On two modified problems of synchronization in cellular automata*

By R. Vollmar

1. A modified firing squad synchronization problem

As Moore (1964) states, the problem to synchronize a finite (but arbitrary long) chain of finite automata was devised about 1957 by Myhill. In the meantime this problem has become well-known as the firing squad synchronization problem (fssp).

Among other people Waksman (1966) and Balzer (1967) have given minimal-time solutions. Moreover there exist some modifications, especially the synchronization of two- and three-dimensional arrays (Shinahr (1974), Nguyen and Hamacher (1974), Grasselli (1975)) and of growing arrays (Herman et al. (1974)) have been investigated. In the "classical" fssp the synchronization process is started by one automaton, the so-called general, at the border. Moore and Langdon (1968) and Varshavsky et al. (1970) have renounced this assumption and they have stated minimal-time solutions for this modification.

We consider a further modification. Starting point is a chain of n automata where each automaton is directly connected with its two neighbours. In the "classical" fssp at time $t=0$ all automata except one of the border automaton are in the quiescent state. This quiescent state is distinguished by the property that an automaton will retain it at time $t+1$ if itself and its two neighbours have been in the quiescent state at time t. Here we will assume that initially k automata, where $1 \leqq k \leqq n$, are allowed to be set to the "general state" — all the other automata assume the quiescent state — and after that it is also possible that automata become generals at later moments.

The problem is to specify the structure of the automata such that independently of the number of automata and generals all automata enter a special state, called "fire" state at exactly the same time and this state may not be assumed at any earlier moment by any automaton.

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This generalization is motivated by the consideration of models of neural layers and their interpretation as cellular automata (an example is given in Vollmar/Spreng (1976)): One of the layers has to detect some changes within an other layer and afterwards it must give simultaneously signals to the following layer. A change is obtained if at least one of the automata in the layer receives a certain number of special signals in a certain interval. It may happen that several automata identify changes and they start independently synchronization processes.

The basis of our solution are time-optimal algorithms of the problem to synchronize chains which contain one general but at an arbitrary place (see e.g. Moore/Langdon (1968) or Varshavsky/Marakhovsky/Peschansky (1970)): The general sends out signals (waves) in the two directions which halve the chain, then halve the two new chains etc. Our concept of the age of signals is applicable to any algorithm of this kind.

Our solution is composed of two independently working procedures; they have been combined in such a manner that the procedure which is the first to end, will cause the synchronization. This is done because the two procedures have incomparable synchronization times.

The synchronization time by one of the procedures, which has been described in Vollmar (1976), is achieved in $<2.5n$.

To find a "good" solution of our problem it is necessary to decide quickly which of two waves coming from distinct generals will survive. We have chosen a strategy such that whenever two signals collide, the signal coming from the "elder" general will survive. This is motivated by the fact that with respect to the synchronization process in some but not in all cases the elder general has "done more" than a younger one. However there are space-time configurations for which this is not valid (see fig. 1). If at the border (or nearby) a general g_1 originates at time t_0 and at time t_0+1 a general g_2 will originate nearby the center of the chain, at time t_c the signals transmitted from g_2 will have passed almost the double number of automata than those ones transmitted from g_1. This disadvantage of our procedure could only be repaired if it were possible to determine the age of the generals *and* their positions relative to the center. Up to now we did not succeed in doing this fast enough.

First we will describe the part of the procedure which has to detect the elder signal. Afterwards it will be shown how the synchronization process can be delayed in dependence on the time needed for this detection.

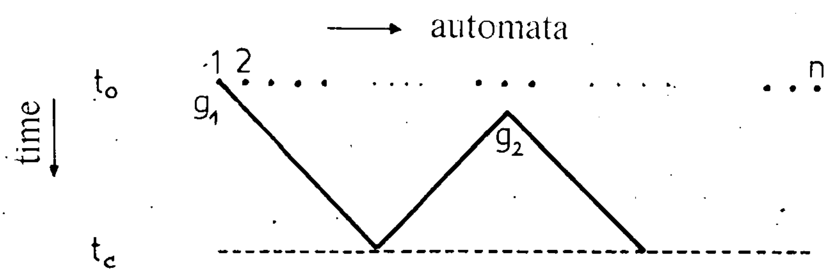

Fig. 1
A "bad" space-time configuration

To be able to classify the signals according to their ages it is necessary that the signals immediately coming from generals "drag along" its ages: For this reason the state set of the automata is increased in such a way that among others the digits of a corresponding number system and some marks can be stored. The age or more precisely the number of automata which have been passed through, is represented in the top automaton in which the signal is arrived and possibly in some automata which have been reached earlier (see fig. 2; only the information relevant for the age is displayed).

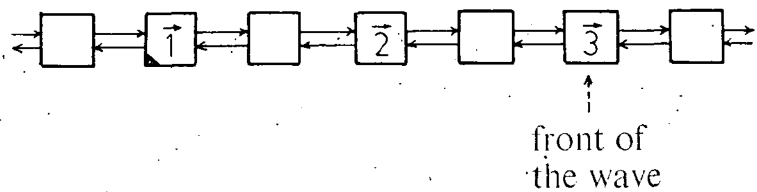

Fig. 2
Configuration with the age of the wave

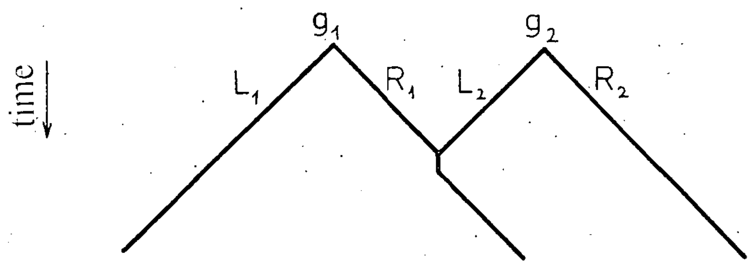

Fig. 3
The problem of overtaking waves

Whenever two signals collide, the distances to the corresponding transmitting generals have to be compared. To do this the propagation of these signals stops and the numbers are subtracted. For this the digits of one of the numbers travel successively to the corresponding place of the other number, i.e. it is stored there in a reversal order. Simultaneously to this shifting process the two numbers are subtracted digit by digit, whenever this is possible. When the first digit of the number — especially marked — has reached the "valid position", the subtraction is finished, and the result is sent out in the corresponding direction to restart the transmission of the "elder" signal. If two signals have the same age by definition the left one will survive. The time needed to make such a comparison is given by $c \log n$ where $c \in \mathbf{N}$.

But still another problem arises (see fig. 3). If the two generals g_1 and g_2 have been originated at the same time, according to our agreement after the collision of R_1 and L_2 R_1 survives. After the comparison it propagates to the right following R_2 and writes its information over that one of R_2. Dependent on the distance between g_1 and g_2 and the running time it is possible that the number representing the age of R_2 is overtaken by R_1. From this time on the age of R_2 is incorrectly represented but as the R_1-signal following from the left (resp. the last of the following signals) is the valid one, there will be no confusion. It is impossible to "inform" the R_2-signal about these occurences because it propagates with unit speed. On the other hand during the comparison of the ages this overtaking will be detected and will cause an interrupt of the subtraction, and the comparison will be done with the following R_1-signal, etc.

If two or more generals exist, it is possible that one of the signals transmitted from a general stops for a certain interval and the other signal moves on (in the opposite direction). To prevent any disturbance of the synchronization process at each step a signal does not propagate a delay signal is transmitted. This signal moves into the opposite direction of the (original) movement of the stopped signal and the transition of each automaton is delayed for one time unit. In fig. 4 the movement and the effect of a delay signal is displayed.

It is clear that the time of the sketched algorithm depends on the number of generals: Since each general causes a delay of about $c \log n$ of the synchronization

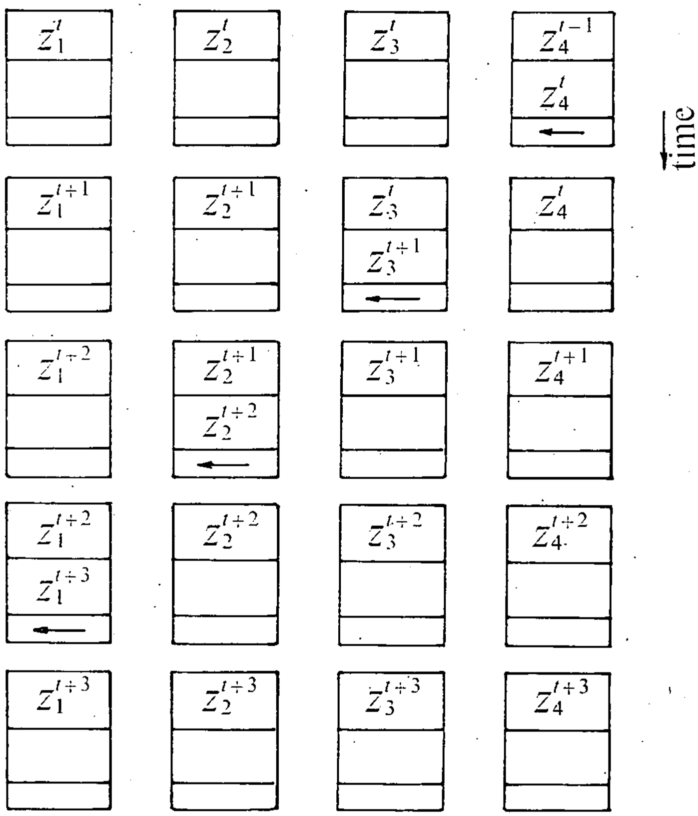

Fig. 4
The movement of a delay signal

time, an upper bound for the total synchronization time is given by

$$2n + cn \log n.$$

The quality of this bound is illustrated by fig. 5.

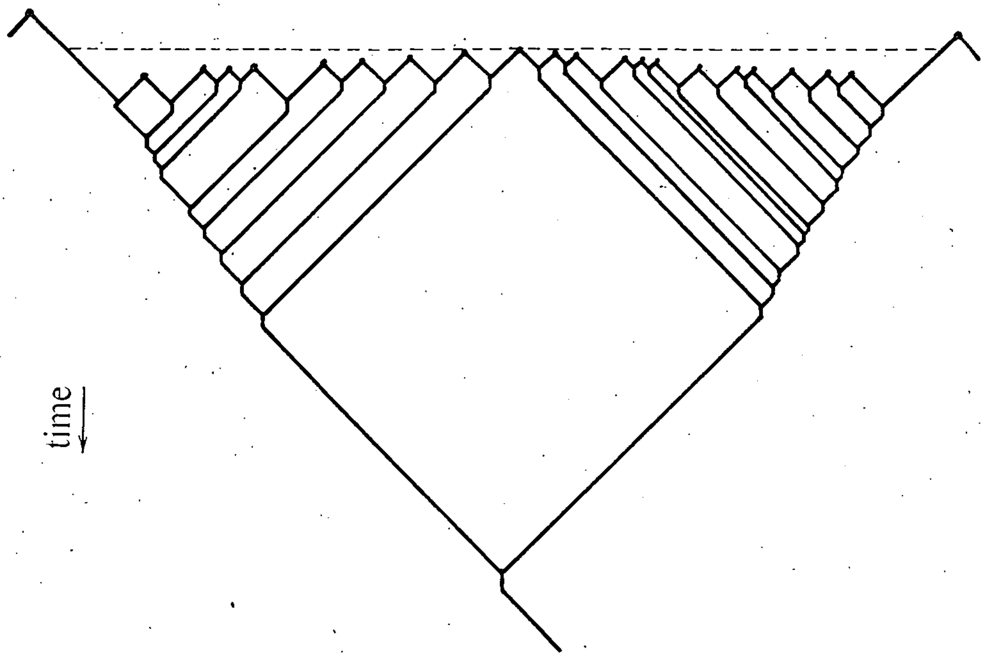

Fig. 5
Space-time diagram of a strongly delayed synchronization
(Between the generals (●) there are other automata.)

On the other hand it should be mentioned that the minimal time is obtained if only one general exists. Moreover there exist configurations for which the sketched algorithm needs a shorter synchronization time as the other procedure mentioned above and vice versa. Therefore we combine the two procedures such that the synchronization time will be

$$< 2.5n.$$

It should be remarked that the method sketched above is also applicable for several generals at arbitrary positions in a rectangular array.

2. A modification of the early bird problem

The method described above is also applicable to a modification of the early bird problem. Rosenstiehl et al. (1972) have described the following problem: To each of the n vertices of an elementary cyclic graph there is assigned an automaton.

These automata may be "excited" at different moments (from the outside); for simplicity we will also say that they assume a "general state". The automata must be designed in such a way that the (single) automaton, which has been excited first, eventually will assume a distinguished state E and all the other automata will assume states I. Rosenstiehl et al. give a solution of this problem, which needs $2n$ steps. They emphasize that this solution does not work if two or more excitations occur at the same moment.

We will discuss a solution of the following problem: At each time an arbitrary number of automata in a chain of n automata may be excited with the only restriction that at time $t=0$ at least one automaton has to be excited and it is not allowed to excite automata which have leaved the quiescent state. After a certain period automata which have been initially excited must assume the state E, and all the other automata assume the state I.

The solution is obtained by the following procedure: Each of the originating generals sends out age signals, as described above. If they collide with other signals a comparison is made. Irrespective of the states in any case the elder signal is transmitted. If two signals of the same age collide, both signals are reflected — with special marks —.

These reflected signals are transmitted backwards, subtracting 1 at each step, until they are decremented to the value 0. In this case, the corresponding automata are marked. An automaton is an early bird (EB) if it is marked by signals from the right and from the left and if the chain of automata has reached a certain age. The last condition is necessary to exclude "local" EBs.

Each automaton contains information about the age of the signal and about the distance to the sending general. In contrast to the procedure in the foregoing paragraph the age is also increased at each step the signal transmission is stopped (because the signals have collided and the comparison takes place).

The number representing the age has to be stored in the automata located between the sending automaton and the automaton where the collision occurs. In general this number will be greater than the number representing the distance, but there are no storage problems if the numbering system is appropriately chosen.

After a collision the numbers representing the *age* are compared: If these numbers are equal, i.e. they come from generals of the same age, or if a signal reaches a border automaton, then the numbers representing the *distance* are reflected. These numbers are transmitted and at each step the value is decremented by 1 until the value 0 is assumed. The corresponding automaton has sent the original signal. It is marked with a label indicating that a reflected signal has arrived. Another label is set if two reflected signals have arrived; in such cases it may be that the marked automaton is *not* an EB (see fig. 6).

Decrementing the numbers a special consideration is necessary if the lowest digit of the number equals 0; but we will not discuss this here. These delays are not illustrated in fig. 6. The total time for the return of these signals depends linearly on the number of the automata.

To compute the total time we have to take into account the following: the time is greater than that one in the paragraph above because the comparisons have to be made with the age numbers (and not with the distances); and those numbers are given as the sum of the distance numbers and the sum of the times for comparisons.

A rough estimation of the total time is given by

$$n+c'\sum_{i=1}^{\lceil n/2\rceil}\log(i\cdot n)$$

if $n>1$, where c' is a constant depending on the algorithm which performs the comparisons. An estimation of the term is given by

$$n\cdot(1+\bar{c}\log n).$$

The age of the signals and the distances to the generals are represented in a polyadic numbering system; therefore the maximum of the values is estimable by $k\cdot\log_B n$. It is possible to give a basis B — which is independent of n — such that the numbers can be stored in the automata between the generals and the collision automata.

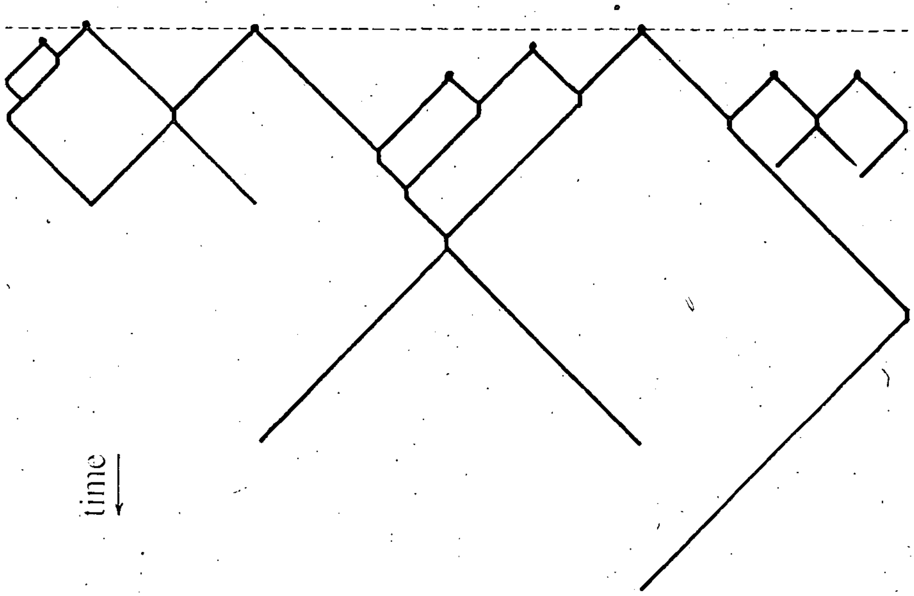

Fig. 6
Space-time diagram of an Early-Bird solution
(without the synchronization of the E- and I-signals)
(Between the generals (●) there are other automata.)

At the time all comparisons have terminated the reflected signals must go back to the corresponding automata. As mentioned above, this time depends linearly on n, and therefore we can find a constant c such that the total amount of time is given by $n\cdot(1+c\log n)$.

To guarantee the synchronized transition to the states E and I, we start at time $t=0$ — independently of the processes described above — a counting procedure which counts (using all the automata for storage) until $\lceil n\cdot(1+c\log n)\rceil$.

If this value is reached, a border automaton starts a synchronization process (following an usual fssp algorithm) such that the states E and I are assumed synchronously.

The solution to the modified EBP needs about $n \cdot (1+\log n)$ time steps. The solution time does not depend on the number of excitations.

It should be noted that this procedure does not solve the modified version of the original problem. As mentioned above in our procedure it is necessary to determine one of the automata as general to start the synchronization process; by reason of the homogeneity of the connections and the determinism of the automata this determination cannot be done in an elementary cyclic graph. On the other side our procedure does not produce either a correct non-synchronous solution because we must wait a certain period — and it is not possible to represent this time in the graph — to make the decision whether doubly marked automata are "real" EBs (see fig. 6).

Abstract

We will introduce the concept of the age of signals which is well-suited for the solution of modifications of the "firing squad synchronization problem" (fssp) and of the "early-bird problem" (ebp).

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