Local and global reversibility of finite inhomogeneous cellular automaton*

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Cellular automata are highly parallel working systems, so they have high importance in computational applications (for example sorting [4], matrix operations, etc.). It seems difficult to apply the classical infinite, homogeneous cellular automata to these purposes [1], [2]. For this reason the classical definitions are modified in this work. In point 1. we introduce the notion of *finite, inhomogeneous* cellular automaton. The reason of first modification (using by many authors, e.g. [7]) is clear: only finite automaton is realisable in practice. Further the second modification (the inhomogeneity) makes the cellular automaton more flexible [11], without excluding the homogeneity in hardware [3].

In the theory of cellular automata there is a very important and interesting question, that how appear the characteristics of local maps in the global map, and conversely. This is the basic conception of present work too, having in the centre the problem of *reversibility*. This subject has been investigated by many authors (in particular by T. Toffoli [8], [9]), but always in the global sense. In this context the reversibility is equivalent to the *bijectivity* of global map.

To the contrary, we mean the reversibility in *local sense*: a cellular automaton we shall call reversible, if its local maps may be changed so, that the new global map is the inverse of the original one.

The bijectivity of global map forms necessary condition for our "strong reversibility". Therefore in point 2. a connection will be proved between the local maps and the number of eden-configurations, from which derives a necessary condition for bijectivity (it is the generalization of results in [5]).

In point 3. a necessary and sufficient condition is presented to the reversibility. With this criterion we can decide the reversibility of a given cellular automaton, and construct its reverse.

The point 4. contains concrete investigations in case of one-dimensional cellular automaton, with the result: only very simple reversible cellular automata exist in this special case.

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1. Basic definitions

(i) *Inhomogeneous cellular automaton* is a (C, A, N, Φ) four-tuple, where
$C=\{c_1, \dots, c_m\}$ is the finite *set of cells*,
$A=\{0, 1, \dots, s-1\}$ is the *set of cell-states*,
N: $c_i \mapsto (c_{i_1}, \dots, c_{i_{n_i}})$ is the *neighbourhood function*, which assigns to each cell its neighbours. (The specification of neighbours may be different cell by cell, i.e. the cellular automaton has totally arbitrary topology.)
Φ: $c_i \mapsto f_i$ is the *function-system*, which assigns to each cell an f_i: $A^{n_i} \to A$ local map. (The local maps also may be different cell by cell.)

(ii) *Configuration* is a map α: $C \to A$, we denote it always with Greek letters.
(iii) *Neighbourhood* of cell c_i in a given configuration is the n_i-tuple of states of its neighbours.
(iv) The *global map* of a cellular automaton is a map F: $\mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is the set of all configurations, and $F(\alpha)=\beta$, if for all i $f_i(a_{i_1}, \dots, a_{i_{n_i}})=\beta(c_i)$ (where $(a_{i_1}, \dots, a_{i_{n_i}})$ is the neighbourhood of c_i in α).
In further we use the abbreviation CA instead of cellular automaton.

2. Relation between the local maps and the number of eden-configurations

We consider a CA (C, A, N, Φ) with the global map F.
The following definition is well-known from the literature:
Definition. A configuration α will be called *garden-of-eden configuration* (in short eden-configuration), if there is no β, for which $F(\alpha)=\beta$.
We have an obvious equivalence:
F is bijective $\Leftrightarrow$ there is no eden-configuration.

Let be c a cell with n neighbours, and f its local map. Suppose, that there are p_a different neighbourhoods of c, where the new cell-state given by f is a. The number of all possible neighbourhoods is s^n, consequently $\sum_{a \in A} p_a = s^n$.

Definition. We say, that the local map f is balanced, if $\forall a$: $p_a=p$, where obviously $p=s^n/s=s^{n-1}$.

When f is unbalanced, the measure of this may be characterized with the quantity $q=\sum_{\substack{a \in A \\ p_a < p}} (p-p_a)$, and we say: f is q-unbalanced.

Theorem. Let be (C, A, N, Φ) an arbitrary CA, c a cell in it, and f its local map. If f is q-unbalanced, then the CA has at least $q \cdot s^{m-n}$ eden-configurations (m is the number of cells, s is the number of cell-sates).

Proof. It is clear, that there are s^{m-n} different configurations, where the neighbourhood of c is a given $(a_1, \dots, a_n)$. So there are exactly $p_a \cdot s^{m-n}$ configurations, where the new cell-state of c is a. At the same time the number of all configurations, where the state of c is a, is $s^{m-1}=p \cdot s^{m-n}$. Consequently if $p_a<p$, then among these $p \cdot s^{m-n}$ configurations there are $(p-p_a) \cdot s^{m-n}$ eden-configurations.

We find the same situation by all state a having the property $p_a<p$, consequently the CA has at least $\sum_{p_a<p} (p-p_a) \cdot s^{m-n}$ eden-configurations. □

Corollaries. (i) If in a CA for any i the local map of c_i is q_i-unbalanced, then the CA has at least $\max\limits_{1 \leq i \leq m} (q_i \cdot s^{m-n})$ eden-configurations.

(ii) To the bijectivity of global map is necessary condition, that all local maps are balanced.

Similar results are published in works [5], [6] on classical infinite, homogeneous CA.

3. The problem of reversibility
Algorithm for decision of reversibility, and construction of the reverse

Definition. A CA (C, A, N, Φ) with a global map F is *reversible*, if there exists another function-system Φ' such, that the CA (C, A, N, Φ') generates the global map F^{-1}.

The first problem in this subject: to decide from a given CA, whether it is reversible. On this purpose we introduce a general algorithm, which is suitable for constructing the reverse, too.

Let be (C, A, N, Φ) a CA, c_i a cell in it. Let's denote with N_1 the neighbours of c_i, and with N_2 the neighbours of neigbours (with a bit incorrect notation $N_1 = N(c_i)$, $N_2 = N(N(c_i))$). It is clear, that the state of N_1 at time $t+1$ is determined by the state of N_2 at time t. If we know the local functions in N_1, we may describe this transition with a table called in following as *inverse-constructing-table* (ICT in short). In case of one-dimensional, two-state CA it is illustrated on figure 1.

If the cell c_i has an f_i' reverse local function, then this function gives back from any N_1-state of column $t+1$ of ICT the state of c_i in column t. Consequently the existence of f_i' has the following necessary condition: if two N_1-states in column $t+1$ of ICT are equal, then the corresponding c_i-states in column t also should be equal. Furthermore this condition is sufficient to the existence of f_i' reverse function, because we may construct it by the ICT.

$$\underbrace{\left| \, c_{i-2} \, \right| \overbrace{c_{i-1} \, \left| \, c_i \, \right| \, c_{i+1}}^{N_1} \left| \, c_{i+2} \, \right|}_{N_2}$$

The local maps in N_1:

	f_{i-1}		f_i		f_{i+1}
000	x_0	000	y_0	000	z_0
001	x_1	001	y_1	001	z_1
010	x_2	010	y_2	010	z_2
011	x_3	011	y_3	011	z_3
100	x_4	100	y_4	100	z_4
101	x_5	101	y_5	101	z_5
110	x_6	110	y_6	110	z_6
111	x_7	111	y_7	111	z_7

The ICT of c_i:

t	$t+1$	t	$t+1$
00000	$x_0 y_0 z_0$	10000	$x_4 y_0 z_0$
00001	$x_0 y_0 z_1$	10001	$x_4 y_0 z_1$
00010	$x_0 y_1 z_2$	10010	$x_4 y_1 z_2$
00011	$x_0 y_1 z_3$	10011	$x_4 y_1 z_3$
00100	$x_1 y_2 z_4$	10100	$x_5 y_2 z_4$
00101	$x_1 y_2 z_5$	10101	$x_5 y_2 z_5$
00110	$x_1 y_3 z_6$	10110	$x_5 y_3 z_6$
00111	$x_1 y_3 z_7$	10111	$x_5 y_3 z_7$
01000	$x_2 y_4 z_0$	11000	$x_6 y_4 z_0$
01001	$x_2 y_4 z_1$	11001	$x_6 y_4 z_1$
01010	$x_2 y_5 z_2$	11010	$x_6 y_5 z_2$
01011	$x_2 y_5 z_3$	11011	$x_6 y_5 z_3$
01100	$x_3 y_6 z_4$	11100	$x_7 y_6 z_4$
01101	$x_3 y_6 z_5$	11101	$x_7 y_6 z_5$
01110	$x_3 y_7 z_6$	11110	$x_7 y_7 z_6$
01111	$x_3 y_7 z_7$	11111	$x_7 y_7 z_7$

Fig. 1

The construction of ICT in case of one-dimensional two-state CA.

So the following in obtained:

Proposition. A CA (C, A, N, Φ) is reversible $\Leftrightarrow$ for each cell c_i, its ICT satisfies: if two N_1-states in column $t+1$ agree, then the corresponding c_i-states in column t must agree too.

If this condition is satisfied, then we can construct the reverse function-system.

4 The reversibility of one-dimensional two-state cellular automaton

The preceding algorithm decides only about a given Φ whether it is reversible, but does not help to find concrete reversible function-systems. It is clear, that there exist trivial ones, for example the identical function-system (where each cell keeps its state, independently of neighbours), or the shift function-system, (where each cell receives the state of the same neighbour).

Nontrivial reversible function-systems have high importance in practice, but to construct them is very difficult. In further we give a necessary condition to the reversibility of one-dimensional two-state CA, from which we shall see, that in one-dimension only very special function-systems are reversible, consequently it is easy to construct them.

So in following the CA (C, A_0, N_0, Φ) will be investigated, where

$C=\{c_1, \ldots, c_m\}$, $m \geqq 5$ is supposed (this assumption makes easier the investigation),

$A_0 = \{0, 1\}$,

N_0: $c_i \mapsto (c_{i-1}, c_i, c_{i+1})$, the indexes are interpreted cyclically (i.e. c_1 and c_m are neighbours). Thus we have a circle-topology.

Φ is arbitrary.

We need the following general definition:

Definition. In a CA (C, A, N, Φ) the cell c_i *depends* on its neighbour c_j, if there are two neighbourhoods of c_i such, that they differ only in state of c_j, and the corresponding new states of c_i are different.

Using this notion we take a remark to the definition of (C, A_0, N_0, Φ): if Φ is such, that c_1 and c_m are independent each of other, then the circle-topology we may replace with a section-topology. So our definition contains the section-topology too.

Two lemmas will be proved in further. In proofs we shall use often the fact, that for reversibility is necessary condition that all local maps are balanced. (It results from the second corollary in point 2.) Moreover we shall use the notation $\bar{a}$, which denotes the opposite of cell-state a.

Lemma 1. Suppose that Φ is reversible, and its reverse is Φ'. In this case if c_{i-1} depends on c_{i-2} by the function-system Φ, then c_i is independent of c_{i-1} by Φ'.

Proof. Suppose, that c_{i-1} depends on c_{i-2}, i.e. there are a, b such, that $f_{i-1}(0, a, b)=x$, and $f_{i-1}(1, a, b)=\bar{x}$.

Now let's consider the function f_{i+1}! We have two different cases:

(i) $\exists y\colon \forall c, d\colon f_{i+1}(b, c, d) = y$.

The function f_{i+1} is balanced, therefore $\forall c, d$: $f_{i+1}(\bar{b}, c, d)=\bar{y}$, that is to say, c_{i+1} depends only on c_i. Thus by the reverse c_i depends only on c_{i+1}.

(ii) $\exists c, d$ and $\exists c', d'\colon f_{i+1}(b, c, d) = y$ and $f_{i+1}(b, c', d') = \bar{y}$.

Let $f_i(a, b, c)=p$, $f_i(a, b, c')=q$. So the ICT of cell c_i contains the following part:

$$\begin{array}{ccccc|ccc} & & t & & & & t+1 & \\ 0 & a & b & c & d & x & p & y \\ 1 & a & b & c & d & \bar{x} & p & y \\ 0 & a & b & c' & d' & x & q & \bar{y} \\ 1 & a & b & c' & d' & \bar{x} & q & \bar{y} \end{array}$$

The four binary triples in column $t+1$ are different, and the reverse function f_i' constructed by the table assigns to each triple the same state b. But f_i' is balanced, so it assigns to the other four triple the state $\bar{b}$. By this the table of f_i' is known. We can see from it, that c_i is independent of c_{i-1}. □

The second lemma needs the following definition:

Definition. Let $c_i, \ldots, c_j$ be a section of cells. We say, that it is *isolated*, if c_i is independent of c_{i-1}, and c_j of c_{j+1}.

Lemma 2. Suppose that Φ is reversible, and its reverse is Φ'. In this case if the section $c_i, \ldots, c_j$ is isolated by Φ, then it is isolated by Φ' too.

Proof. Two configurations will be called equivalent (with respect to the section $c_i, \ldots, c_j$), if their sections corresponding to the $c_i, \ldots, c_j$ are equal. So a classification is obtained on the set $\mathcal{A}$.

It is easy to prove the following chain: $c_i, \ldots, c_j$ is isolated by $\Phi \Rightarrow$ the previous classification is F-compatible (i.e. $\forall \alpha, \beta$: $\alpha \sim \beta \Rightarrow F(\alpha) \sim F(\beta)$) $\Rightarrow$ it is F^{-1}-compatible too (because F is one-to-one) $\Rightarrow$ $c_i, \ldots, c_j$ is isolated by Φ. □

Definition. A function-system we call a *shift function-system*, if each cell depends only on its left (or only on its right) neighbour.

Theorem. If the CA (C, A_0, N_0, Φ) is reversible, then there exists one of the following two cases:

(i) Each cell stands in an isolated section containing maximum three cells.

(ii) Φ is a shift function-system.

Proof. (i) Suppose that there are c_i and c_j such, that c_i is independent of its left neighbour, and c_j is independent of its right neighbour. The cellular automaton has circle-topology, consequently the section $c_i, \ldots, c_j$ always exists. Furthermore this section is isolated, and — having applied the lemma 2. — it is isolated by Φ' too.

Now let's consider an arbitrary cell c_k. According to the lemma 1. either c_{k-1} is independent of c_{k-2}, or by the reverse c_k is independent of c_{k-1}. In the first case the section $c_{k-1}, \ldots, c_j$, in the second case the section $c_k, \ldots, c_j$ is isolated. Applying the geometrical inverse of lemma 1. we get: either $c_i, \ldots, c_{k+1}$ or $c_i, \ldots, c_k$ is isolated. The common part of two isolated sections is isolated too, so we have: c_k stands in an isolated section containing maximum three cells.

(ii) Suppose the negation of the previous case, that is each cell depends (for example) on the left neighbour. We shall prove, that in this case each cell is independent of the right neighbour: suppose, that for any k c_{k+1} depends on c_{k+2}. At the same time c_{k-1} depends on c_{k-2}, and from the lemma 1. we get, that c_k is an isolated cell. This fact contradicts to the original assumption.

So each cell has only two real neighbours: the left cell and itself. We may classify the balanced local maps for two neighbours in three types:

$$
\text{I.}\ \begin{array}{cc|c} 0 & 0 & a \\ 0 & 1 & a \\ 1 & 0 & b \\ 1 & 1 & b \end{array}
\qquad
\text{II.}\ \begin{array}{cc|c} 0 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & a \\ 1 & 1 & b \end{array}
\qquad
\text{III.}\ \begin{array}{cc|c} 0 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & b \\ 1 & 1 & a \end{array}
$$

In our case each cell depends on the left neighbour, so the type II. is out of the question. If all functions have the type III., then $\forall\alpha$: $F(\alpha)=F(\bar{\alpha})$, thus the global map is not one-to-one.

If there are functions type I. and type III. at the same time, then there exists a cell c_i such, that f_i has the type I., and f_{i+1} has the type III. Therefore the ICT of c_i contains the following part:

$$
\begin{array}{ccccc|ccc} & & t & & & & t+1 & \\ a & b & c & d & e & x & y & z \\ a & b & \bar{c} & \bar{d} & e & x & y & z \end{array}
$$

These two lines exclude the reversibility.

So we get: all local maps have the type I., i.e. Φ is a shift function-system.

Corollaries. 1. If (C, A_0, N_0, Φ) has section topology, then each reversible Φ has the type (i).

2. If (C, A_0, N_0, Φ) is homogeneous, then we have only the six trivial reversible function-systems: the identical one, and its contrary (where each cell alters its state independently of neighbours), the left and right shift function-systems, and their contrary.

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