Processing of random sequences with priority

By A. IVÁNYI and I. KÁTAI

To the memory of Professor László Kalmár

Introduction

This paper is devoted to study of processing random sequences with priority. At first we formulate the general problem (§ 1.), later we show: the state sequence characterizing the course of the processing — as processing of independent homogeneous Markov-chains — is also a homogeneous Markov-chain (§ 2.).

We deal with characterizing the processing speed (§ 3.). Since the stationary initial distribution plays a main role, therefore we give a simple algorithm to determine it: when the transition probability matrix is the simplest (§ 4.) and for two sequences (§ 5.).

Finally we investigate the asymptotic behaviour of the speed (§ 6.).

Our work has practical importance e.g. in computer performance analysis, more precisely in modelling of multiprogrammed computers with one processor and interleaved memory [1]. In this case the programs are modelled by sequences (the program with the greatest priority by the first sequence etc.), the chosen measure of the speed corresponds to the average number of the executed operations in a time unit, the transition probability matrix with the same elements corresponds to the random program behaviour model and the asymptotic problem is connected with the great number of memory moduls.

§ 1. Formulation of the problem

Let $\mathscr{A}_N$ denote the set $\{1, 2, \ldots, N\}$, and

$$\begin{matrix} f_1^{(1)}, f_2^{(1)}, \ldots \\ \vdots \\ f_1^{(r)}, f_2^{(r)}, \ldots \end{matrix} \tag{1.1}$$

r infinite sequences consisting of the elements of $\mathscr{A}_N$. We process the elements in the sequences according to the following rules:

1. Processing proceeds in the points of time $1, 2, \ldots$; let i be equal to 1.

2. Let k_1 denote the greatest positive integer for which the elements $f_1^{(1)}, \ldots, f_{k_1}^{(1)}$ are mutually distinct. If $k_1, \ldots, k_{t-1}$ have been defined, then let k_t $(t=2, \ldots, r)$ denote the greatest nonnegative integer for which

$$\bigcup_{j=1}^{t-1} \{f_1^{(j)}, \ldots, f_{k_j}^{(j)}\} \cap \{f_1^{(t)}, \ldots, f_{k_t}^{(t)}\} = \emptyset \tag{1.2}$$

holds.

3. In the i-th point of time we process the first k_t elements of the t-th $(t=1, \ldots, r)$ sequence. We omit the processed elements from the sequences, and reduce the lower index of the remaining elements by k_t in the t-th $(t=1, \ldots, r)$ sequence.

4. We add 1 to i and continue the processing from the rule 2.

For a more precise characterizing of the processing we register the first and last processed and the first nonprocessed elements for every point of time. Therefore the processing in the first point of time is characterized by the array

$$\begin{array}{l} f_1^{(1)}, |\ldots, f_{k_1}^{(1)}, \| f_{k_1+1}^{(1)} \\ \vdots \\ f_1^{(t)}, |\ldots, f_{k_t}^{(t)}, \| f_{k_t+1}^{(t)} \\ \vdots \\ f_1^{(r)}, |\ldots, f_{k_r}^{(r)}, \| f_{k_r+1}^{(r)}. \end{array} \tag{1.3}$$

If $k_t=0$ holds for a given t, then we have $*, \| f_1^{(t)}$ in the t-th line of (1.3). The star shows that none of the elements has been processed. For the sake of brevity let

$$A_t = \langle f_1^{(t)}, |\ldots, f_{k_t}^{(t)}, \| f_{k_t+1}^{(t)} \rangle \quad \text{or} \quad A_t = \langle *, \| f_{k_t+1}^{(t)} \rangle, \tag{1.4}$$

resp. By using this notation, the processing in the first point of time is characterized by

$$\vartheta = (A_1, \ldots, A_r). \tag{1.5}$$

Let $\mathscr{D}_r$ denote the set of all possible ϑ's. In other words $\mathscr{D}_r$ is the set of all ϑ's that are representable in form (1.3) giving suitable values to the elements $f_i^{(t)}$. It is clear, that $(A_1, \ldots, A_r)$ belongs to $\mathscr{D}_r$ if and only if the following conditions hold:

1. $A_1 = \langle i_1, |i_2, \ldots, i_k, \| j \rangle$; $i_1, \ldots, i_k, j \in \mathscr{A}_N$; $i_1, \ldots, i_k$ are mutually distinct, $j \in \{i_1, \ldots, i_k\}$.

2. Let $A_1, \ldots, A_{t-1}$ be defined, then

$$A_t = \langle s_1, |s_2, \ldots, s_m \| l \rangle, \quad s_1, \ldots, s_m, \ l \in \mathscr{A}_N,$$

and

a) $\{s_1, \ldots, s_m\} \cap \bigcup_{n=1}^{t-1} A_n = \emptyset,$

b) $s_1, s_2, \ldots, s_m$ are mutually distinct,

c) $l \in \{s_1, \ldots, s_m\} \cup \left(\bigcup_{n=1}^{t-1} A_n \right)$

or

$$A_t = \langle *, \| l \rangle, \quad \text{and} \quad l \in \bigcup_{n=1}^{t-1} A_n.$$

After this the processing of a given array of type (1.1) can be described by the state sequence

$$\vartheta^{(1)}, \vartheta^{(2)}, \ldots, \vartheta^{(s)}, \ldots$$
$$\vartheta^{(s)} \in D_r \quad (s = 1, 2, \ldots). \tag{1.6}$$

It is obvious that there are pairs $C_1, C_2 \in \mathcal{D}_r$ that cannot occur as consecutive states, i.e. for which $\vartheta^{(s)} = C_1$, $\vartheta^{(s+1)} = C_2$.

Let

$$\vartheta^{(s)} = (A_1^{(s)}, \ldots, A_r^{(s)}), \tag{1.7}$$

where

$$A_t^{(s)} = \langle i_{1,t}^{(s)}, |\ldots, i_{k_t^{(s)},t}^{(s)}, \| j_t^{(s)} \rangle \tag{1.8}$$

or

$$A_t^{(s)} = \langle *, \| j_t^{(s)} \rangle. \tag{1.9}$$

Let *in* $\vartheta^{(s)}$ and *fin* $\vartheta^{(s)}$ denote the initial and final elements of $\vartheta^{(s)}$, i.e.

$$in\, \vartheta^{(s)} = (i_{1,1}^{(s)}, \ldots, i_{1,r}^{(s)});\ fin\, \vartheta^{(s)} = (j_1^{(s)}, \ldots, j_r^{(s)}), \tag{1.10}$$

remarking that if $A_t^{(s)} = \langle *, \| j_t^{(s)} \rangle$, then in *in* $\vartheta^{(s)}$ we put $*j_t^{(s)}$ instead of $i_{1,t}^{(s)}$. It is clear, that the transition $\vartheta^{(s)} \to \vartheta^{(s+1)}$ is realisable if and only if *fin* $\vartheta^{(s)} =$ *in* $\vartheta^{(s+1)}$ holds. Deciding about this equality we do not take into account whether the components of *in* $\vartheta^{(s+1)}$ contain stars or not.

§ 2. Processing of independent Markov-chains

Let $\xi_i^{(l)}$ $(l = 1, \ldots, r;\ i = 1, 2, \ldots)$ be random variables with values from $\mathcal{A}_N$ for which the following conditions hold:

1. The sequences $\xi_i^{(l)}$ $(i = 1, 2, \ldots)$ for every l form a homogeneous Markov-chain with an initial distribution π_l and transition probability matrix Π_l, i.e.

$$\pi_l(p(1, l), \ldots, p(N, l)), \quad \text{where} \quad p(k, l) = P(\xi_1^{(l)} = k)$$

and

$$\Pi_l = [p(x, y, l)], \quad \text{where} \quad p(x, y, l) = P(\xi_{v+1}^{(l)} = x | \xi_v^{(l)} = y).$$

2. The sequences $\xi_i^{(l)}$ $(i = 1, 2, \ldots)$ are mutually independent.
3. The elements of the matrices Π_l are positive.

Our job is to process the array of random variables

$$\begin{matrix} \xi_1^{(1)}, \xi_2^{(1)}, \ldots \\ \vdots \\ \xi_1^{(r)}, \xi_2^{(r)}, \ldots \end{matrix} \tag{2.1}$$

by using the algorithm defined in § 1.

Let

$$\mathcal{B}^{(1)}, \mathcal{B}^{(2)}, \ldots, \mathcal{B}^{(s)}, \ldots$$
$$\mathcal{B}^{(s)} \in \mathcal{D}_r \quad (s = 1, 2, \ldots) \tag{2.2}$$

denote the state sequence of type (1.5).

We prove the following

Theorem 1. Under the previous conditions the sequence (2.2) represents a homogeneous Markov-chain.

Proof. Let us compute the probabilities

$$P(\mathscr{B}^{(1)} = \vartheta^{(1)}) = q(\vartheta^{(1)}),$$
$$P(\mathscr{B}^{(s+1)} = \vartheta^{(s+1)} | \mathscr{B}^{(1)} = \vartheta^{(1)}, \ldots, \mathscr{B}^{(s)} = \vartheta^{(s)}).$$

We shall use the notations (1.8) and (1.9).

Let

$$\tau(A_t^{(1)}) = p(i_{1,t}^{(1)}; t) \cdot p(i_{2,t}^{(1)}, i_{1,t}^{(1)}; t) \ldots p(j_t^{(1)}, i_{k_t^{(1)},t}^{(1)}; t), \tag{2.3}$$

if $A_t^{(1)}$ has the form (1.8) and

$$\tau(A_t^{(1)}) = p(j_t^{(1)}; t), \tag{2.4}$$

if $A_t^{(1)}$ has the form (1.9).

It is clear, that

$$q(\vartheta^{(1)}) = \prod_{t=1}^{r} \tau_t(A_t^{(1)}).$$

Let

$$\lambda_t(A_t^{(s)}) = p(i_{2,t}^{(s)}; i_{1,t}^{(s)}; t) \ldots p(j_t^{(s)}; i_{k_t,t}^{(s)}; t), \tag{2.5}$$

if $A_t^{(s)}$ has the form (1.8) and let

$$\lambda_t(A_t^{(s)}) = 1, \tag{2.6}$$

if $A_t^{(s)}$ has the form (1.9). Further let

$$Q(\vartheta^{(s)}) = \prod_{t=1}^{r} \lambda_t(A_t^{(s)}). \tag{2.7}$$

Since the sequences $\xi_i^{(l)}$ form homogeneous Markov-chains, therefore

$$P(\mathscr{B}^{(1)} = \vartheta^{(1)}, \ldots, \mathscr{B}^{(s+1)} = \vartheta^{(s+1)}) = q(\vartheta^{(1)}) Q(\vartheta^{(2)}) \ldots Q(\vartheta^{(s+1)}),$$

if $\vartheta^{(1)}, \ldots, \vartheta^{(s+1)}$ is a realisable sequence. It is clear, that for a nonrealisable sequence

$$P(\mathscr{B}^{(1)} = \vartheta^{(1)}, \ldots, \mathscr{B}^{(s+1)} = \vartheta^{(s+1)}) = 0.$$

So we have proved that (2.2) is a homogeneous Markov-chain with initial distribution (2.4) and with the following transition probabilities:

$$P(\mathscr{B}^{(s+1)} = \vartheta^{(s+1)} | \mathscr{B}^{(s)} = \vartheta^{(s)}, \ldots, \mathscr{B}^{(1)} = \vartheta^{(1)}) =$$
$$= \begin{cases} Q(\vartheta^{(s+1)}), & \text{if } \ \mathit{in}\, \vartheta^{(s+1)} = \mathit{fin}\, \vartheta^{(s)} \\ 0, & \text{if } \ \mathit{in}\, \vartheta^{(s+1)} \neq \mathit{fin}\, \vartheta^{(s)}. \end{cases} \tag{2.8}$$

Now we shall prove, that under a suitable positive k all of the conditional probabilities

$$P(\mathscr{B}^{(s+k)} = C_2 | \mathscr{B}^{(s)} = C_1)$$

are positive for every $C_1, C_2 \in \mathscr{D}_r$.

Since $Q(C)$ are positive for every $C \in \mathcal{D}_r$, thus we have to show that there exists a realisable sequence

$$C_1 = \vartheta_1, \vartheta_2, \ldots, \vartheta_{k+1} = C_2.$$

Let $fin\, C_1 = (j_1, \ldots, j_r) = \beta_1$, $in\, C_2 = (i_1, i_2, \ldots, i_r) = \alpha_2$, where α_2 may have stars. It is clear that there is a realisable sequence starting with C_1 and ending with ϑ_{u-1}, where $fin\, \vartheta_{u-1} = \alpha_2$. Since the number of possible states is finite, we can find a bound d with $u \leqq d$. Let $k > d$, and

$$\vartheta^{(s)} = \begin{bmatrix} i_1, | i_2, \ldots, i_r, \| i_1 \\ *, \qquad\qquad \| i_2 \\ \vdots \\ *, \qquad\qquad \| i_r \end{bmatrix} \quad (s = u, \ldots, k).$$

In this case the subsequence $\vartheta^{(k)}, \vartheta^{(k+1)}$ is realisable.

Hence immediately follows the following

Theorem 2. Under the conditions of Theorem 1 the sequence (2.2) is an ergodic Markov-chain.

§ 3. Determination of the processing speed

Let $l(\vartheta)$ (interpreted for every $\vartheta \in \mathcal{D}_r$) be an arbitrary function having complex values.

Since any given array (1.1) determines uniquely the sequence (1.5), therefore the sequence

$$l(\vartheta^{(1)}), l(\vartheta^{(2)}), \ldots \tag{3.1}$$

is determined too. We are interested in such functions l that characterize the speed of the processing. Assuming that the conditions stated for $\xi_i^{(l)}$ in § 2. are satisfied, we shall show that the mean values and other moments of the random variables

$$\eta_t(l) = \sum_{j=1}^{t} l(\mathcal{B}^{(j)}) \tag{3.2}$$

can be computed by using known theorems.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, $\varrho_1, \varrho_2, \ldots$ a homogeneous Markov-chain with a finite set of possible states $\{1, 2, \ldots, n\}$. Let

$$\pi = (p_1, \ldots, p_n) \tag{3.3}$$

denote the initial distribution and

$$\Pi = [p_{ij}]_{i,j=1,\ldots,n} \tag{3.4}$$

the matrix of transition probabilities.

Let

$$p_{ij}^{(k)} = P(\varrho_{t+k} = j \,|\, \varrho_t = i) \quad (i = 1, 2, \ldots).$$

The following wellknown assertion is due to Markov.

Lemma 1. Let us suppose that there exist j and k such that $p_{ij}^{(k)}>0$ for $i=1, \dots, n$. Then

$$\lim_{r\to\infty} p_{ij}^{(r)} = x_j, \quad \sum_{j=1}^{n} x_j = 1, \tag{3.5}$$

further

$$|p_{ij}^{(r)} - x_j| \leqq C \cdot \varphi^r, \tag{3.6}$$

where $C>0$ and φ $(0<\varphi<1)$ are suitable constants.

Let f be a function having complex values defined on the set $\{1, \dots, n\}$. Let $M_\pi f(\varrho_t)$ denote the mean value of $f(\varrho_t)$ supposing that ϱ_t has an initial distribution π. Let $\theta_1, \theta_2, \dots$ be a stationary Markov-chain on the set $\{1, \dots, n\}$ with a transition probability matrix (3.4). Therefore the Markov-chain $\theta_1, \theta_2, \dots$ has an initial distribution $x=(x_1, \dots, x_n)$. As an immediate consequence of Lemma 1 we get

$$|M_\pi f(\varrho_t) - M_x f(\theta_t)| \leqq C_1 \varphi^t, \tag{3.7}$$

where $C_1>0$, $0<\varphi<1$ are constants. Since $\theta_1, \theta_2, \dots$ is stationary, therefore

$$M_x f(\theta_t) = M_x f(\theta_1), \tag{3.8}$$

and from (3.7) it follows that

$$M_\pi\left(\sum_{j=1}^{t} f(\varrho_j)\right) = t\big(M_x f(\theta_1)\big) + O(1). \tag{3.9}$$

Theorem 2 guarantees the fulfilment of Lemma 1 for the sequence (2.2). The approximate determination of $M\eta_t(l)$ is simple, if the stationary initial distribution belonging to the chain (2.2) is known.

The explicite calculation of the stationary values is in general a cumbersome matter, since the number of elements in $\mathcal{D}_r$ is about n^3 even for $r=1$.

Now we give a simple algorithm to compute it in a special case.

§ 4. Algorithm for the computation of the stationary distribution

Let the random variable sequences (2.1) be mutually independent with the distribution

$$P(\xi_i^{(l)} = k) = \frac{1}{N} \quad (l = 1, \dots, r;\ k = 1, \dots, N;\ i = 1, 2, \dots). \tag{4.1}$$

Let $\vartheta=(A_1, \dots, A_r)$ denote the processing in the first point of time, and $b(A_j)$ denote the number of processed elements of the chain $\xi_i^{(j)}$ (at this time), and $b(\vartheta)$ denote

$$b(\vartheta) \doteq \big(b(A_1), \dots, b(A_r)\big). \tag{4.2}$$

For given integers $k_1 \geqq 1$, $k_i \geqq 0$ $(i=2, \dots, r)$ let $p(k_1, \dots, k_r)$ denote the probability of the event $b(\vartheta)=(k_1, \dots, k_r)$, i.e.

$$p(k_1, \dots, k_r) = P\big(b(\vartheta) = (k_1, \dots, k_r)\big). \tag{4.3}$$

Let $s_0=0$, $s_t=k_1+\ldots+k_t$ $(t=1, \ldots, r)$. It is clear that $p(k_1, \ldots, k_r)=0$ unless

$$1 \leqq k_1 \leqq N, \quad 0 \leqq k_t \leqq N-1, \quad s_t \leqq N \quad (t=2, \ldots, r). \tag{4.4}$$

Let

$$\gamma(t, N) = \prod_{\nu=1}^{t} \left(1-\frac{\nu}{N}\right) \tag{4.5}$$

and let V_m^k denote the number of k-variations of m elements.

It follows from simple combinatorial considerations that in the cases (4.4)

$$\begin{aligned} p(k_1, \ldots, k_r) &= \frac{1}{N^{s_r+r}} V_N^{k_1} V_{N-s_1}^{k_2} \ldots V_{N-s_{r-1}}^{k_r} \cdot s_1 s_2 \ldots s_r = \\ &= \frac{N!}{(N-s_r)!} \cdot \frac{s_1 s_2 \ldots s_r}{N^{s_r+r}} = \gamma(s_r-1, N) \frac{s_1}{N} \cdot \frac{s_2}{N} \ldots \frac{s_r}{N}. \end{aligned} \tag{4.6}$$

From this representation we can easily get the limit distributions of s_t's as $N\to\infty$ for a fixed r. We are going to devote an other paper to compute the distribution and moments of k_r's under various conditions.

§ 5. Processing of two sequences

Let $r=2$. Suppose that the conditions stated for $\xi_i^{(j)}$ in the previous paragraph are fulfilled. We wish to determine the mean speed of the processing. Using the notations (2.2) the speed is determined by the sequence of random vectors

$$l(\mathscr{B}^{(1)}), l(\mathscr{B}^{(2)}), \ldots, l(\mathscr{B}^{(s)}), \ldots .$$

Due to the independence of $\xi_i^{(j)}$'s

$$P(\xi_{i+1}^{(j)} = u | \xi_i^{(j)} = v) = \frac{1}{N}.$$

By using notations (2.5), (2.6) and (2.7) we get

$$Q(\vartheta^{(s)}) = N^{-(k_1^{(s)}+\ldots+k_r^{(s)})}. \tag{5.1}$$

So $Q(\vartheta^{(s)})$ depends only on $l(\vartheta^{(s)})$. It is clear, that the condition for the realisability of $\vartheta^{(s)}$, $\vartheta^{(s+1)}$ is $fin(\vartheta^{(s)})=in(\vartheta^{(s+1)})$.

For given $i=(i_1, i_2)$ [or $*i_2$ instead of i_2], $j=(j_1, j_2)$, $k=(k_1, k_2)$ let

$$\mathscr{B}(i, j, k) = \bigcup_{\substack{in\,\vartheta=i \\ fin\,\vartheta=j \\ l(\vartheta)=k}} \vartheta. \tag{5.2}$$

Let $\mathscr{E}$ be the set of all elements $\mathscr{B}(i, j, k)$.

The sequences (2.1) and (2.2) determine the sequence

$$\alpha_1, \alpha_2, \dots, \quad \alpha_j \in \mathcal{E} \quad (j = 1, 2, \dots) \tag{5.3}$$

uniquely, where α_j $(j=1, 2, \dots)$ denotes that element of $\mathcal{E}$ for which $\mathcal{B}^{(j)} \in \alpha_j$ $(j=1, 2, \dots)$.

It is clear that the sequence (5.3) is a homogeneous Markov-chain with an initial distribution

$$P(\alpha_1 = \mathcal{B}(i, j, k)) = \frac{1}{N^{k_1+k_2+2}} \nu(i, j, k), \tag{5.4}$$

where $\nu(\alpha)$ or $\nu(i, j, k)$ denotes the number of elements of $\mathcal{D}_r$ belonging to α.

It is clear that

$$P(\alpha_{s+1} = \mathcal{B}(i, j, k) | \alpha_s = C) = \begin{cases} \frac{1}{N^{k_1+k_2}} \nu(i, j, k), \\ \text{if} \quad C, \mathcal{B} \quad \text{is realisable} \\ 0, \quad \text{otherwise.} \end{cases} \tag{5.5}$$

For the computation of $\nu(i, j, k)$ we have to distinguish the following cases:

$\langle 1 \rangle: k_2 \neq 0, \quad$ then $\quad i_2 \neq * j_2, \; i_2 \neq i_1, j_1$

$\langle 1.1 \rangle: j_1 \neq i_1$ —
- $\langle 1.1.1 \rangle: j_2 \neq i_1, j_1, i_2$
- $\langle 1.1.2 \rangle: j_2 = i_1$
- $\langle 1.1.3 \rangle: j_2 = j_1$
- $\langle 1.1.4 \rangle: j_2 = i_2$

$\langle 1.2 \rangle: j_1 = i_1$ —
- $\langle 1.2.1 \rangle: j_2 \neq i_2, i_1$
- $\langle 1.2.2 \rangle: j_2 = i_1$
- $\langle 1.2.3 \rangle: j_2 = i_2$

$\langle 2 \rangle: k_2 = 0, \quad$ then $\quad i_2 = * j_2$

$\langle 2.1 \rangle: j_1 \neq i_1$ —
- $\langle 2.1.1 \rangle: j_2 \neq j_1, i_1$
- $\langle 2.1.2 \rangle: j_2 = j_1$
- $\langle 2.1.3 \rangle: j_2 = i_2$

$\langle 2.2 \rangle: j_1 = i_1$ —
- $\langle 2.2.1 \rangle: j_2 \neq i_1$
- $\langle 2.2.2 \rangle: j_2 = i_1$.

We summarize the types, the number of possible different pairs of i's and j's, the corresponding $\nu(i, j, k)$ and $G(\text{type}\,\langle ., ., . \rangle)$ values in the following table, where

$$G(\text{type}\,\langle ., ., . \rangle) = \sum_{\mathcal{B}(i, j, k)} \frac{1}{N^{k_1+k_2}} \nu(i, j, k),$$

and we summarize for the $\mathcal{B}$'s of given type.

Type	$v(i, j, k)$	The number of possible pairs i, j	G(type $\langle .,.,. \rangle$)
$\langle 1.1.1 \rangle$	$\frac{(N-4)!}{(N-k_1-k_2)!}(k_1-1)(k_1+k_2-3)$	$N(N-1)(N-2)(N-3)$	$\gamma(k_1+k_2, N)(k_1-1)\cdot$ $\cdot(k_1+k_2-3)$
$\langle 1.1.2 \rangle$	$\frac{(N-3)!}{(N-k_1-k_2)!}(k_1-1)$	$N(N-1)(N-2)$	$\gamma(k_1+k_2, N)(k_1-1)$
$\langle 1.1.3 \rangle$	s. $\langle 1.1.2 \rangle$	s. $\langle 1.1.2 \rangle$	s. $\langle 1.1.2 \rangle$
$\langle 1.1.4 \rangle$	s. $\langle 1.1.2 \rangle$	s. $\langle 1.1.2 \rangle$	s. $\langle 1.1.2 \rangle$
$\langle 1.2.1 \rangle$	$\frac{(N-3)!}{(N-k_1-k_2)!}(k_1+k_2-2)$	s. $\langle 1.1.2 \rangle$	$\gamma(k_1+k_2, N)(k_1+k_2-2)$
$\langle 1.2.2 \rangle$	$\frac{(N-2)!}{(N-k_1-k_2)!}$	$N(N-1)$	$\gamma(k_1+k_2, N)$
$\langle 1.2.3 \rangle$	s. $\langle 1.2.2 \rangle$	s. $\langle 1.2.2 \rangle$	s. $\langle 1.2.2 \rangle$
$\langle 2.1.1 \rangle$	$\frac{(N-3)!}{(N-k_1)!}(k_1-1)(k_1-2)$	$N(N-1)(N-2)$	$\gamma(k_1, N)(k_1-1)(k_1-2)$
$\langle 2.1.2 \rangle$	$\frac{(N-2)!}{(N-k_1)!}(k_1-1)$	$N(N-1)$	$\gamma(k_1, N)(k_1-1)$
$\langle 2.1.3 \rangle$	s. $\langle 2.1.2 \rangle$	s. $\langle 2.1.2 \rangle$	s. $\langle 2.1.2 \rangle$
$\langle 2.2.1 \rangle$	s. $\langle 2.1.2 \rangle$	s. $\langle 2.1.2 \rangle$	s. $\langle 2.1.2 \rangle$
$\langle 2.2.2 \rangle$	$\frac{(N-1)!}{(N-k_1)!}$	N	$\gamma(k_1, N)$

We have got the values in this table by using simple combinatorial considerations. Let us consider e.g. the case $\langle 1.1.1 \rangle$. Let $k_1, k_2, i_1, i_2, j_1, j_2$ be fixed, i_1, i_2, j_1, j_2 be mutually distinct.

We need to enumerate the number of arrays

$$\begin{bmatrix} i_1, & |u_2, & \dots, & u_{k_1}, & \| \\ i_2, & |v_2, & \dots, & v_{k_2}, & \| \end{bmatrix}.$$

It is clear, that $j_1 \in \{u_2, \dots, u_{k_1}\}$. On the other hand $j_2 \in \{u_2, \dots, u_{k_1}\} \cup \{v_2, \dots, v_{k_2}\}$. Let us consider the subcase $j_2 \in \{u_2, \dots, u_{k_1}\}$. We can arrange the elements j_1 and j_2 among $u_2, \dots, u_{k_1}$ in $(k_1-1)(k_1-2)$ different ways. Then we choose the remaining (k_1-3) u's in $\frac{(N-4)!}{(N-4-(k_1-3))!} = \frac{(N-4)!}{(N-k_1-1)!}$ ways. The sequences $v_1, \dots, v_{k_2}$ and $i_1, u_2, \dots, u_{k_1}, i_2$ have no common elements, otherwise they are arbitrary. Therefore we can choose the sequence $v_2, \dots, v_k$ in $\frac{(N-k_1-1)!}{(N-k_1-1-(k_2-1))!} = \frac{(N-k_1-1)!}{(N-k_1-k_2)!}$ ways. Therefore $\frac{(N-4)!}{(N-k_1-k_2)!}$ different processing states belong to this subcase.

Let now $j_2 \in \{v_2, \ldots, v_{k_2}\}$. The set $\{u_2, \ldots, u_{k_1}\}$ contains j_1, but it does not contain i_1, i_2, j_2. Therefore we can choose this set in

$$(k_1-1)\frac{(N-4)!}{(N-4-(k_1-2))!} = (k_1-1)\frac{(N-4)!}{(N-k_1-2)!}$$

ways. The set $\{v_2, \ldots, v_{k_2}\}$ contains j_2, but it doesn't contain the different elements $i_2, i_1, u_2, \ldots, u_{k_1}$, therefore the number of such sets is

$$(k_2-1)\frac{(N-k_1-2)!}{(N-k_1-2-(k_2-2))!} = (k_2-1)\frac{(N-k_1-2)!}{(N-k_1-k_2)!}.$$

Therefore

$$(k_1-1)(k_2-1)\frac{(N-4)!}{(N-k_1-k_2)!}$$

different processing states belong to the second subcase. From here

$$\nu(i, j, k) = (k_1-1)(k_2-1)\frac{(N-4)!}{(N-k_1-k_2)!}.$$

The proof of the remaining cases is a bit easier.

Due to Lemma 1 the stationary distribution is constructable for the Markov-chain (5.3). Let $u(\alpha)$ denote it.

Let us introduce the notation

$$F(x, y) = \sum_{\substack{\mathrm{fin}\,\alpha=(x,y) \\ \alpha\in\mathscr{E}}} u(\alpha). \tag{5.6}$$

Due to the symmetry

$$F(x, y) = \begin{cases} F(1, 2), & \text{if} \quad x \neq y \\ F(1, 1), & \text{if} \quad x = y. \end{cases} \tag{5.7}$$

Let

$$G(x, y) = \sum_{\substack{\mathrm{in}\,\alpha=(x,y) \\ \alpha=\mathscr{E}}} u(\alpha). \tag{5.8}$$

For the stationary distribution we have

$$\sum_{\alpha_1\in\mathscr{E}} P(\alpha_2|\alpha_1)u(\alpha_1) = u(\alpha_2), \tag{5.9}$$

where

$$\sum_{\alpha\in\mathscr{E}} u(\alpha) = 1. \tag{5.10}$$

Introducing the notation

$$Q(\alpha) = \sum_{\mathscr{B}\in\alpha} Q(\mathscr{B}) \quad (\alpha\in\mathscr{E})$$

we get

$$P(\alpha_2|\alpha_1) = \begin{cases} Q(\alpha_2), & \text{if} \quad in\,\alpha_2 = fin\,\alpha_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then due to (5.9)

$$u(\alpha_2) = Q(\alpha_2) \sum_{\text{fin}\,\alpha_1=\text{in}\,\alpha_2} u(\alpha_1) = Q(\alpha_2)F(in\,\alpha_2). \tag{5.11}$$

Let x and y be arbitrary integers $(1 \leqq x, y \leqq N)$, and $fin\,\alpha_1=(x, y)$. Then

$$1 = \sum_{\alpha_2 \in \mathscr{E}} P(\alpha_2|\alpha_1) = \sum_{\text{in}\,\alpha_2=(x,y)} Q(\alpha_2). \tag{5.12}$$

Hence, by using (5.11) and (5.12) we get

$$G(x, y) = \sum_{\text{in}\,\alpha=(x,y)} u(\alpha) = F(x, y) \sum_{\text{in}\,\alpha=(x,y)} Q(\alpha) = F(x, y). \tag{5.13}$$

Let

$$\lambda = F(1, 2), \quad \mu = F(1, 1). \tag{5.14}$$

If λ and μ are known, then the stationary distribution can be computed easily by using (5.11).

From (5.10) we have

$$N(N-1)\lambda + N\mu = 1. \tag{5.15}$$

To determine λ and μ we shall give another relation between them.

Let

$$\mathscr{E}_2 = \{\alpha | in\,\alpha = (h, h),\ (h = 1, 2, \ldots, N)\}, \tag{5.16}$$

$$\mathscr{E}_1 = \mathscr{E} \setminus \mathscr{E}_2. \tag{5.17}$$

It is clear that

$$\mu = F(1, 1) = \sum_{\text{fin}\,\alpha=(1,1)} u(\alpha) = \lambda \sum_{\substack{\text{fin}\,\alpha=(1,1)\\ \alpha\in\mathscr{E}_1}} Q(\alpha) + \mu \sum_{\substack{\text{fin}\,\alpha=(1,1)\\ \alpha\in\mathscr{E}_2}} Q(\alpha) = \lambda\alpha + \mu\beta.$$

Let us observe that the α's having the form

$$\begin{bmatrix} h, & |(\ldots)\|1 \\ *, & \|h = 1 \end{bmatrix} = \begin{bmatrix} 1, & |(\ldots)\| \\ & \| \end{bmatrix}$$

are belonging to $\mathscr{E}_2$. These α's are belonging to $\langle 2.2.2 \rangle$. Therefore

$$\beta = \frac{1}{N} \sum_{k=1}^{N} \gamma(k-1, N). \tag{5.18}$$

Let us consider the sum α. We classify the α's in $\mathscr{E}_1$ according to $in\,\alpha=(x, y)$, where $x \neq y$:

Class 1: $y=1$, then $x \neq 1$. This is the case $\langle 2.1.2 \rangle$.

Class 2: $x=1$, then $y \neq 1$. This is the case $\langle 1.2.2 \rangle$.

Class 3: $x \neq 1$, $y \neq 1$. This is the case $\langle 1.1.3 \rangle$. Let α_1, α_2, α_3 denote the corresponding sums and let

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3.$$

From the table we can see easily that

$$\alpha_1 = \frac{1}{N}\sum_{t=1}^{N-1}\gamma(t, N)t, \quad \alpha_2 = \alpha_1, \quad \alpha_3 = \frac{1}{N}\sum_{t=1}^{N-1}\gamma(t, N)\frac{t^2-3t+2}{2},$$
$$\alpha = \frac{1}{N}\sum_{t=1}^{N-1}\gamma(t, N)\frac{t^2+t+2}{2}. \tag{5.19}$$

From the system of linear equations

$$\begin{cases} N(N-1)\lambda+N\mu = 1 \\ \lambda\alpha+\mu(\beta-1) = 0 \end{cases} \tag{5.20}$$

we can compute λ and μ.

Let now $f(\alpha)$ be a function depending only on the length of processing (number of processed elements). Let us compute $M_u f(\alpha)$, i.e. supposing the stationarity of (5.3).

Then we get

$$M_u f(\alpha) = \lambda \sum_{\alpha\in\mathcal{E}_1} f(\alpha)Q(\alpha)+\mu \sum_{\alpha\in\mathcal{E}_2} f(\alpha)Q(\alpha). \tag{5.21}$$

Those processing states

$$\begin{bmatrix} i_1, | \dots \| j_1 \\ i_2, | \dots \| j_2 \end{bmatrix}$$

belong to the elements $\alpha\in\mathcal{E}_2$, for which $i_1=j_2$, $i_2=*i_1$, i.e. the cases (2.1.3), (2.2.2).

From here

$$\sum_{\alpha\in\mathcal{E}_2} f(\alpha)Q(\alpha) = \sum_{k_1=1}^{N} f(k_1, 0)\cdot k_1\cdot\gamma(k_1, N), \tag{5.22}$$

$$\sum_{\alpha\in\mathcal{E}_1} f(\alpha)Q(\alpha) = \sum_{k_1=1}^{N} f(k_1, 0)k_1(k_1-1)\gamma(k_1, N)+$$
$$+\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-k_1} f(k_1, k_2)k_1(k_1+k_2)\gamma(k_1+k_2, N). \tag{5.23}$$

Substituting (5.22) and (5.23) into (5.21) we get

$$M_u f(\alpha) = \lambda \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-k_1} f(k_1, k_2)k_1(k_1+k_2)\gamma(k_1+k_2, N)+$$
$$+\sum_{k_1=1}^{N} f(k_1, 0)\gamma(k_1, N)[\lambda k_1(k_1-1)+\mu k_1]. \tag{5.24}$$

§ 6. Asymptotic behaviour of the speed

We compute the asymptotic value of the expression (5.24) as $N\to\infty$ for $f(k_1, k_2)=s_2$. Let M denote the left hand side of (5.24). Then

$$M=\lambda \sum_{1\leq s_1<s_2\leq N} s_1 s_2^2 \gamma(s_2, N)+\lambda \sum_{s_1=1}^{N} s_1^2(s_1-1)\gamma(s_1, N)+\mu \sum_{s_1=1}^{N} s_1^2\gamma(s_1, N), \quad (6.1)$$

where λ, μ is the solution of

$$\begin{cases} N(N-1)\lambda+N\mu=1 \\ \alpha\lambda+(\beta-1)\mu=0 \end{cases} \quad (6.2)$$

and α and β are defined by

$$\alpha=\frac{1}{N}\sum_{t=0}^{N-1}\frac{t^2+t+2}{2}\gamma(t, N), \quad (6.3)$$

$$\beta=\frac{1}{N}\sum_{t=0}^{N-1}\gamma(t, N). \quad (6.4)$$

(6.1) is easily computable approximately from the original expression. We shall give M as a simple function of N. Let

$$\tau_k=\sum_{t=0}^{N-1} t^k\gamma(t, N), \quad (k=1, 2, 3, 4) \quad (6.5)$$

and

$$\varrho_j=\sum_{t=j}^{N-1}\gamma(t, N) \quad (j=0, 1, \ldots, 4). \quad (6.6)$$

It is clear that

$$\begin{aligned} \varrho_1 &= \varrho_0-1, \\ \varrho_2 &= \varrho_0-1-\left(1-\frac{1}{N}\right)=\varrho_0-2-\frac{1}{N}, \\ \varrho_3 &= \varrho_2-\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)=\varrho_0-3-\frac{4}{N}-\frac{2}{N^2}; \\ \varrho_4 &= \varrho_3-\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right)=\varrho_0-4-\frac{10}{N}-\frac{13}{N^2}-\frac{6}{N^3}. \end{aligned} \quad (6.7)$$

Now we compute τ_k's as functions of $\varrho_0, \ldots, \varrho_k$. Because of the definition of $\gamma(t, N)$ we have

$$\gamma(t+1, N)=\gamma(t, N)\left[1-\frac{t+1}{N}\right] \quad (t=0, 1, \ldots),$$

i.e.

$$\gamma(t+1, N)N=\gamma(t, N)[N-(t+1)], \quad (t=0, 1, \ldots). \quad (6.8)$$

Hence

$$\gamma(t, N)t=(N-1)\gamma(t, N)-N\gamma(t+1, N) \quad (6.9)$$

and therefore, by using $\gamma(k, N)=0$ (if $k \geqq N$), we get

$$\tau_1 = (N-1)\varrho_0 - N\varrho_1.$$

Let us compute now the polynomial t^k as the sum of the basic functions

$$p_0(t, N) = 1, \quad p_j(t, N) = \prod_{h=1}^{j} (N-(t+h)) \quad (j = 1, \ldots, 4).$$

By simple operations we get

$$t^2 = p_2(t, N) - (2N-3)p_1(t, N) + (N-1)^2 p_0(t, N), \tag{6.10}$$

$$t^3 = -p_3(t, N) + (3N-6)p_2(t, N) - (3N^2-9N+7)p_1(t, N) + (N-1)^3 p_0(t, N), \tag{6.11}$$

$$t^4 = p_4(t, N) + Ep_3(t, N) + Fp_2(t, N) + Gp_1(t, N) + Hp_0(t, N), \tag{6.12}$$

where

$$E = -4N+10, \quad F = 6N^2-24N+25,$$

$$G = -4N^3+18N^2-28N+15, \quad H = (N-1)^4. \tag{6.13}$$

On the other hand because of (6.8)

$$\gamma(t, N)p_k(t, N) = N^k \gamma(t+k, N), \quad (k = 0, \ldots, 4). \tag{6.14}$$

So we have

$$\gamma(t, N) \cdot t^2 = N^2 \gamma(t+2, N) - (2N-3)N\gamma(t+1, N) + (N-1)^2 \gamma(t, N),$$

$$\gamma(t, N) \cdot t^3 = -N^3 \gamma(t+3, N) + (3N-6)\gamma(t+2, N) - (3N^2-9N+7)N\gamma(t+1, N) + \\ + (N-1)^3 \gamma(t, N),$$

$$\gamma(t, N) \cdot t^4 = N^4 \gamma(t+4, N) + E \cdot N^3 \gamma(t+3, N) + FN^2 \gamma(t+2, N) + \\ + GN\gamma(t+1, N) + H\gamma(t, N),$$

and hence

$$\begin{aligned} \tau_2 &= N^2 \varrho_2 - (2N-3)N\varrho_1 + (N-1)^2 \varrho_0 = (N+1)\varrho_0 - 2N, \\ \tau_3 &= -N^3 \varrho_3 + (3N-6)N^2 \varrho_2 - (3N^2-9N+7)\varrho_1 + (N-1)^3 \varrho_0 = \\ &= (2N^2+3N) - (4N+1)\varrho_0, \\ \tau_4 &= N^4 \varrho_4 + EN^3 \varrho_3 + FN^2 \varrho_2 + GN_1 \varrho_1 + H\varrho_0 = (3N^2+11N)\varrho_0 - (11N^2+4N). \end{aligned} \tag{6.15}$$

Now it follows from (6.3) and (6.4)

$$\beta = \frac{1}{N}\varrho_0, \quad \alpha = \frac{1}{2N}(\tau_2+\tau_1+2\varrho_0) = \left(\frac{1}{2}+\frac{1}{N}\right)\varrho_0 - \frac{1}{2}. \tag{6.16}$$

By substituting (6.16) into (6.2) we get

$$\mu = \frac{\alpha}{N\alpha + N(N-1)(1-\beta)} = \frac{(N+2)\varrho_0 - N}{(2N^3-3N^2)+(4N-N^2)\varrho_0} \tag{6.17}$$

and

$$\lambda = \frac{1-\beta}{N\alpha + N(N-1)(1-\beta)} = \frac{2N-2\varrho_0}{(2N^3-3N^2)+(4N-N^2)\varrho_0}. \tag{6.18}$$

Let us observe that

$$M = \lambda \sum_{s_2=2}^{N} \frac{s_2(s_2-1)s_2^2}{2} \gamma(s_2, N) + \lambda \sum_{s_1=1}^{N} s_1^2(s_1-1)\gamma(s_1, N) +$$

$$+\mu \sum_{s_1=1}^{N} s_1^2 \gamma(s, N) = \lambda\left[\frac{1}{2}(\tau_4-\tau_3)+(\tau_3-\tau_2)\right] + \mu\tau_2 = \lambda\left[\frac{1}{2}(\tau_4+\tau_3)-\tau_2\right] + \mu\tau_2. \tag{6.19}$$

Substituting (6.15), (6.17) and (6.18) into (6.19) we get M as a function of N and ϱ_0:

$$M = \frac{N^3(3\varrho_0-9) - N^2(2\varrho_0^2-11\varrho_0+9) - N(2\varrho_0^2+7\varrho_0)+5\varrho_0^2}{2N^3-N^2(\varrho_0+3)+4N\varrho_0}. \tag{6.20}$$

To estimate this expression we need the following

Lemma 2.

$$\varrho_0 = \sum_{t=0}^{N-1} \gamma(t, N) = \sqrt{\frac{\pi N}{2}} + O(1) \quad (N \to \infty). \tag{6.21}$$

Proof. Since $1-x \leqq e^{-x}$, we get

$$\gamma(t, N) = \prod_{\nu=1}^{t}\left(1-\frac{\nu}{N}\right) \leqq e^{-\frac{1}{N}\sum_{\nu=1}^{t}\nu} < e^{-\frac{t^2}{2N}}. \tag{6.22}$$

Therefore

$$\sum_{H \leqq t < N} \gamma(t, N) < \sum_{t \geqq H} e^{-\frac{t^2}{2N}} < \int_{H-1}^{N} e^{-\frac{t^2}{2N}} dt = \frac{1}{2}\sqrt{2N} \int_{\frac{(H-1)^2}{2N}}^{\frac{N}{2}} e^{-\lambda}\lambda^{1/2} d\lambda <$$

$$< \frac{1}{2}\sqrt{2N}\frac{\sqrt{2N}}{H-1} \int_{\frac{(H-1)^2}{2N}}^{\infty} e^{-\lambda} d\lambda = \frac{N}{H-1} e^{-\frac{(H-1)^2}{2N}}. \tag{6.23}$$

On the other hand

$$\sum_{t \leqq H} e^{-\frac{t^2}{2N}} \leqq \int_0^H e^{-\frac{t^2}{2N}} dt < \sqrt{2N}\frac{1}{2}\int_0^{\infty} e^{-\lambda}\lambda^{1/2} d\lambda \leqq C_1\sqrt{N}. \tag{6.24}$$

It is clear that

$$\varrho_0 = \sum_{t \leqq N^{0,6}} \gamma(t, N) + \sum_{N^{0,6} < t \leqq N} \gamma(t, N) = \Sigma_A + \Sigma_B. \tag{6.25}$$

Because of (6.23)

$$\Sigma_B \leqq C_2 N^{0,4} \cdot e^{-\frac{1}{3}N^{0,2}} = o(1). \tag{6.26}$$

On the other hand by using Stirling-formula for $\gamma(t, N)$ in the interval $1 \leqq t \leqq \leqq N^{0,6}$ we get

$$\log \gamma(t, N) = \left(N - \frac{1}{2} - t\right) \log \frac{N}{N-t} - t + O\left(\frac{1}{N}\right). \tag{6.27}$$

Since

$$\log \frac{N}{N-t} = -\log\left(1 - \frac{N}{t}\right) = \frac{t}{N} + \frac{t^2}{2N^2} + O\left(\left(\frac{t}{N}\right)^3\right), \tag{6.28}$$

therefore

$$\log \gamma(t, N) = -\frac{t^2}{2N} + O\left(\frac{t}{N} + \frac{t^3}{N^2}\right), \tag{6.29}$$

and so

$$\Sigma_A = \sum_{t \leqq N^{0,6}} e^{-\frac{t^2}{2N}} \left(1 + O\left(\frac{t}{N} + \frac{t^3}{N^2}\right)\right) = \sum_{t \leqq N^{0,6}} e^{-\frac{t^2}{2N}} + \frac{1}{N} O(\Sigma_C) + \frac{1}{N^2} O(\Sigma_D), \tag{6.30}$$

where

$$\Sigma_C = \sum_{t \leqq N^{0,6}} t \cdot e^{-\frac{t^2}{2N}}, \tag{6.31}$$

$$\Sigma_D = \sum_{t \leqq N^{0,6}} t^3 e^{-\frac{t^2}{2N}}. \tag{6.32}$$

Since

$$\begin{aligned} \sum_{t=1}^{\infty} t^{\alpha} e^{-\frac{t^2}{2N}} < \int_0^{\infty} t^{\alpha} e^{-\frac{t^2}{2N}} dt = (2N)^{\frac{\alpha+1}{2}} \frac{1}{2} \int_0^{\infty} \lambda^{\frac{\alpha-1}{2}} e^{-\lambda} d\lambda = \\ = (2N)^{\frac{\alpha+1}{2}} \frac{1}{2} \Gamma\left(\frac{\alpha+1}{2}\right), \end{aligned} \tag{6.33}$$

therefore

$$\Sigma_C = O(N), \quad \Sigma_D = O(N^2), \tag{6.34}$$

and so

$$\varrho_0 = \sum_{1 \leqq t \leqq N^{0,6}} e^{-\frac{t^2}{2N}} + O(1) = \sum_{t=1}^{\infty} e^{-\frac{t^2}{2N}} + O(1). \tag{6.35}$$

Since $e^{-\frac{t^2}{2N}}$ is a monoton decreasing function of t, therefore

$$\int_0^{\infty} e^{-\frac{t^2}{2N}} dt < \sum_{t=1}^{\infty} e^{-\frac{t^2}{2N}} < \int_1^{\infty} e^{-\frac{t^2}{2N}} dt, \tag{6.36}$$

and so

$$\varrho_0 = \int_0^{\infty} e^{-\frac{t^2}{2N}} dt + O(1). \tag{6.37}$$

Since

$$\int_0^\infty e^{-\frac{t^2}{2N}}\,dt = \frac{1}{2}\sqrt{2N}\int_0^\infty e^{-\lambda}\lambda^{1/2}\,d\lambda = \frac{\sqrt{2}}{2}\Gamma\left(\frac{1}{2}\right)\sqrt{N} = \sqrt{\frac{\pi}{2}}\sqrt{N}, \tag{6.38}$$

therefore

$$\varrho_0 = \sqrt{\frac{\pi}{2}}\sqrt{N} + O(1). \tag{6.39}$$

By substituting (6.39) into (6.20) we get the following

Teorem 3. Let $f(k_1, k_2) = k_1 + k_2$. Then under the assumptions of § 5. we have

$$\lim_{N\to\infty} N^{-0,5} M_u f(\alpha) = \frac{3}{2}\sqrt{\frac{\pi}{2}}. \tag{6.40}$$

In a previous paper [2] we have proved that the similar limit is $\sqrt{\frac{\pi}{2}}$ for the processing speed of one sequence. Comparing the results we get that the processing speed of the second sequence is half of the speed of the first one.

DEPT. OF COMPUTER SCIENCE
OF L. EÖTVÖS UNIVERSITY
MÚZEUM KRT. 6—8.
BUDAPEST, HUNGARY
H—1088

References

[1] Burnett, G. J. and E. G. Coffman, Combinatorial problem related to interleaved memory systems, *J. Assoc. Comput. Mach.*, v. 20, 1973, pp. 39—46.

[2] Iványi, A. and I. Kátai, Estimates for speed of computers with interleaved memory systems, *Ann. Univ. Sci. Budapest. Eötvös Sect. Mat.*, v. 19, 1976, pp. 159—164.

(Received August 7, 1978)