On the formal definition of VDL-objects

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Originally the VDL (Vienna Definition Language) was designed for defining programming languages [1], [2], [3], but recently it has been used as a general technique of defining data structures and algorithms [4].

The VDL is a definition system. This system consists of objects, a machine operating on objects and a programming language.

The VDL-objects are abstractions of data structures of a certain type. In this paper we deal with the objects and the basic operators of VDL manipulating on objects.

The VDL-objects form a set with the elements of which there are associated selection and construction operators. The basic properties of the operators are taken as axioms and their main properties are proved. A complet formal system of VDL-objects is given, which can be regarded as a detailed elaboration of the axiomatic definition of VDL data structures given in [4] and [5].

Definition 1. The elements of the non empty set OB are called objects, if there exists a finite set S of selectors and a construction function k such that

$$s: OB \to OB \quad \text{for all} \quad s \in S, \quad \text{and}$$

$$k: OB \times S \times OB \to OB.$$

It is assumed the validity of the following:

Axiom 1. If $t \in OB$, $s \in S$, $t_1 \in OB$, then

$$s(k(t, s, t_1)) = t_1,$$

and

$$s'(k(t, s, t_1)) = s'(t) \quad \text{for all} \quad s' \in S \quad \text{and} \quad s' \neq s.$$

The "fixed point" of the system, i.e. the null object of the set OB is defined as follows:

Definition 2. A $t \in OB$ is called the *null object* if and only if

$$(\forall s \in S)(s(t) = t$$

Axiom 2. There is exatly one null object.

In the following we denote the null object by Ω.

The objects can be classified according to their "distance" from the null object. The so called elementary objects are "nearest" to the null object, and they can be defined in the following way:

Definition 3. A $t \in OB$ is called *elementary object* if and only if

$$(\forall s \in S)(s(t) = \Omega).$$

Let EO be the set of elementary objects.

Definition 4. The elements of the set

$$CO = OB \setminus EO$$

are called *composite objects.*

Axiom 3. If $t \in OB$ then there exists an integer N_t such that for any sequence $s_1 \in S, s_2 \in S, \dots, s_n \in S, (n \geqq N_t)$

$$s_n(\dots(s_2(s_1(t))\dots) = \Omega.$$

COROLLARY 1. There is no $t \in OB, t \neq \Omega$ for which

$$s_m(\dots(s_2(s_1(t))\dots) = t.$$

Axiom 4. Elementary objects are regarded as different, that is if

$$EO = \{\dots, t_i, \dots, t_j, \dots\}$$

then $t_i \neq t_j$.

Definition 5. The objects $t_1 \in CO$ and $t_2 \in CO$ are *equal* if and only if

$$(\forall s \in S)(s(t_1) = s(t_2)).$$

Lemma 1. Ω is an elementary object.

Proof. This results from Definitions 2 and 3.

Theorem 1. If EO has at least two elements, then CO is a non empty set.

Proof. Let $t \in OB, t \neq \Omega$ and $s \in S$. Then by Axiom 1.

$$s(k(t, s, t)) = t,$$

and hence

$$k(t, s, t) \in CO.$$

Theorem 2. If CO is a non empty set, then EO has at least two elements.

Proof. Let us suppose, that $EO = \{\Omega\}$. Let $t \in CO$. Then, by definition,

$$(\exists s_1 \in S)(s_1(t) \neq \Omega).$$

But the set EO has only one element. Therefore

$$t_1 = s_1(t) \in CO.$$

Hence

$$(\exists s_i \in S)(s_2(t_1) \neq \Omega).$$

The repeated application of this procedure leads to a contradiction to Axiom 3.
Now let us consider the "structure" of the objects. First of all we define the immediate components of the object.

Definition 6. If $t \in OB$ and $s \in S$, then the object $s(t)$ is called the *immediate component* of the object t.

COROLLARY 2. All immediate components of an elementary object are the null object.

Definition 7. Let $t \in CO$. The *immediate characteristic set* of t is defined as

$$\{\langle s_1 : s_1(t)\rangle, \langle s_2 : s_2(t)\rangle, \ldots, \langle s_m : s_m(t)\rangle\}$$

where

$$s_i(t), (s_i \in S), \quad i = 1, 2, \ldots, m$$

are all non null immediate components of the object t.

Lemma 2. Any composite object can be uniquely represented with its immediate characteristic set.

Proof. This follows from Definition 5 immediately.

Definition 8. Let $t \in CO$ and let

$$t_1, t_2, \ldots, t_m$$

be every non null immediate component of the object t such that

$$s_i(t) = t_i, \quad i = 1, 2, \ldots, m,$$

then the symbol

$$\mu_0(\langle s_1 : t_1\rangle, \langle s_2 : t_2\rangle, \ldots, \langle s_m : t_m\rangle)$$

will stand for the object t.
By Lemma 2, this is an unambigouos representation of the object t.
The composite object can be represented also by a tree as shown in Fig. 1.

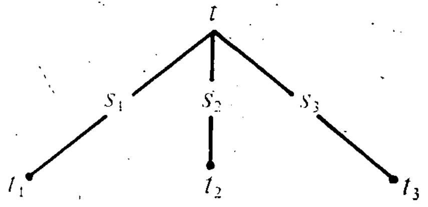

Fig. 1
The tree of the object

$$t = \mu_0(\langle s_1 : t_1\rangle, \langle s_2 : t_2\rangle, \langle s_3 : t_3\rangle)$$

Theorem 3. If

$$t = \mu_0(\langle s_1 : t_1 \rangle, \langle s_2 : t_2 \rangle, \ldots, \langle s_m : t_m \rangle)$$

then the object t can be constructed from the objects $t_1, t_2, \ldots, t_m$ by applying the operation k m times.

Proof. Let us consider the following sequence

$$y_1 = k(\Omega, s_1, t_1),$$
$$y_2 = k(y_1, s_2, t_2),$$
$$\vdots$$
$$y_m = k(y_{m-1}, s_m, t_m).$$

Then, by Axiom 1, we have

$$s_m(y_m) = t_m,$$
$$s_{m-1}(y_m) = s_{m-1}(y_{m-1}) = t_{m-1},$$
$$\vdots$$
$$s_1(y_m) = s_1(y_{m-1}) = \ldots = s_1(y_1) = t_1,$$

and for every $s \in S$, $s \neq s_i$, $i = 1, 2, \ldots, m$,

$$s(y_m) = \Omega.$$

Hence, by Lemma 2, we have the result $y_m = t$.

Definition 9. The composition

$$\chi = s_m \circ s_{m-1} \circ \ldots \circ s_1, \quad s_i \in S, \quad i = 1, 2, \ldots, m$$

is called *composite selector*. The result of applying a composite selector χ to an objects $t \in CO$ is defined as follows

$$\chi(t) = s_m(\ldots(s_2(s_1(t)))\ldots).$$

Let S^* be the set of all the composite selectors constructed by the elements of S and all the simple selectors.

Definition 10. If $\chi = s \circ \chi'$, $\chi' \in S^*$, $s \in S$, $t \in OB$, $t_1 \in OB$ then

$$k(t, \chi, t_1) = k(t, \chi', k(\chi'(t), s, t_1)).$$

Theorem 4. The objects $t_1 \in CO$, $t_2 \in CO$ are equal if and only if

$$(\forall \chi \in S^*)(\chi(t_1) = \chi(t_2)).$$

Proof. Let $\chi = s_m \circ \ldots \circ s_2 \circ s_1$.
If $t_1 = t_2$, then, by Definition 5, we have

$$s_1(t_1) = s_2(t_2)$$
$$s_2 \circ s_1(t_1) = s_2 \circ s_1(t_2)$$
$$\vdots$$
$$\chi(t_1) = \chi(t_2)$$

for every composite selector in S^*.

On the other hand, if

$$(\forall \chi \in S^*)(\chi(t_1) = \chi(t_2))$$

then

$$(\forall s \in S)(s(t_1) = s(t_2)),$$

i.e., the immediate characteristic sets of t_1 and t_2 are equal. Therefore, $t_1 = t_2$.

Theorem 5. For every object $t \in CO$ there exists a composite selector χ such that

$$\chi(t) \in EO \setminus \{\Omega\}.$$

Proof. By Definition 4, we have

$$(\exists s_1 \in S)(s_1(t) \neq \Omega).$$

If $t_1 = s_1(t) \in EO$, then the assertion follows immediately. Otherwise $t_1 \in CO$ and as above we have

$$(\exists s_2 \in S)(s_2(t_1) = t_2 \neq \Omega),$$

and so forth, by virtue of Axiom 3,

$$(\exists n \geqq 1)(s_n(t_{n-1}) \in EO \setminus \{\Omega\}).$$

Therefore, for $\chi = s_n \circ \ldots \circ s_2 \circ s_1$,

$$\chi(t) \in EO \setminus \{\Omega\}.$$

Definition 11. Let

$$H_1(t) = \{\langle s_1 : t_1 \rangle, \langle s_2 : t_2 \rangle, \ldots, \langle s_m : t_m \rangle\}$$

be the immediate characteristic set of $t \in CO$. Let us introduce the notation

$$H_1(t) = \{\langle \chi_1^{(1)} : t_1^{(1)} \rangle, \langle \chi_2^{(1)} : t_2^{(1)} \rangle, \ldots, \langle \chi_m^{(1)} : t_m^{(1)} \rangle\},$$

where

$$\chi_i^{(1)} = s_i, \; t_i^{(1)} = t_i, \quad i = 1, 2, \ldots, m.$$

If

$$(\exists j, 1 \leqq j \leqq m)(t_j^{(1)} \in CO)$$

choose the smallest j such that $t_j^{(1)} \in CO$ and let

$$\bar{H}(t_i) = \{\langle s_1' : z_1 \rangle, \langle s_2' : z_2 \rangle, \ldots, \langle s_n' : z_n \rangle\}$$

be the immediate characteristic set of t_i, where $s_i' \in S$, $i = 1, 2, \ldots, n$.

Substituting $\overline{H}(t_i)$ into $H_1(t)$, the following set may be derived

$$H_2(t) = \{\langle\chi_1^{(1)}:t_1^{(1)}\rangle, \ldots, \langle s_1'\circ\chi_i^{(1)}:z_1\rangle, \ldots, \langle s_n'\circ\chi_i^{(1)}:z_n\rangle, \ldots, \langle\chi_m^{(1)}:t_m^{(1)}\rangle\} =$$
$$= \{\langle\chi_1^{(2)}:t_1^{(2)}\rangle, \ldots, \langle\chi_M^{(2)}:t_M^{(2)}\rangle\}.$$

Iterating the preceding procedure, we can generate the sequence of sets

$$H_1(t), H_2(t), H_3(t), \ldots$$

The elements of this sequence are called *characteristic sets* of t.

Definition 12. Let $t\in CO$ and let

$$H_i(t) = \{\langle\chi_1^{(i)}:t_1^{(i)}\rangle, \ldots, \langle\chi_{m_i}^{(i)}:t_{m_i}^{(i)}\rangle\}, \quad i = 1, 2, \ldots$$

be all the characteristic sets of t. The characteristic set

$$H_N(t) = \{\langle\chi_1^{(N)}:t_1^{(N)}\rangle, \ldots, \langle\chi_{m_N}^{(N)}:t_{m_N}^{(N)}\rangle\}$$

for which

$$(\forall j, 1 \leqq j \leqq m_N)(t_j^{(N)}\in EO)$$

is called the *elementary characteristic set* of t.

Theorem 6. Let $t\in CO$, then the sequence of characteristic sets

$$H_1(t), H_2(t), \ldots$$

is finite, and its last element is the elementary characteristic set of t.

Proof. By Axiom 3, the procedure given in Definition 11 terminates after a finite number of steps, and the last element of the sequence obviously satisfies the criteria of the elementary characteristic set of t.

Theorem 7. A composite object t can be uniquely represented with its any characteristic set.

Proof. On the base of the procedure given in Definition 11, by Lemma 2, Theorem 7 follows for t and $H_1(t)$. Similarly it is also true for t_i and $\overline{H}(t_i)$. Hence t can be uniquely represented with $H_2(t)$. Similarly we may show that t can be uniquely represented with $H_3(t)$, $H_4(t)$, ...

COROLLARY 3. It follows from the Theorems 6 and 7 that any $t\in CO$ can be unambigouosly represented by an elementary characteristic set.

Definition 13. Let $t\in CO$ and let

$$H(t) = \{\langle\chi_1:t_1\rangle, \ldots, \langle\chi_m:t_m\rangle\}$$

be a characteristic set of t. Then the object t can be notified by the symbol

$$\mu_0(\langle\chi_1:t_1\rangle, \ldots, \langle\chi_m:t_m\rangle).$$

Based on Theorem 6, every composite object can be represented by a tree in which there are only elementary objects as terminal nodes. For example the composite object

$$t = \mu_0(\langle s_1 : a\rangle, \langle s_1 \cdot s_2 : b\rangle, \langle s_3 \cdot s_2 : c\rangle)$$

where

$$\{a, b, c\} \subseteqq EO$$

can be represented by the tree shown in Fig. 2.

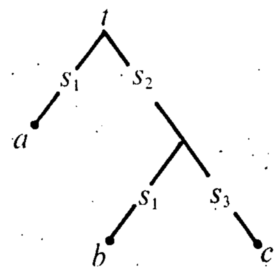

Fig. 2

The tree of the object

$$t = \mu_0(\langle s_1 : a\rangle, \langle s_1 \circ s_2 : b\rangle, \langle s_3 \circ s_2 : c\rangle)$$

Definition 14. A composite selector χ is said to be dependent on a composite selector χ' if and only if

$$\chi' = \chi'' \circ \chi \quad \text{or} \quad \chi' = \chi \quad \text{for some} \quad \chi'' \in S^*.$$

Definition 15. The selectors χ_i and χ_j are said to be *independent* if and only if neither χ_i is dependent on χ_j nor χ_j is dependent on χ_i.

Theorem 8. Let

$$H(t) = \{\langle \chi_1 : t_1\rangle, \langle \chi_2 : t_2\rangle, \ldots, \langle \chi_m : t_m\rangle\}$$

be a characteristic set of $t \in CO$. Then for all $1 \leqq i, j \leqq m$, $i \neq j$ implies that χ_i and χ_j are independent.

Proof. The proof is by induction on k in $H_k(t)$. Based on definition 14, every pair $\chi_i^{(1)}, \chi_j^{(1)}$ is independent in $H_1(t)$.

Assume that Theorem 8 holds for every $H_i(t)$, $1 \leqq i \leqq k$ and prove it for $H_{k+1}(t)$. Let

$$H_k(t) = \{\langle \chi_1^{(k)} : t_1^{(k)}\rangle, \ldots, \langle \chi_{m_k}^{(k)} : t_{m_k}^{(k)}\rangle\},$$

and

$$t_1^{(k)} \in EO, \ldots, t_{j-1}^{(k)} \in EO, \quad \text{but} \quad t_j^{(k)} \in CO. \quad \text{Let}$$

$$\{\langle s_1 : z_1\rangle, \ldots, \langle s_N : z_N\rangle\}$$

be the immediate characteristic set of $t_j^{(k)}$. Then

$$H_{k+1}(t) = \{\langle \chi_1^{(k)} : t_1^{(k)}\rangle, \ldots, \langle s_1 \circ \chi_j^{(k)} : z_1\rangle, \ldots, \langle s_N \circ \chi_j^{(k)} : z_N\rangle, \ldots$$
$$\ldots, \langle \chi_{m_k}^{(k)} : t_{m_k}^{(k)}\rangle\}$$

Here, by our assumption, every pair $\chi_p^{(k)}$, $\chi_q^{(k)}$ is independent. We now have to show that

$$\chi_p = s_p \circ \chi_j^{(k)} \quad (1 \leqq p \leqq N)$$

and

$$\chi_q = s_q \circ \chi_j^{(k)} \quad (1 \leqq q \leqq N)$$

$(p \neq q)$ are also independent. For example, we can easily see, that there exists no χ such that

$$\chi_p = \chi \circ \chi_q,$$

because

$$s_p \neq \chi \circ s_q.$$

Similarly, it is also easy to show that any $s_p \circ \chi_j^{(k)}$ is not dependent on $\chi_i^{(k)}$, $i \neq j$.

Theorem 9. If $\chi \in S^*$, $t \in OB$, $t_1 \in OB$, then

$$\chi(k(t, \chi, t_1)) = t_1$$

and

$$\chi'(k(t, \chi, t_1)) = x'(t)$$

provided that χ is not dependent on χ', $\chi' \in S^*$.

Proof. We prove the theorem by induction. Consider the selector

$$\chi_m = s_m \circ s_{m-1} \circ \ldots \circ s_1 \quad (s_i \in S).$$

If $\chi = \chi_1$ and $\chi' = \chi_1$ then the Theorem is true by Axiom 1.

The principle of the induction states:

If our Theorem holds for any $\chi = \chi_k$ and $\chi' = \chi_j'$ with $1 \leqq k, j \leqq m$ then it also holds for any $\chi = \chi_k$ and $\chi' = \chi_j'$ with $1 \leqq k, j \leqq m+1$.

Assume the Theorem is true for all

$$\chi = \chi_k \quad \text{and} \quad \chi' = \chi_j' \quad \text{with} \quad 1 \leqq k, j \leqq m$$

and prove it for all $1 \leqq k, j \leqq m+1$. By Definition 10,

$$k(t, \chi_{m+1}, t_1) = k(t, \chi_m, k(\chi_m(t), s, t_1)),$$

where $\chi_{m+1} = s \circ \chi_m$. Furthermore, by our assumption,

$$s \circ \chi_m(k(t, \chi_m, k(\chi_m(t), s, t_1))) = s(k(\chi_m(t), s, t_1)),$$

and, by Axiom 1,

$$s(k(\chi_m(t), s, t_1)) = t_1.$$

Consider the second equation in the Theorem. If χ is not dependent on χ' then

$$\chi' = \bar{\chi}' \circ s', \chi = \bar{\chi} \circ s, \quad \text{where} \quad s \neq s'$$

or

$$\chi' = \chi_1' \circ \chi_2, \quad \chi = \chi_1 \circ \chi_2$$

where

$$\chi_1' = s_1' \circ s_2' \circ \ldots \circ s_i',$$

$$\chi_1 = s_1 \circ s_2 \circ \ldots \circ s_j$$

and

$$s_i' \neq s_j.$$

Hence

$$\bar{\chi}' \circ s'(k(t, \chi, t_1)) = \bar{\chi}' \circ s'(k(t, s, k(s(t), \bar{\chi}, t_1))) = \bar{\chi}' \circ s'(t)$$

or

$$\chi_1' \circ \chi_2(k(t, \chi, t_1)) = \chi_1' \circ \chi_2(k(t, \chi_2, k(\chi_2(t), \chi_1, t_1))) =$$
$$\chi_1'(k(\chi_2(t), \chi_1, t_1)) = \chi_1' \circ \chi_2(t).$$

This completes the proof.

Theorem 10. Let

$$t = \mu_0(\langle \chi_1 : t_1 \rangle, \ldots, \langle \chi_m : t_m \rangle).$$

Then the object t can be constructed from the objects $t_1, t_2, \ldots, t_m$ by applying the operation k m times.

Proof. The proof is analogous to that of Theorem 3.

Due to this Theorem, every composite object can be constructed from elementary objects too. Hence each composite object is a structure of elementary objects.

Definition 16.

$$k(t, \chi_1, t_1, \chi_2, t_2, \ldots, \chi_n, t_n) = k(k(t, \chi_1, t_1), \chi_2, t_2, \ldots, \chi_n, t_n)$$

Theorem 11. If χ_1 and χ_2 are independent, then for arbitrary objects t_1 and t_2

$$k(t, \chi_1, t_1, \chi_2, t_2) = k(t, \chi_2, t_2, \chi_1, t_1).$$

Proof. Consider the right side of the equation. It follows from Theorem 9 that

$$\chi_1(k(k(t, \chi_2, t_2), \chi_1, t_1)) = t_1,$$

and for every χ' which is not dependent on χ_1,

$$\chi'(k(k\ (t, \chi_2, (t_2), \chi_1,)\ (t_1) = \chi' k(t, \chi_2, t_2).$$

Hence for $\chi' = \chi_2$,

$$\chi_2(k(k(t, \chi_2, t_2), \chi_1, t_1)) = \chi_2(k(t, \chi_2, t_2)) = t_2,$$

and for every χ'' which is not dependent on χ_2,

$$\chi''(k(k(t, \chi_2, t_2), \chi_1, t_1)) = \chi''(k(t, \chi_2, t_2)) = \chi''(t).$$

Similarly, it can be shown that

$$\chi_1(k(t, \chi_1, t_1, \chi_2, t_2)) = t_1,$$
$$\chi_2(k(t, \chi_1, t_1, \chi_2, t_2)) = t_2,$$

and for every $\bar{\chi}$ which is not dependent on χ_1 and χ_2

$$\bar{\chi}(k(t, \chi_1, t_1, \chi_2, t_2)) = \bar{\chi}(t).$$

This completes the proof.

In the VDL the following notation is used:

$$\mu(t; \langle \chi_1 : t_1 \rangle, \ldots, \langle \chi_n : t_n \rangle) \equiv k(t, \chi_1, t_1, \ldots, \chi_n, t_n).$$

О формальном определении VDL-объектов

Первоначально VDL был предназначен для определения языков программирования [1, 2, 3], но в последнее время применяется и как общий метод определения структуры данных и алгоритмов [4].

VDL является системой определения. Эта система состоит из объектов, машины десйтвующей над объектами, и из языка программирования.

VDL-объекты представляют собой структуры данных определенного типа. В данной работе изучаются объекты и основные VDL-операторы, действующие над объектами.

С элементами множества VDL-объектов связаны операторы выбора и конструирования. Основные свойства этих операторов излошены в виде аксиом, а дальнейшие свойства доказаны. Таким образом, задана польная формальная система VDL-объектов, которую можно рассматривать как подробную разработку аксиоматического определения структуры данных VDL, предложенного в [4] и [5].

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