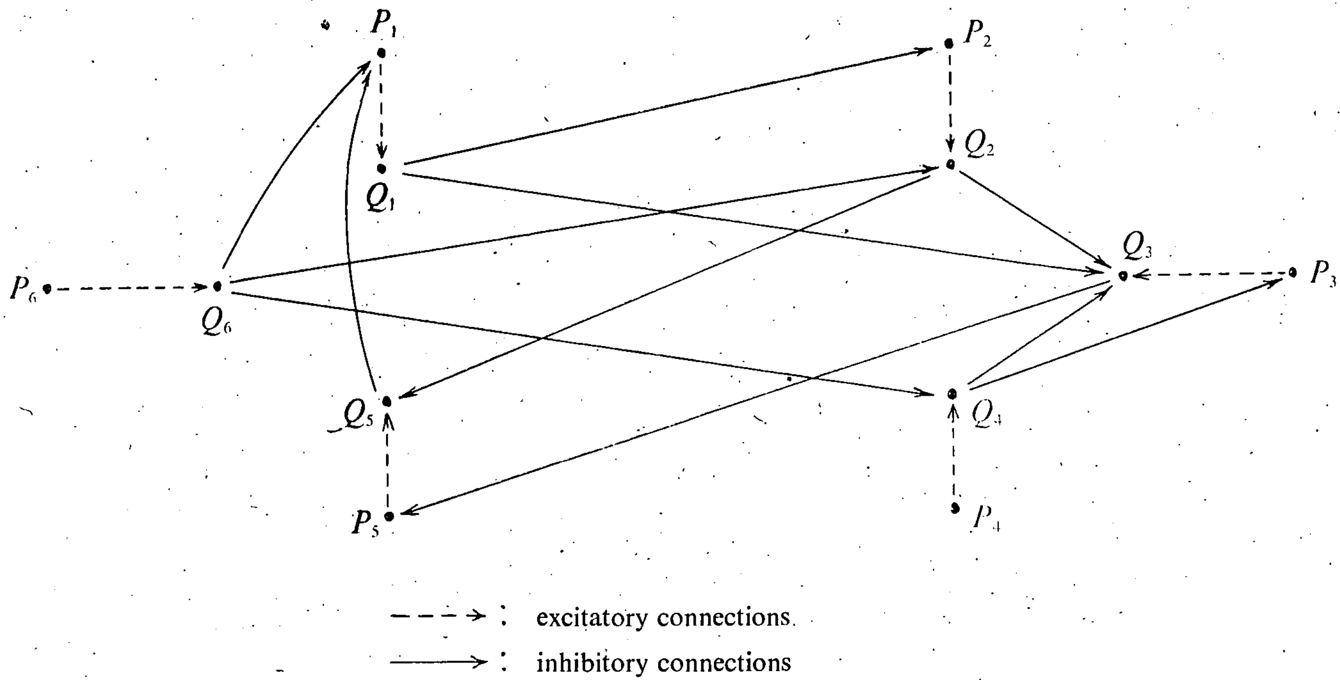

Fig. 5. A complex network.

Since the above definition of complex networks is awfully intricate, the reader may expect eagerly the usual elucidating considerations. The present notion is a modified (and, we can say, improved) version of the simple networks (in § 10). There are two types of neurons (denoted by P's and Q's with subscripts, resp.) and the neurons constitute pairs of form (P_i, Q_i). The edges of form $\overrightarrow{P_i Q_i}$ express *excitatory* connections, the remaining ones (each starting from a Q_i) express inhibiting ones. The inhibited state is denoted by 0 and the recovery state is by numbers in the open interval (0, 1) (without any change). In contrast to the simple networks, now *two* kinds of firing state are defined for the vertices Q_i, they are denoted by the numbers[18] 1 and 2 (for the P_i's only the firing state 1 is permitted). A neuron Q_i is in the state 2 if and only if both of Q_i and P_i have finished a recovery phase (and they did not get inhibition in the meantime). If Q_i is in the state 1, then Q_i does not really produce an inhibiting effect (but it is ready to produce the effect instantaneously when it gets an excitation from its pair P_i). The "background effect" acts similarly to the simple networks (but this effect alone is unable to produce the state 2 for a Q_i). The recovery durations τ_x and τ_z (for the P_i's and Q_i's, resp.) may differ from each other. The detailed prescriptions in (IV) are elaborated in analogy with Section 3 of [2]. Finally we specify the meaning of some formulae in (IV). $\chi_p(\Gamma^{q,2})$ is the of P_i's being inhibited by a Q_i at the initial instant 0. $\chi_p(\Delta^p \cap \Delta^q)$ is the set of P_i's for which an inhibition sets in at the instant σ. $\Gamma^{q,1} - (\Gamma^p \cup \Delta^p)$ is the set of subscripts i such that (1) the state of Q_i was 1 during the interval $[0, \sigma)$ and (2) the state of P_i does not reach 1 at σ. $\Delta^q \cap (\Gamma^p \cup \Delta^p)$ is the set of i's such that (1) the state of Q_i converges to 1 (from below) if t approximates σ (from the left) and (2) the state of P_i reaches 1 at σ. $(\Gamma^{q,2} \cup \Gamma^{q,1}) \cap (\Gamma^p \cup \Delta^p)$ is the set of i's such that (1) the state of Q_i is 1 or 2 during $[0, \sigma)$ and (2) the state of P_i reaches 1 at σ.

III. On the interconnections between structure and function

§ 12.

The exact mathematical treatment of the behaviour of networks dealt with in Chapter 2 of [14] has been done in the article [7]. We applied in [7] the considerations of [2] to the more particular type of networks whose structure is $G(n; 1, 2, \ldots, k)$ with some n and k where $1 \leqq k < n$, $n \geqq 3$ (cf. § 3 and § 10 of this paper); this specialization enables us to deduce more explicit assertions on the behaviour in comparison to the case when (in [2]) we did not restrict ourselves to any special graph structure.

We have introduced in [7] the notion of regular state in terms of certain equalities and inequalities between the values $x_1, x_2, \ldots, x_n$. The main consequences are:

any regular state is periodic and its period is a divisor of the number[19] $\dfrac{\tau[n, k+1]}{k+1}$ (if it is properly periodic),

any non-regular state is either pre-periodic or bounded aperiodic and — respectively to these cases — the corresponding number c_x or b_x does not exceed 2τ.

§§ 6—8 of [3] have been devoted to an extension of the results of [7] to the networks the structure of which belongs to the graph class $\mathfrak{A}(C_\beta)$ where C_β is the family of (β)-graphs (see § 1 and § 5). It was proved that — after a suitable defini-

[18] The occurrence of the isolated number 2 in (II) means that we have given up certain continuity properties of the mathematical model.

[19] $[a, b]$ denotes here the least common multiple of a and b. We say that a is a divisor of b if $ac = b$ with some positive integer c (a and b are not necessarily integers). The word "period" is now meant as it was defined in § 8. (This is the same as "smallest period" in the terminology of the previous articles. Other changes in the terminology are that we say now "periodic" and "non-periodic" instead of "cyclic" and "acyclic", resp.)

tion of regularity — each regular state is periodic; I did not succeed, however, in showing the expected converse of this statement. Let therefore the conjecture that terminates the article [3] be recalled as follows:

PROBLEM 14. *Decide whether or not there exists a properly periodic non-regular state of a network lying in* $\mathfrak{A}(C_\beta)$.

The next question is of comprehensive nature, it proposes further investigations in analogy with [7] (the activity of a network is thought again as it was studied in [2]):

PROBLEM 15. *For network classes corresponding to various graph types affected in Chapter I of this paper, determine the sets of periodic states and the other periodicity properties.*

§ 13.

Let henceforward the network type of [2] be considered. We mention a mathematical formulation of the question how rapidly the effects can be propagated in a network.

Whenever a network and a permitted state $x=(x_1, x_2, \ldots, x_n)$ of it is given and the edge $\overrightarrow{P_jP_i}$ exists, then we call $\overrightarrow{P_jP_i}$ a red edge or a green edge accordingly to which of $x_j<x_i$, $x_j>x_i$ is true[20]. For emphasizing the role of x we can speak of x-red and x-green edges.

We say that the independence statement $I(r, s, t)$ holds if

$$\alpha_i(x, t) = \alpha_i(y, t)$$

is true for every possible choice of G, i, x, y where the occurring symbols have the following meanings:

(i) r and s are non-negative integers, at least one of them is positive,

(ii) t is a positive real number,

(iii) (G, Σ, φ) is a network (as in § 10),

(iv) the number of vertices of G is denoted by n,

(v) i is an integer such that $1 \leqq i \leqq n$,

(vi) $x=(x_1, x_2, \ldots, x_n)$ and $y=(y_1, y_2, \ldots, y_n)$ are states of (G, Σ, φ), each edge of G is supposed to be red or green concerning any of x, y,

(vii) any integer j $(1 \leqq j \leqq n)$ fulfils the condition: either $x_j=y_j$ or each path from P_j to P_i (in G) contains at least r x-red edges, at least r y-red ones, at least s x-green ones and at least s y-green ones.

Conditions (i), (vii) imply $x_i=y_i$ since $x_j \neq y_j$ is possible only if the number of edges of an arbitrary path from P_j to P_i is at least $r+s (>0)$.

We define $\pi(r, s)$ as the largest number u (possibly ∞) possessing the following property: the independence statement $I(r, s, t)$ is true for every t such that $0<t<u\tau$.

[20] The edge is not coloured if $x_i=x_j$.

PROBLEM 16. *Let the function* $\pi(r, s)$ *be studied.*

In the particular case $s=0$, I conjecture $\pi(r, 0)=\left[\frac{r-1}{2}\right]$ ($[a]$ denotes here the integer fulfilling $a-1<[a]\leqq a$).

Example. Let the graph with the four vertices P_1, P_2, P_3, P_4 and three edges $\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_3P_4}$ be considered with the initial states $x=(0.96, 0.97, 0.98, 0.99)$ and $y=(0.95, 0.97, 0.98, 0.99)$. An easy discussion shows that

(1) $$\alpha_2(x, t) = \alpha_2(y, t) \quad \text{if} \quad t < 0.04\tau \quad \text{but}$$
$$\alpha_2(x, 0.04\tau) = 0 \neq 1 = \alpha_2(y, 0.04\tau),$$

(2) $$\alpha_3(x, t) = \alpha_3(y, t) \quad \text{if} \quad t \leqq 0.04\tau \quad \text{but e.g.}$$
$$\alpha_3(x, 0.05\tau) = 0.01 \neq 0 = \alpha_3(y, 0.05\tau),$$

(3) $$\alpha_4(x, t) = \alpha_4(y, t) \quad \text{if} \quad t < 1.04\tau \quad \text{but}$$
$$\alpha_4(x, 1.04\tau) = 0 \neq 1 = \alpha_4(y, 1.04\tau).$$

Hence this example implies $\pi(1, 0)\leqq 0.04$, $\pi(2, 0)\leqq 0.04$, $\pi(3, 0)\leqq 1.04$.

Remark. It may seem to be curious that red and green edges were distinguished in the above definition of π. Now we want to explain why this was done. In fact, one can introduce $I^*(q, t)$ and $\pi^*(q)$ in an analogous (but simpler) manner; this function π^* is, however, identically zero. E.g. the discussion of the network having the edges $\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_3P_4}, \overrightarrow{P_4P_5}$ with the initial states $x=(0, 0.98, 0.97, 0.96, 0.95)$ and $y=(0.99, 0.98, 0.97, 0.96, 0.95)$ shows $\pi^*(4)\leqq 0.04$.

§ 14.

Let the so-named complex networks be considered the formalized definition of which was contained in § 11.

Suppose that H_p and H_q are non-empty finite sets of positive integers such that

$$H_p \cup H_q = \{1, 2, 3, \ldots, |H_p|+|H_q|\}.$$

(This equality implies $H_p \cap H_q)=\emptyset$). Let n, k fulfil $n>k=|H_p|+|H_q|$; we denote by $G(n; H_p, H_q)$ the graph whose vertices are $P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n$ and the edges of which are determined in the following way:

$\overrightarrow{P_iQ_i}$ exists for every i $(1\leqq i\leqq n)$,
$\overrightarrow{Q_iP_j}$ exists precisely when there is an $h(\in H_p)$ such that $i-j\equiv h \pmod{n}$,
$\overrightarrow{Q_iQ_j}$ exists precisely when there is an $h(\in H_q)$ such that $i-j\equiv h \pmod{n}$.
Chapter 3 of [14] gives a suggestion for raising the following question:

PROBLEM 17. *Let the complex networks (in sense of* [14]*) built up over the class of graphs expressible in the form* $G(n; H_p, H_q)$ *be studied (in a manner analogous to* [7]*).*

IV. On the stochastic behaviour of networks

§ 15.

In the network type considered in § 10, the recovery time τ was supposed to be (common for the vertices and) strictly determined. It may be a better simulation of real processes if the value of τ is randomly chosen (at every occasion when a recovery phase takes place) according to some probabilistic distribution. E.g. the following question may be raised:

PROBLEM 18. *Let a (stochastic) analogon of the investigations mentioned in § 10 and § 12 be given when τ is a logarithmically normally distributed stochastic variable (i.e. $\tau = e^{\tau'}$ where τ' is distributed normally).*

§ 16.

Another possibility for probabilistic considerations arises if (τ is deterministic but) the initial state of a network is not fixed. Namely, let the case be considered when

(a) the components of the initial state are numbers chosen randomly (e.g. in sense of the uniform distribution) between 0 and 1,

(b) the class of all (logically possible) manners of behaviour is partitioned to some subclasses $K_1, K_2, \ldots$ by virtue of some simple properties, and

(c) we are interested in the probability $P(K_i)$ of the event that an initial state will lead to a behaviour belonging to K_i.

In § 5 of [7] we have proposed a problem of this type when the graph structure of the networks is $G(n; 1)$ and a behaviour is defined to belong to the class K_i if it leads to a regular state with precisely $i(\leqq n/2)$ vertices being in the maximal state 1.

In §§ 4—5 of the paper [6] an attempt was made for showing how a particular problem, being similar to the mentioned type, can be studied. The following question was discussed:

(i) we consider the networks built up on finite trees with a distinguished vertex (called *root* of the tree) such that every edge is directed towards the root,

(ii) the components of the initial state of a network are chosen randomly (like in (a)),

(iii) we separate five types of the behaviour of the networks (each starting from an initial state) according to which of the following statements are fulfilled:[21]

the state of the root is 1 somewhere in the open interval $(0, \tau)$,

the state of the root is 0 at τ,

the state of the root is 1 at τ,

(iv) the probabilities of the occurrence of the types are calculated such that the number of vertices of the tree is fixed and the trees (having this size) are chosen equiprobably with respect to an isomorphism notion.

[21] These three statements can be combined with each other in eight manners. From among these (logically imaginable) cases, two ones are impossible, a third one is of zero probability.

§ 17.

In the last section of this paper a glance is thrown at the problem of genesis of networks, i.e. how a graph (underlying a network) may develope in a stochastic manner. In the article [11] the following version of this question is thoroughly studied (concerning non-directed graphs):

at the beginning of the process there are given n isolated vertices (this set of vertices remains unchanged),

after i steps we start with a graph having (the n vertices and) i edges from among the $\frac{n(n-1)}{2}$ possible ones, the $(i+1)$-th step is that we draw a further edge e such that e joins one of the $\frac{n(n-1)}{2}-i$ non-adjacent vertex pairs and is chosen equiprobably,

the procedure terminates after k steps.

Erdős and Rényi have determined in [11] which graphs may be typically formed in this manner (depending on the order of magnitude of k). It seems that these typical graphs are quite dissimilar from the graphs belonging to the classes dealt with in our Chapter I. Therefore, if one shares the opinion that the elements of our graph classes are particularly able as carriers of networks producing reasonable activities, then he must seek other principles in addition to the above principle of „inserting a new edge" in sense of Erdős and Rényi.

Such an additional principle may be that an edge ceases to exist under certain circumstances. If some condition, implying that edges are destroyed, is fixed, then one can study e.g. what happens typically when he starts with a complete graph (with n vertices and all the possible $\frac{n(n-1)}{2}$ edges, being the edges oriented randomly and independently of each other) and applies the "edge-destroying" principle in k steps. Of course, the principles of inserting and destroying may also be combined with each other.

Let us return again to the manner of functioning introduced in § 10. A concrete possibility how an edge may cease is illustrated in the following example. Let a network have four vertices P_1, P_2, P_3, P_4, four edges $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$, $\overrightarrow{P_2P_4}$, $\overrightarrow{P_3P_4}$ and the initial state (0.6, 0.7, 0.8, 0.9). A discussion of the activity of this network shows that it is defined only for the instants t smaller than 1.4τ. When the instant 1.4τ is approximated, then both of $\alpha_3(t)$ and $\alpha_4(t)$ converge to 1; this fact and the existence of the edge $\overrightarrow{P_3P_4}$ imply that the working of this network remains undefined if $t \geqq 1.4\tau$. It seems advisable to supplement the network definition by demanding that an edge e is deleted at such an instant when both vertices incident to e reach the value 1 simultaneously.

PROBLEM 19. *Let the activity of a network be understood as in § 10. Which typical graph structures (depending on the order of magnitude of k) arise if we start with a complete graph with n vertices and we apply the edge-destroying principle, mentioned above, till when k edges had been deleted?*

It is expectable that the stochastic considerations proposed in § 16 are closely related to the study of Problem 19.

I wish to express my thanks to Dr. E. Lábos and Dr. L. Lovász (the official referees of this article) for a number of useful advices.

О некоторых открытых проблемах прикладной теории автоматов и теории графов (возбуждаемых математическим моделированием неких нейрональных сетей)

Статья [14] U. Kling-a и Gy. Székely-a брала инициативу к математическому моделированию строении и функционировании некоторых нейрональных сетей, см. работы [2], [7], [3] и т. д. В настоящей статье даётся краткий обзор этих исследований и специфицировается ряд дальнейших нерешенных математических вопросов. В первой из четырёх глав работы содержаются проблемы относительно конечных ориентированных графов.

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(Received June 10, 1976)