Truth functions and problems in graph colouring

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To the memory of Professor LÁSZLÓ KALMÁR

Introduction

The aim of this paper is to introduce some truth functions, which seem to be useful in the theory of graph colouring, and to study their basic properties and their interrelations.

It can be hoped that a future article will contain a some more detailed analysis of these functions and some applications of the results presented now.

Theorem 3 includes (somewhat implicitly) a purely graph-theoretical assertion. In fact, a simple representation of the maximal v-critical graphs may be given: these can be produced as the intersection of N graphs each of which is the complement of a partition graph[1].

§ 1. Concepts and notations for graphs

1.1. By a graph, always a non-directed finite graph is meant without loops and parallel edges. Later the vertex set of any graph will be viewed to be labelled, a vertex will be identified with the corresponding number (except when it is emphasized explicitly that a graph is considered abstractly, i.e., apart from isomorphy).

If a natural number is denoted by a letter N, then denote by $\mathcal{N}$ (the script form of the *same* latter) the set $\{1, 2, \dots, N\}$; furthermore, we define $\mathcal{N}_i$ by

$$\mathcal{N}_i = \{1, 2, \dots, i-1, i+1, \dots, N\}$$

for an arbitrary i ($1 \leqq i \leqq N$). The letter $\mathcal{H}$ denotes an arbitrary set of natural numbers (not necessarily of form $\{1, 2, \dots, H\}$). The cardinality of a set $\mathcal{H}$ is denoted by $|\mathcal{H}|$. $\mathfrak{C}_{\mathcal{H}}$ is the complete graph with the vertex set $\mathcal{H}$. If $\mathcal{H}$ is a subset of $\mathcal{N}$, then we put $\overline{\mathcal{H}} = \mathcal{N} - \mathcal{H}$.

[1] The notions occuring here will be defined later.

For a graph $\mathfrak{G}$, $\mathscr{V}(\mathfrak{G})$ is the set of vertices of $\mathfrak{G}$ and $\Gamma(\mathfrak{G})$ is the set of edges of $\mathfrak{G}$. If the number N is fixed and $\mathscr{V}(\mathfrak{G})\subseteqq\mathscr{N}$ for a graph $\mathfrak{G}$, then we denote the complement of $\mathfrak{G}$ (with respect to $C_{\mathscr{N}}$) by $\overline{\mathfrak{G}}$.

The isomorphy of graphs is denoted by $\approx$. The sign $\subseteqq$ can express both subset and subgraph; we write $\subset$ if the inclusion is proper. $\varkappa(\mathfrak{G})$ is the chromatic number of $\mathfrak{G}$.

A graph $\mathfrak{G}$ is called *partition graph* if each connected component of $\mathfrak{G}$ is complete.

Let us fix the set $\mathscr{H}$. By $\mathscr{P}^c_{\mathscr{H}}$ the set of all partition graphs $\mathfrak{G}$ is meant such that $\mathscr{V}(\mathfrak{G})=\mathscr{H}$ and the number of connected components of $\mathfrak{G}$ is c.

1.2. Let $\Lambda=\|\lambda_{ij}\|$ be a symmetric matrix of size $N\times N$ such that the entries of Λ are truth values and $\lambda_{11}=\lambda_{22}=\ldots=\lambda_{NN}=\downarrow$. Let the function Φ assign to Λ the graph $\mathfrak{G}=\Phi(\Lambda)$ with $V(\mathfrak{G})=\mathscr{N}$ such that the edge $\overline{ij}$ exists in $\mathfrak{G}$ if and only if $\lambda_{ij}=\uparrow$. Φ is obviously a one-to-one mapping and the range of Φ exhausts the set of all graphs on the vertex set V.[2]

1.3. An abstract graph $\mathfrak{G}$ is called *edge-critical* (or *e-critical)* if $\varkappa(\mathfrak{G}')<\varkappa(\mathfrak{G})$ for every $\mathfrak{G}'$ such that $\mathfrak{G}'$ results from $\mathfrak{G}$ by deleting one edge.

Analogously, $\mathfrak{G}$ is called *vertex-critical* (or *v-critical*) if $\varkappa(\mathfrak{G}')<\varkappa(\mathfrak{G})$ holds for any $\mathfrak{G}'$ such that may be obtained from $\mathfrak{G}$ by deleting one vertex (and the edges incident to it). Any *e*-critical graph is evidently *v*-critical.

A *v*-critical graph $\mathfrak{G}$ is called *maximal v-critical* if $\varkappa(\mathfrak{G}^*)>\varkappa(\mathfrak{G})$ holds for every choice of $\mathfrak{G}^*$ such that $\mathfrak{G}^*$ is *v*-critical and $\mathfrak{G}$ is a subgraph of $\mathfrak{G}^*$.

If $\mathfrak{G}$ is *e*-critical and $\varkappa(\mathfrak{G})=c$, then $\mathfrak{G}$ is called *c*-edge-critical.

Let the natural numbers c, N be fixed $(c<N)$. Denote by $\mathscr{K}^c_N$ the set of all *c*-edge-critical abstract graphs such that $\mathscr{V}(\mathfrak{G})\leqq\mathscr{N}$.

We get the graph class $\mathscr{V}^c_N$ or $\mathscr{M}^c_N$ in a similar manner if "edge-critical" is replaced by "vertex-critical" or "maximal vertex-critical" (respectively) in the above definition. And, moreover, if $|\mathscr{V}(\mathfrak{G})|\leqq N$ is replaced by $|\mathscr{V}(\mathfrak{G})|=N$, then the resulting graph classes are denoted by $\hat{\mathscr{K}}^c_N$, $\hat{\mathscr{V}}^c_N$ and $\hat{\mathscr{M}}^c_N$ (respectively, in analogy to how $\mathscr{K}^c_N$, $\mathscr{V}^c_N$, $\mathscr{M}^c_N$ have been defined).

§ 2. Introduction of truth functions defined on graphs

2.1. Consider a number N and the vertex set $\mathscr{N}$, let a graph $\mathfrak{G}_0$ be fixed with $V(\mathfrak{G}_0)=\mathscr{N}$. Define a truth function $\chi_{\mathfrak{G}_0}[\Lambda]$ by

$$\chi_{\mathfrak{G}_0}[\Lambda]=\bigwedge_{\lambda^0_{ij}=\uparrow}\lambda_{ij} \tag{2.1}$$

where

Λ is a symmetric matrix of size $N\times N$ (as in Section 1.2.),[3]
the variables of Λ are the entries λ_{ij} of Λ fulfilling $i<j$,

[2] Cf. the first sentence of 1.1. $\Phi(\Lambda)$ can be viewed as a non-directed graph because of the symmetry of Λ.

[3] Hence $\Phi(\Lambda)$ is a graph whose vertex set is $\mathscr{N}$.

on the right-hand side of (2.1) the conjunction is taken for all pairs (i, j) such that $\lambda_{ij}^0 = \uparrow$ where λ_{ij}^0 is the entry of $\Phi^{-1}(\mathfrak{G}_0)$ being in crossing of the i-th row and j-th column.

An obvious consequence of the above definition is:

Proposition 1. *The value $\chi_{\mathfrak{G}_0}[\Lambda]$ is $\uparrow$ if and only if any edge of $\mathfrak{G}_0$ is an edge of $\Phi(\Lambda)$, too.*

2.2. Let c be a natural number $(c<N)$. In analogy to the above definition of $\chi_{\mathfrak{G}_0}$, we define the truth function D^c by

$$D^c[\Lambda] = \bigvee_{\mathfrak{G}^* \in \mathscr{P}_{\mathscr{N}}^c} \chi_{\mathfrak{G}^*}[\Lambda]$$

where the disjunction is taken for all elements $\mathfrak{G}^*$ of the set $\mathscr{P}_{\mathscr{N}}^c$. The meaning of D^c is expressed in the following evident assertion:

Proposition 2. *The following three statements are equivalent for any matrix Λ:*
(i) $D^c[\Lambda] = \uparrow$,
(ii) $\mathfrak{G} = \Phi(\Lambda)$ *contains a partition graph consisting of c connected components,*
(iii) *the complement of $\Phi(\Lambda)$ is c-colourable* (*i.e.*, $\varkappa(\overline{\mathfrak{G}}) \leqq c$).

2.3. In the particular case when $\mathfrak{G}_0$ has only one edge e, the function $\chi_{G_0}[\Lambda]$ expresses whether this edge e is present in $\Phi(\Lambda)$ or not. In this special case we write also $\chi_e[\Lambda]$.

Let an abstract graph $\mathfrak{K}$ with at most N vertices be chosen. Define the function $L_{\mathfrak{K}}$ by

$$L_{\mathfrak{K}}[\Lambda] = \bigwedge_{\mathfrak{K}'} \bigvee_{e} \chi_e[\Lambda]$$

where $\mathfrak{K}'$ runs through all graphs such that
$\mathscr{V}(\mathfrak{K}') \subseteqq \mathscr{N}$ and
$\mathfrak{K}'$ is isomorphic to $\mathfrak{K}$;
for any choice of $\mathfrak{K}'$, e runs through the edges of $\mathfrak{K}'$.

The next result follows easily from this definition:

Proposition 3. $L_{\mathfrak{K}}[\Lambda] = \uparrow$ *if and only if no subgraph of the complement of $\Phi(\Lambda)$ is isomorphic to $\mathfrak{K}$.*

2.4. Let the functions E^c and F^c be defined by

$$E^c[\Lambda] = \bigwedge_{\mathfrak{K} \in \mathscr{K}_{N-1}^{c+1}} L_{\mathfrak{K}}[\Lambda]$$

and

$$F^c[\Lambda] = \bigwedge_{\mathfrak{K} \in \hat{\mathscr{K}}_N^{c+1}} L_{\mathfrak{K}}[\Lambda]$$

The following two assertions follow easily from these definitions and from Proposition 3.

Proposition 4. $E^c[\Lambda] = \uparrow$ *if and only if the complement of $\Phi(\Lambda)$ has no $(c+1)$-edge-critical subgraph with at most $N-1$ vertices.*

Proposition 5. *$F^c[A]=\uparrow$ if and only if the complement of $\Phi(A)$ has no $(c+1)$ edge-critical subgraph containing each of the N vertices.*

Proposition 6. *The equality*

$$D^c[A]=E^c[A]\wedge F^c[A]$$

holds for any matrix A.

Proof. Let us consider four assertions:
(i) $E^c[A]\wedge F^c[A]=\uparrow$,
(ii) the complement of $\Phi(A)$ has no $(c+1)$-edge-critical subgraph,
(iii) the complement of $\Phi(A)$ is c-colourable,
(iv) $D^c[A]=\uparrow$.
Propositions 4, 5 imply the equivalence of (i) and (ii). Proposition 2 has stated that (iii), (iv) are equivalent. If (ii) is false then $\varkappa(\overline{\Phi(A)})>c$, this means the falsity of (iii). As it was shown in [2], the falsity of (iii) implies the falsity of (ii).

2.5. We mention some obvious consequences of the definitions occuring in this §. $\chi_{\mathfrak{G}_0}$ is an elementary conjunction. D^c was defined in a disjunctive normal form. Each of $L_{\mathfrak{K}}$, E^c, F^c was introduced as the conjunction of functions expressed in disjunctive normal form. All these functions are isotonic.

In what follows we shall write e.g. $D^c_{\mathcal{N}}$ instead of D^c if we want to emphasize that graphs with the vertex set $\mathcal{N}$ are considered.

§ 3. Results

The most important interrelation concerning the defined truth functions is expressed by

Theorem 1. *For any matrix A we have*

$$E^c_{\mathcal{N}}[A]=\bigwedge_{i=1}^{N} D^c_{\mathcal{N}_i}[A].$$

From Theorem 1 we shall infer to

Theorem 2. *There is exactly one truth function $A^c_{\mathcal{N}}[A]$ such that*
(i) *$A^c_{\mathcal{N}}[A]$ is isotonic*
(ii) *any matrix A fulfils the equality $E^c_{\mathcal{N}}[A]=D^c_{\mathcal{N}}[A]\vee A^c_{\mathcal{N}}[A]$,*
and
(iii) *$A^c_{\mathcal{N}}[A]$ and $D^c_{\mathcal{N}}[A]$ have no prime implicant in common.*

Remark. $A^c_{\mathcal{N}}$ is identically true if and only if

$$\mathfrak{G}\in\mathcal{K}^c_N\Rightarrow|\mathcal{V}(\mathfrak{G})|\leqq N-1.$$

In the next assertion $A^c_{\mathcal{N}}$ is characterized by means of vertex-critical graphs.

Theorem 3. *Suppose that the numbers N, c are such that there is a $(c+1)$-v-critical graph with N vertices. $\chi_0=\chi_{\mathfrak{G}_0}[A]$ is a prime implicant of $A^c_{\mathcal{N}}[A]$ if and only if*
(a) *$\overline{\mathfrak{G}}_0\in\mathcal{M}^{\hat{c}+1}_N$ and*
(b) *$\mathcal{V}(\overline{\mathfrak{G}}_0)=\mathcal{N}$.*

We are able to give for a disjunctive normal form of F^c a characterization which is somewhat less explicit in comparison to how E^c has been characterized in Theorem 2.

Theorem 4. *Let N, c be numbers as in Theorem 3. $\chi_0 = \chi_{\mathfrak{G}_0}[\Lambda]$ is a prime implicant of $F^c_{\mathcal{N}}[\Lambda]$ if and only if*

(a) *$\overline{\mathfrak{G}}_0$ has no subgraph $\mathfrak{G}_1$ such that all the vertices of $\overline{\mathfrak{G}}_0$ are contained in $\mathfrak{G}_1$ and $\mathfrak{G}_1 \in \hat{\mathcal{K}}^{c+1}_N$ and*

(b) *whenever $\mathfrak{G}_2$ is a subgraph of $\mathfrak{G}_0$ then there is a subgraph $\mathfrak{G}_3$ of $\overline{\mathfrak{G}}_2$ such that $\mathfrak{G}_3 \in \hat{\mathcal{K}}^{c+1}_N$.*

Moreover, if χ_0' is a prime implicant of $F^c_{\mathcal{N}}[\Lambda]$, then either $\mathfrak{G}_0 \in \mathcal{P}^c_{\mathcal{N}}$ or $\mathfrak{G}_0$ contains a $(c+1)$-critical graph $\mathfrak{G}_4$ such that $\mathfrak{G}_4$ has at most $N-1$ vertices.

§ 4. Proofs

We shall use the following well-known fact (see [1], p. 40):

Lemma 1. *An isotonic truth function has a single irredundant disjunctive normal form, this form consists of all its prime implicants.*

Proof of Theorem 1. Since $D^c_{\mathcal{N}}[\Lambda]$ is isotonic, we can use Lemma 1. By Proposition 6 and the definitions of E^c, F^c, we have

$$D^c_{\mathcal{N}_i}[\Lambda] = \bigwedge_{\mathfrak{K} \in \mathcal{K}^{c+1}_{\mathcal{N}_i}} L_{\mathfrak{K}}[\Lambda]$$

for any i $(1 \leqq i \leqq N)$. If we form the conjunction of these N equalities (in such a manner that the conjunction of the left-hand sides and the conjunction of the right-hand sides is taken, with an equality sign between them), then the right-hand side can be simplified to $E^c_N[\Lambda]$, thus we get the assertion of Theorem 1.

Proof of Theorem 2. Let us distinguish three cases. If $N < c+1$, $\mathcal{P}^c_{\mathcal{N}} = \emptyset$ and so D^c is undefined. If $N = c+1$, then, by Proposition 4, $E^c_N \equiv \uparrow$, as there exists no $(c+1)$-edge-critical graph with at most c vertices. So $D^c[\Lambda] \equiv F^c[\Lambda]$ whence follows the existency and unicity (in the sense of the assertion) of $A^c_N[\Lambda]$, namely $A^c_N[\Lambda] \equiv \uparrow$. If $N > c+1$, the proof runs as follows.

Our first aim is to verify that each prime implicant χ_0 of $D^c_N[\Lambda]$ is a prime implicant of $E^c_N[\Lambda]$. By Proposition 6, any implicant χ_0 of $D^c_N[\Lambda]$ is an implicant of $E^c_N[\Lambda]$. Let χ_0' be a prime implicant of $E^c_N[\Lambda]$ such that χ_0' is a subconjunction of χ_0. By the definition of D^c, there is a graph $\mathfrak{G}_0(\in \mathcal{P}^c_{\mathcal{N}})$ such that $\chi_0 = \chi_{\mathfrak{G}_0}[\Lambda]$. Let $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_c$ be the connected components of $\mathfrak{G}_0$ (any of them is a complete graph). As $N > c+1$, $|\mathcal{V}(\mathfrak{T}_k)| > 1$ for at least one k $(1 \leqq k \leqq c)$. Fixing such a k, let r be an arbitrary element of $\mathcal{V}(\mathfrak{T}_k)$. Let an edge e be chosen in $\mathfrak{G}_0$ such that r, e are not incident. We have $\chi_0' = \chi_{G_0'}[\Lambda]$ for a suitable subgraph $\mathfrak{G}_0'$ of $\mathfrak{G}_0$. Let $\mathfrak{G}_{or}'$ be defined by $\mathfrak{G}_{or}' = \mathfrak{G}_0' \cap \mathfrak{C}_{N_r}$. By Theorem 1, there is a partition graph $\mathfrak{G}_P'(\in \mathcal{P}^c_{\mathcal{N}_r})$ such that $\mathfrak{G}_P' \subseteq \mathfrak{G}_{or}'$. If $\mathfrak{G}_P(\in \mathcal{P}^c_{\mathcal{N}_r})$ is defined by $\mathfrak{G}_P = \mathfrak{G}_0 \cap \mathfrak{C}_{N_r}$, then we have $\mathfrak{G}_P' \subseteq \mathfrak{G}_{or}' \subseteq \mathfrak{G}_P$. Since $\mathfrak{G}_P'$, $\mathfrak{G}_P$ are partition graphs on the same vertex set and the number of their connected components coincide, $\mathfrak{G}_P' \subset \mathfrak{G}_P$ is impossible, hence $\mathfrak{G}'_P=\mathfrak{G}'_{or}=\mathfrak{G}_P$. The (arbitrarily chosen) edge e of $\mathfrak{G}_0$ belongs to $\mathfrak{G}'_P(\subseteqq\mathfrak{G}_0)$, thus $\chi'_0=\chi_0$.

$A^c_N[\Lambda]$ is defined as the disjunction of the prime implicants φ of $E^c_N[\Lambda]$ such that φ is not a prime implicant of $D^c_N[\Lambda]$.

Lemma 2. *Let $\mathfrak{G}$ be a graph such that $\mathscr{V}(\mathfrak{G})=\{1, 2, \dots, N\}$. The following three assertions are equivalent for $\mathfrak{G}$:*

(i) $\mathfrak{G}$ *is $(c+1)$-vertex-critical,*

(ii) $\mathfrak{G}\cap\mathfrak{C}_{\mathscr{N}_i}$ *is c-chromatic for any i $(1\leqq i\leqq N)$,*

(iii) $\overline{\mathfrak{G}}\cap\mathfrak{C}_{\mathscr{N}_i}$ *includes a partition graph with connected components.*

Remarks. $\mathfrak{G}\cap\mathfrak{C}_{\mathscr{N}_i}$ results from $\mathfrak{G}$ by deleting the vertex i and the edges incident to it. $\overline{\mathfrak{G}}\cap\mathfrak{C}_{\mathscr{N}_i}$ is the complement of $\mathfrak{G}\cap\mathfrak{C}_{\mathscr{N}_i}$ with respect to $\mathfrak{C}_{\mathscr{N}_i}$.

Proof of Lemma 2. (i) and (ii) are equivalent in consequence of the definition of vertex-critical graphs. The equivalence of (ii), (iii) is obvious (cf. the statements (ii), (iii) in Proposition 2).

Proof of Theorem 3. Assume that the first sentence of Theorem 3 holds for N, c.

Necessity. Let $\chi_{\mathfrak{G}_0}[\Lambda]$ be a prime implicant of $A^c_N[\Lambda]$.

First we prove that condition (ii) of Lemma 2 holds for $\overline{\mathfrak{G}}_0$. Let k be an arbitrary element of $\mathscr{N}$. χ_0 is an implicant of $D^c_{\mathscr{N}_k}$ because of Theorem 1. So $\mathfrak{G}_0$ includes an element of $\mathscr{P}^c_{\mathscr{N}_k}$, say $\mathfrak{P}_k$. For this element $\mathfrak{P}_k\subseteqq\mathfrak{G}_0\cap\mathfrak{C}_{\mathscr{N}_k}$, thus $\mathfrak{G}_0$ satisfies condition (iii) of Lemma 2, and so — by the lemma — conditions (i) and (ii) too.

Hence $\overline{\mathfrak{G}}_0$ is $(c+1)$-vertex-critical (by Lemma 2). The necessity will completely be proved if we show the maximality of $\overline{\mathfrak{G}}_0$.

Let $e=\overline{ij}$ be an arbitrary edge of $\mathfrak{G}_0$. Define the graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ (again on the vertex set $\{1, 2, \dots, N\}$ such that

the edges of $\mathfrak{G}_1$ are the edges of $\overline{\mathfrak{G}}_0$ and e,

the edges of $\mathfrak{G}_2$ are the edges of $\mathfrak{G}_0$ except e.

It is clear that $\mathfrak{G}_1, \mathfrak{G}_2$ are complements of each other, and

$$(\chi_0=)\chi_{\mathfrak{G}_0}[\Lambda]=\chi_{\mathfrak{G}_2}[\Lambda]\wedge\lambda_{ij}.$$

Let the short notation χ_2 be used for $\chi_{\mathfrak{G}_2}[\Lambda]$. χ_2 is not an implicant of $A^c_N[\Lambda]$, consequently there exists a $k(\in\mathscr{N})$ such that χ_2 is not an implicant of $D_{\mathscr{N}_k}[\Lambda]$ (by Theorem 1).

If $\mathfrak{G}_3$ is defined by $\mathfrak{G}_3=\mathfrak{G}_2\cap\mathfrak{C}_{\mathscr{N}_k}$, it is clear that $\chi_{G_3}[\Lambda]$ is not an implicant of $D_{\mathscr{N}_k}[\Lambda]$.

By Proposition 2 this means that $\mathfrak{G}_3$ has no subgraph $\mathfrak{P}$ such that $\mathfrak{P}\in\mathscr{P}^c_{\mathscr{N}_k}$. From Lemma 2 it follows that $\overline{\mathfrak{G}}_2\notin\hat{\mathscr{V}}^c_N$. As $\overline{\mathfrak{G}}_2=\overline{\mathfrak{G}}_0\cup\{e\}$ and e is an arbitrary edge of $\mathfrak{G}_0$, $\overline{\mathfrak{G}}_0$ is maximal v-critical indeed, which completes the necessity proof.

Sufficiency. If conditions (a) and (b) are fulfilled by $\mathfrak{G}_0$, then

(1) $\chi_{\mathfrak{G}_0}[\Lambda]=\chi_0$ is an implicant of A^c_N.

This can be shown in two steps.

(1.1) χ_0 is an implicant of E^c_N. Indeed, $\mathfrak{G}_0$ satisfies condition (i) of Lemma 2, and so also condition (iii) of this lemma. This implies that the graph $\overline{\mathfrak{G}}_0\cap\mathfrak{C}_{N_i}$ includes an element $\mathfrak{P}$ of $\mathscr{P}^c_{\mathscr{N}_i}$ and therefore χ_0 is an implicant of $D^c_{\mathscr{N}_i}$ (for every $i(\in N)$) by Proposition 2.

Now from Theorem 1 it follows that χ_0 is an implicant of E_N^c.

(1.2) χ_0 is not an implicant of D_N^c. By Proposition 2 this is true if and only if $\mathfrak{G}_0$ includes no element of $\mathscr{P}_{\mathscr{N}}^c$, that is $\varkappa(\overline{\mathfrak{G}}_0) > c$. But this is now in consequence of condition (a) of our theorem.

From (1.1) and (1.2) we conclude that (1) is true. It remains to prove that (2) χ is a prime implicant of A_N^c.

To prove this chose an arbitrary edge e of $\mathfrak{G}_0$. Let us introduce a new graph $\mathfrak{G}_1$ by $\mathfrak{G}_1 = \overline{\mathfrak{G}}_0 \cup \{e\}$. As $\mathfrak{G}_0$ is maximal $(c+1)$-vertex-critical, $\mathfrak{G}_1$ is not $(c+1)$-vertex-critical. By Lemma 2, there exists an $r(\in \mathscr{N})$ such that the graph $\overline{\mathfrak{G}}_1 \cap \mathfrak{C}_{\mathscr{N}_r}$ includes no partition graph $\mathfrak{P} \in P_{\mathscr{N}_r}^c$. By Proposition 2, for this r $\chi_{\mathfrak{G}_1}[\Lambda]$ is not an implicant of $D_{\mathscr{N}_r}^c[\Lambda]$.

By Theorem 1, $\chi_{\mathfrak{G}_1}$ is not an implicant of $E_N^c[\Lambda]$, thus we have proved assertion 2. This completes the sufficiency proof.

Proof of Theorem 4. The first part of the assertions — the sufficient and necessary condition — is equivalent to Proposition 5; so it does not require any proof. To prove the last sentence of the theorem, let us distinguish two cases: (i) $\varkappa(\overline{\mathfrak{G}}_0) \geqq c+1$ and (ii) $\varkappa(\overline{\mathfrak{G}}_0) < c$. In case (i) by the first part of this theorem $|V(\overline{\mathfrak{G}}_0)| \leqq N-1$, which is the second alternative of the assertion to be proved. In case (ii) there exists a graph $\mathfrak{P} \in \mathscr{P}_{\mathscr{N}}^c$ such that $\mathfrak{G}_0 \supseteqq \mathfrak{P}$ and so $\chi_{\mathfrak{P}}[\Lambda]$ is a subconjunction of $\chi_{\mathfrak{G}_0}$. But $\chi_{\mathfrak{P}}[\Lambda]$ is an implicant of $F_N^c[\Lambda]$ because it is an implicant of $D_N^c[\Lambda]$. As $\chi_{\mathfrak{G}_0}[\Lambda]$ is a prime implicant of F_N^c, it cannot include $\chi_{\mathfrak{P}}[\Lambda]$ properly, therefore $\chi_{\mathfrak{G}_0}[\Lambda] = \chi_{\mathfrak{P}}[\Lambda]$, that is $\mathfrak{G}_0 = \mathfrak{P}$, proving the second part of the theorem. Thus Theorem 4 is proved.

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Acknowledgement. The author wishes to thank to A. Ádám for his many valuable comments and criticisms. He has rewritten the original paper to a great extent, introducing a number of notions and propositions to ease the reading.

References

[1] Ádám, A., *Truth functions and the problem of their realization by two-terminal graphs*, Akadémiai Kiadó, Budapest, 1968.

[2] Dirac, G. A., Some theorems on abstract graphs, *Proc. London Math. Soc.*, v. 2, 1952, pp. 69—81.

(Received Sept. 29, 1977)