On some types of incompletely specified automata

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1. Preliminaries

In this paper the most general definition of an incompletely specified (or *partial*) finite automaton (generalized, probabilistic and deterministic) is proposed and some special classes of such automata are introduced. The conceptions of this paper are the further development of the author's ideas, stated in the book [1]. The known notions of partial finite automata (for example [1], [2], [3] and [4]) are included in the proposed definitions as exeptional cases. For the notations and notions that will not be defined here, the author refers to the books [1] and [5].

First of all it is useful to recall some definitions of the completely specified finite automata theory [1] and [5], and introduce some further notations.

By an *alphabet* X we mean a finite non-empty ordered set of elements. A finite sequence $X^{(t)}=X_{s_1}X_{s_2}\dots X_{s_t}$ ($X_{s_i}\in X$, $t\geqq 0$) is called a *word* over X, and $t=|X^{(t)}|$ is the *length* of $X^{(t)}$. We use the notations X^* and X^t for the set of all words over X and for the set of all words of length t over X, respectively. Besides the following notations are used for the sets of all real numbers, vectors and matrices:

$$\mathscr{R}=(-\infty,\infty),\quad \mathscr{R}^m=\{r|r=(r_1,r_2,\dots,r_m),\ r_i\in\mathscr{R},\ i=\overline{1,m}\},$$

$$\mathscr{R}^{m,n}=\{R|R=(r_{ij})_{m,n},\ r_{ij}\in\mathscr{R},\ i=\overline{1,m},\ j=\overline{1,n}\}.$$

A vector is called *stochastic* (or *probabilistic*) if all its entries are non-negative and the sum of its entries is equal to 1. A matrix is called *stochastic* (or *probabilistic*) if all its rows are stochastic vectors. A stochastic vector is called *degenerate* if one of its entries is 1 and the other are equal to 0. A stochastic matrix is *degenerate* if all its rows are degenerate stochastic vectors. The following notations are used for the sets of all stochastic (degenerate stochastic) m-dimensional vectors and $(m\times n)$-matrices:

$$\mathscr{P}^m=\{p|p=(p_1,p_2,\dots,p_m),\ p_i\in[0,1],\ i=\overline{1,m},\ \sum_i p_i=1\},$$

$$\mathscr{D}^m=\{d|d=(d_1,d_2,\dots,d_m),\ d_i\in\{0,1\},\ i=\overline{1,m},\ \sum_i d_i=1\},$$

$$\mathscr{P}^{m,n}=\{P|P=(p_{ij})_{m,n},\ p_{ij}\in[0,1],\ \sum_j p_{ij}=1,\ i=\overline{1,m},\ j=\overline{1,n}\},$$

$$\mathscr{D}^{m,n}=\{D|D=(d_{ij})_{m,n},\ d_{ij}\in\{0,1\},\ \sum_j d_{ij}=1,\ i=\overline{1,m},\ j=\overline{1,n}\}.$$

Let $X=\{X_1, X_2, \ldots, X_n\}$, $A=\{A_1, A_2, \ldots, A_m\}$, $Y=\{Y_1, Y_2, \ldots, Y_k\}$ be the alphabets of inputs, states and outputs, respectively. Then a *finite generalized automaton* is a system

$$\mathbf{A}_{gen} = \langle X, A, Y, r^{(0)}, R\rangle \tag{1}$$

where $r^{(0)} \in \mathcal{R}^m$ is the initial vector and R $(\in \mathcal{R}^{nm,km})$ is the transition-output matrix, which presents a mapping of $X \times A \times Y \times A$ into the set of real numbers $\mathcal{R}$. The matrix R is usually represented by a combination of its nk square submatrices $\{R(X_s, Y_l)\}$ such that

$$R = \begin{pmatrix} R(X_1, Y_1) & R(X_1, Y_2) & \ldots & R(X_1, Y_k) \\ \cdots & \cdots & \cdots & \cdots \\ R(X_n, Y_1) & R(X_n, Y_2) & \ldots & R(X_n, Y_k) \end{pmatrix}.$$

In this case it may be said that R presents a mapping of $X \times Y$ into $\mathcal{R}^{m,m}$. The domain of this mapping is extended from $X \times Y$ to $(X \times Y)^t$ $(t=1, 2, \ldots)$, where

$$(X \times Y)^t = \{(X^{(t)}, Y^{(t)}) | X^{(t)} \in X^t,\ Y^{(t)} \in Y^t\}$$

and

$$R(X^{(t)}, Y^{(t)}) = \prod_{i=1}^{t} R(X_{s_i}, Y_{l_i}),$$

with

$$X^{(t)} = X_{s_1} X_{s_2} \ldots X_{s_t}, \quad Y^{(t)} = Y_{l_1} Y_{l_2} \ldots Y_{l_t}.$$

The *generalized mapping* Φ induced by a generalized automaton $\mathbf{A}_{gen}$ (in notation: $\Phi \vdash \mathbf{A}_{gen}$) is the mapping of

$$(X \times Y)^* = \{(X^{(t)}, Y^{(t)}) | X^{(t)} \in X^t,\ Y^{(t)} \in Y^t,\ t = 0, 1, \ldots\},$$

into $\mathcal{R}$ defined by

$$\Phi(X^{(t)}, Y^{(t)}) = r^{(0)} \prod_{i=1}^{t} R(X_{s_i}, Y_{l_i}) e,$$

where e is the m-dimensional column vector whose each entry is 1.

Hereafter we use the term automaton to mean a finite automaton.

A *probabilistic automaton*

$$\mathbf{A}_{pr} = \langle X, A, Y, p^{(0)}, P\rangle \tag{2}$$

is a generalized automaton (1) such that $r^{(0)} = p^{(0)} \in \mathcal{P}^m$ and $R \doteq P \in \mathcal{P}^{nm,km}$. $p^{(0)}$ is called the *initial probabilistic distribution* on the state set A and P is called the *transition-output probability matrix* of the automaton $\mathbf{A}_{pr}$. The elements of P are treated as

$$p_{si,lj} = Pr(Y_l A_j | X_s A_i).$$

A probabilistic automaton $\mathbf{A}_{pr}$ induces the *probabilistic mapping* Φ of $(X \times X)^*$ into the closed real interval $[0, 1]$ defined by

$$\Phi(X^{(t)}, Y^{(t)}) = Pr(Y^{(t)} | X^{(t)}) = p^{(0)} \prod_{i=1}^{t} P(Y_{l_i} | X_{s_i}) e,$$

where $P(Y_{l_i} | X_{s_i})$ is the proper square submatrix of R.

A *deterministic automaton*

$$\mathbf{A}_{det} = \langle X, A, Y, d^{(0)}, D \rangle$$

is a probabilistic automaton (2) such that $p^{(0)} = d^{(0)} \in \mathcal{D}^m$ and $P = D \in \mathcal{D}^{nm,km}$. If $d^{(0)} = (d_1, d_2, \dots, d_m)$, $d_j = 1$, $d_i = 0$, $i \neq j$, then A_j is called the initial state of $\mathbf{A}_{det}$. A deterministic automaton $\mathbf{A}_{det}$ with the initial state A_j induces the *deterministic mapping*

$$\Phi_j \colon X^* \to Y^*$$

given by

$$\Phi_j(X^{(t)}) = Y^{(t)} \Leftrightarrow d^{(0)} \prod_{i=1}^{t} D(X_{s_i}, Y_{l_i}) e = 1.$$

2. Partial vectors, matrices and automata

Hereafter we use the term "partial" to mean "incompletely specified". In accordance with the classical automata theory an automaton $\mathbf{A}_{gen}$ ($\mathbf{A}_{pr}$ or $\mathbf{A}_{det}$) is partial if some of the elements of $r_1^{(0)}$, R ($p^{(0)}$, P or $d^{(0)}$, D) are undefined and represented by "—" ([2], [3] and [4]). The conditions under which this occurs are usually treated as "don't care conditions" when either some combinations of input and present state never occur or the output (the next state) is of no concern for some combinations of input and present state. Such an incomplete specification is usually interpreted to mean that the designer may use these incomplete specifications in arbitrary way to his advantage in obtaining a completely specified automaton. It is clear that such an interpretation of partial automata is not universal and does not embrace many interesting (as theoretical, so practical) cases. For example, there are many such problems that an incomplete specification of an automaton is the result of our ignorance of its exact structure or is the effect of the opportunity to choose its structure from a certain restricted class of structures. As a rule in practice there are not free choises of the indeterminate elements of $r^{(0)}$, R ($p^{(0)}$, P or $d^{(0)}$, D) and the various ways of their specification are closely interdependent. Thus it will be usefull to offer the most general interpretation of partial automata.

Some more general classes of partial probabilistic vectors, matrices and automata were proposed and studied by the author in the book [1]. Now we are going to make the furthermost generalization of the concept of partial vectors, matrices and automata. The main idea of this generalization is that any partial object (vector matrix, automaton) may be treated as a set of completely specified objects (vectors, matrices, automata) which are the results of various ways of its specification. Thus it is possible to describe this partial object by means of a set of objects and to investigate this set.

We shall now introduce the following general definitions. Any non-empty subset $\tilde{r}$ of the set $\mathcal{R}^m$ is called a *partial m-dimensional vector.* Any non-empty subset $\tilde{R}$ of the set $\mathcal{R}^{m,n}$ is called a *partial* $(m \times n)$*-matrix.* For instance, the partial $(m \times m)$-matrix

$$\tilde{R} = \{R \mid R \in \mathcal{R}^{m,m},\ |R| \in (0, 2]\}$$

is the subset of those $(m \times m)$-matrices whose determinants have values lying in the interval (0, 2].

A *partial generalized automaton* is a system

$$\tilde{\mathbf{A}}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle \tag{3}$$

where X, A, Y are as usual the alphabets of inputs, states and outputs, $\tilde{r}^{(0)}$ ($\subseteq \mathcal{R}^m$) is a partial initial vector and $\tilde{R}$ ($\subseteq \mathcal{R}^{nm,km}$) is a partial transition-output matrix. A partial generalized automaton (3) defines the set of completely specified generalized automata (1) such that

$$\mathbf{A}_{\text{gen}} \in \tilde{\mathbf{A}}_{\text{gen}} \Leftrightarrow r^{(0)} \in \tilde{r}^{(0)} \quad \& \quad R \in \tilde{R}.$$

By the *partial generalized mapping* $\tilde{\Phi}$ induced by $\mathbf{A}_{\text{gen}}$ we mean the following set of mappings of $(X \times Y)^*$ into $\mathcal{R}$:

$$\tilde{\Phi} = \{\Phi | \Phi \vdash A_{\text{gen}}, A_{\text{gen}} \in \tilde{A}_{\text{gen}}\}.$$

3. Partial p-vectors, p-matrices, p-automata

In accordance with above definitions any non-empty subset $\tilde{p}$ of the set $\mathcal{P}^m$ is called a *partial probabilistic vector*, or shortly, a *partial p-vector*. Any non-empty subset $\tilde{P}$ of the set $\mathcal{P}^{m,n}$ is called a *partial probabilistic* $(m \times n)$*-matrix*, or shortly, a *partial p-matrix*. Thus, any partial vector $\tilde{r}$ (matrix $\tilde{R}$) is a partial p-vector (p-matrix) if and only if all $r \in \tilde{r}$ ($R \in \tilde{R}$) are stochastic.

A *partial probabilistic automaton* (a *partial p-automaton*) is a system

$$\tilde{\mathbf{A}}_{pr} = \langle X, A, Y, \tilde{p}^{(0)}, \tilde{P} \rangle$$

where $\tilde{p}^{(0)} \subseteq \mathcal{P}^m$, $\tilde{P} \subseteq \mathcal{P}^{nm,km}$ and

$$\mathbf{A}_{pr} \in \tilde{\mathbf{A}}_{pr} \Leftrightarrow p^{(0)} \in \tilde{p}^{(0)} \quad \& \quad P \in \tilde{P}.$$

So far we have said nothing about methods of specification of $\tilde{r}^{(0)}$, $\tilde{p}^{(0)}$, $\tilde{R}$, $\tilde{P}$. As it was shown in [1] some problems of abstract theory of partial automata may be investigated without indication of such a concrete specification method. But there are many problems which may be solved only if this method is given. Many different types of partial vectors, matrices and automata may be constructed by various methods of specification of $\tilde{r}^{(0)}$, $\tilde{p}^{(0)}$, $\tilde{R}$ and $\tilde{P}$. Some of them will be introduced hereinafter.

4. Partial f-vectors, f-matrices, f-automata

Let $\xi_1, \xi_2, \ldots, \xi_q$ be q independent parameters and $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_q$ be their domains. Let $f_i(\xi_1, \xi_2, \ldots, \xi_q)$ $(i = \overline{1, m})$ be real single-valued functions. Then a partial vector

$$\tilde{r} = \{r | r = (r_1, r_2, \ldots, r_m),\ r_i = f_i(\xi_1, \xi_2, \ldots, \xi_q),\ i = \overline{1, m},\ \xi_\nu \in \tilde{\sigma}_\nu,\ \nu = \overline{1, q}\} \tag{4}$$

is called a *partial f-vector* and is presented as

$$\tilde{r} = (f_1(\{\xi_\nu\}), f_2(\{\xi_\nu\}), \ldots, f_m(\{\xi_\nu\})) \quad (\xi_\nu \in \tilde{\sigma}_\nu,\ \nu = \overline{1, q})$$

where

$$f_i(\{\xi_\nu\}) = f_i(\xi_1, \xi_2, \ldots, \xi_q).$$

Accordingly, a partial matrix

$$\tilde{R} = \{R \mid R = (r_{ij})_{m,n},\ r_{ij} = f_{ij}(\{\xi_\nu\}),\ i = \overline{1, m},\ j = \overline{1, n},\ \xi_\nu \in \tilde{\sigma}_\nu,\ \nu = \overline{1, q}\},$$

where f_{ij} is a real single-valued function ($i=\overline{1, m}$, $j=\overline{1, n}$), is called a *partial f-matrix* and is presented as

$$\tilde{R} = (f_{ij}(\{\xi_\nu\}))_{m,n}\ (\xi_\nu \in \tilde{\sigma}_\nu,\ \nu = \overline{1, q}). \tag{5}$$

For example,

$$\tilde{R} = \begin{pmatrix} \xi_2 + \sin \xi_1 & \sqrt{1+\xi_1\xi_2} \\ \xi_1 - \xi_2^2 & 2 \end{pmatrix} \quad \left(\xi_1 \in \left[\frac{1}{2}, 1\right),\quad \xi_2 \in \{0, 1, 2\}\right)$$

is a partial square f-matrix of order 2.

By substituting the different values of the parameters into f_i or f_{ij}, the various completely specified vectors or matrices of $\tilde{r}$ or $\tilde{R}$ may be found.

We say that a function $f(\{\xi_\nu\})$ *essentially depends* on the parameter ξ_ν if there exist $b_1, b_2 \in \tilde{\sigma}_\nu$ such that

$$f(\xi_1, \ldots, \xi_{\nu-1}, b_1, \xi_{\nu+1}, \ldots, \xi_q) \not\equiv f(\xi_1, \ldots, \xi_{\nu-1}, b_2, \xi_{\nu+1}, \ldots, \xi_q)$$

holds.

A partial f-vector (4) *essentially depends* on ξ_ν if some of its elements essentially depends on ξ_ν. Two partial f-vectors are called *indepedent* if there is no such parameter on which both f-vectors essentially depend.

If every two rows of a partial f-matrix are independent partial f-vectors then this matrix is called a *partial f-matrix with independent rows* and it may be represented in the form

$$\tilde{R} = (f_{ij}(\{\xi_\nu^{(i)}\}))_{m,n}\quad (\xi_\nu^{(i)} \in \tilde{\sigma}_\nu^{(i)},\ i = \overline{1, m},\ \nu = \overline{1, q_i}),$$

where all parameters are independent.

If every two columns of a partial f-matrix are independent partial f-vectors then this matrix is called a *partial f-matrix with independent columns.* Such a matrix may be represented in the form

$$\tilde{R} = (f_{ij}(\{\xi_\nu^{(j)}\}))_{m,n}\quad (\xi_\nu^{(j)} \in \tilde{\sigma}_\nu^{(j)},\ \nu = \overline{1, q_j},\ j = \overline{1, m}).$$

For example,

$$\tilde{R} = \begin{pmatrix} \xi_2^{(1)} + \sin \xi_1^{(1)} & \sqrt{1+\xi_1^{(1)}\xi_2^{(1)}} \\ \xi_1^{(2)} + \cos \xi_2^{(2)} & 3\,\xi_1^{(2)}\xi_2^{(2)} \end{pmatrix},$$

where

$$\xi_1^{(1)} \in \left[\frac{1}{2}, 1\right),\quad \xi_2^{(1)} \in \{0, 1, 2\},$$

$$\xi_1^{(2)} \in \left[2, 3\frac{1}{4}\right),\quad \xi_2^{(2)} \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right],$$

is a partial square f-matrix of order 2 with independent rows.

In accordance with above definitions a *partial generalized f-automaton* is a system

$$\tilde{\mathbf{A}}_{gen} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle, \tag{6}$$
$$\tilde{r}^{(0)} = (f_1(\{\xi_\nu\}), f_2(\{\xi_\nu\}), \dots, f_m(\{\xi_\nu\})),$$
$$\tilde{R} = (f_{si,lj}(\{\xi_\nu\}))_{nm,km}, \quad \xi_\nu \in \tilde{\sigma}_\nu, \quad \nu = \overline{1, q}$$

where $f_i, f_{si,lj}$ are real single-valued functions defined on all $\xi_\nu (\in \tilde{\sigma}_\nu, \nu = \overline{1, q})$ and $\tilde{\sigma}_\nu$ $(\nu = \overline{1, q})$ are specified.

Let $\tilde{R}(X_s, Y_l)$ be a partial square f-submatrix of $\tilde{R}$ defined by

$$\tilde{R}(X_s, Y_l) = (f_{si,lj}(\{\xi_\nu\}))$$
$$i = \overline{1, m}, \quad j = \overline{1, m}.$$

Then the partial generalized mapping $\tilde{\Phi}$ (the set of mappings of $(X \times Y)^*$ into $\mathscr{R}$ induced by the partial generalized f-automaton (6) may be defined by

$$\tilde{\Phi}(X^{(t)}, Y^{(t)}) = \tilde{r}^{(0)} \prod_{i=1}^{t} \tilde{R}(X_{s_i}, Y_{l_i}) e \quad (\xi_\nu \in \tilde{\sigma}_\nu, \ \nu = \overline{1, q}).$$

5. Partial *pf*-vectors, *pf*-matrices, *pf*-automata

A partial f-vector (4) is probabilistic if and only if

$$0 \leqq f_i(\{\xi_\nu\}) \leqq 1 \quad \text{and} \quad \sum_i f_i(\{\xi_\nu\}) = 1 \quad (\xi_\nu \in \tilde{\sigma}_\nu, \ \nu = \overline{1, q}). \tag{7}$$

Such a partial f-vector is called a *partial pf-vector*. A partial f-matrix (5) is a *partial pf-matrix* if

$$0 \leqq f_{ij}(\{\xi_\nu\}) \leqq 1 \quad \text{and} \quad \sum_j f_{ij}(\{\xi_\nu\}) = 1 \quad (\xi_\nu \in \tilde{\sigma}_\nu, \ \nu = \overline{1, q}, \ i = \overline{1, m}). \tag{8}$$

For example,

$$\tilde{P} = \begin{pmatrix} \sin^2 \xi & \cos^2 \xi \\ \frac{2\xi}{\pi} & 1 - \frac{2\xi}{\pi} \end{pmatrix} \quad \left(\xi \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right) \right)$$

is a partial square pf-matrix of order 2.

It is clear that there are no partial pf-matrices with independent columns, but we shall say that a partial pf-matrix $\tilde{P}$ is a *partial pf-matrix with minimal dependent columns* if there is a partial f-matrix $\tilde{R}$ with independent columns such that for every completely specified stochastic $(m \times n)$-matrix P,

$$P \in \tilde{P} \Leftrightarrow P \in \tilde{R}$$

holds.

A *partial probabilistic f-automaton* (i. e., a *partial pf-automaton*) is a system

$$\tilde{\mathbf{A}}_{pr} = \langle X, A, Y, \tilde{p}^{(0)}, \tilde{P} \rangle,$$

where

$$\tilde{p}^{(0)} = (f_1(\{\xi_v\}),\ f_2(\{\xi_v\}),\ \dots, f_m(\{\xi_v\})),$$
$$0 \leqq f_i(\{\xi_v\}) \leqq 1, \quad \sum_i f_i(\{\xi_v\}) = 1 \tag{9}$$

and

$$\tilde{P} = (f_{si,lj}(\{\xi_v\}))_{nm,km},$$
$$0 \leqq f_{si,lj}(\{\xi_v\}) \leqq 1, \quad \sum_{lj} f_{si,lj}(\{\xi_v\}) = 1, \tag{10}$$
$$\xi_v \in \tilde{\sigma}_v, \quad v = \overline{1,q}, \quad s = \overline{1,n}, \quad i,j = \overline{1,m}, \quad l = \overline{1,k}.$$

6. Partial *l*-vectors, *l*-matrices, *l*-automata

A partial *f*-vector defined as

$$\tilde{r} = (f_1, f_2, \dots, f_m), \quad f_i = \sum_v a_i^{(v)} \xi_v,$$
$$i = \overline{1,m}, \quad \xi_v \in \tilde{\sigma}_v, \quad v = \overline{1,q}$$

where $a_i^{(v)}$ ($v=\overline{1,q}$, $i=\overline{1,m}$) are real coefficients, is called a *partial l-vector*. A partial *f*-matrix defined as

$$\tilde{R} = (f_{ij})_{m,n}, \quad f_{ij} = \sum_v a_{ij}^{(v)} \xi_v,$$
$$i = \overline{1,m}, \quad j = \overline{1,n}, \quad \xi_v \in \tilde{\sigma}_v, \quad v = \overline{1,q},$$

is called a *partial l-matrix*. For example,

$$\tilde{R} = \begin{pmatrix} \xi_1 + 2\xi_2 & \xi_2 - \xi_1 & 3\xi_2 \\ 4 & 2\xi_1 + 1 & \xi_2 \end{pmatrix},$$
$$\xi_1 \in \left(\frac{3}{4}, 2\frac{1}{2}\right], \quad \xi_2 \in \left[2, 7\frac{1}{3}\right).$$

A *partial generalized l-automaton* is a system

$$\tilde{\mathbf{A}}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle,$$

where

$$\tilde{r}^{(0)} = \left(\sum_v a_1^{(v)} \xi_v, \sum_v a_2^{(v)} \xi_v, \dots, \sum_v a_m^{(v)} \xi_v\right),$$
$$\tilde{R} = \left(\sum_v a_{si,lj}^{(v)} \xi_v\right)_{nm,km},$$
$$\xi_v \in \tilde{\sigma}_v, \quad v = \overline{1,q}.$$

Accordingly, a partial *l*-vector (*l*-matrix, generalized *l*-automaton) is a partial *pl*-vector (*pl*-matrix, *pl*-automaton) if for all its entries $f_i (f_{ij}, f_i, f_{si,lj})$ the conditions (7) ((8), (9), (10)) hold. Some examples of partial *pl*-automata may be found in [1].

7. Partial $\tilde{\sigma}$-vectors, $\tilde{\sigma}$-matrices, $\tilde{\sigma}$-automata

A partial l-vector in form

$$\tilde{r} = (\xi_1, \xi_2, \ldots, \xi_q) \quad (\xi_\nu \in \tilde{\sigma}_\nu, \ \nu = \overline{1, q})$$

where $\tilde{\sigma}_\nu$ $(\nu = \overline{1, q})$ are defined subsets of $\mathcal{R}$, is called a *partial vector with independent elements*, or more briefly, a *partial $\tilde{\sigma}$-vector* and is specified as

$$\tilde{r} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_m). \tag{11}$$

A partial l-matrix in form

$$\tilde{R} = (\xi_{ij})_{m,n} \quad (\xi_{ij} \in \tilde{\sigma}_{ij}, \ i = \overline{1, m}, \quad j = \overline{1, n}),$$

where $\tilde{\sigma}_{ij}$ $(i = \overline{1, m}, j = \overline{1, n})$ are defined subsets of $\mathcal{R}$, is called a *partial $\tilde{\sigma}$-matrix* and is specified as

$$\tilde{R} = (\tilde{\sigma}_{ij})_{m,n}, \tag{12}$$

i.e., in form of matrix whose elements are defined subsets of $\mathcal{R}$. For example,

$$\tilde{R} = \begin{Vmatrix} \left[-\frac{1}{2}, 1\right] & \{0, 2\} & \left\{\frac{1}{4}, \frac{1}{2}\right\} \cup \left(\frac{2}{3}, 1\right] \\ \frac{3}{4} & \left(-\frac{1}{2}, \frac{1}{2}\right) & \left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right\} \\ 1 & 0 & \left\{\xi \mid \xi = \frac{1}{2^t}, \quad t = 1, 2, \ldots\right\} \end{Vmatrix}.$$

It is useful to notice that each partial f-matrix with independent rows and columns may be represented in form of a partial $\tilde{\sigma}$-matrix.

Accordingly with these definitions a *partial generalized $\tilde{\sigma}$-automaton* is a system

$$\tilde{\mathbf{A}}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle,$$

$$\tilde{r}^{(0)} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_m), \quad \tilde{R} = (\tilde{\sigma}_{si,lj})_{nm,km}$$

where $\tilde{\sigma}_i$, $\tilde{\sigma}_{si,lj}$ are defined subsets of $\mathcal{R}$. If

$$\begin{gathered} \mathbf{A}_{\text{gen}} = \langle X, A, Y, r^{(0)}, R \rangle \\ r^{(0)} = (r_1, r_2, \ldots, r_m), \quad R = (r_{si,lj})_{nm,km} \end{gathered} \tag{13}$$

is a completely specified generalized automaton then

$$\mathbf{A}_{\text{gen}} \in \tilde{\mathbf{A}}_{\text{gen}} \Leftrightarrow r_i \in \tilde{\sigma}_i \quad \& \quad r_{si,lj} \in \tilde{\sigma}_{si,lj} \quad \text{for all } s, i, l, j.$$

8. Partial $p\tilde{\sigma}$-vectors, $p\tilde{\sigma}$-matrices, $p\tilde{\sigma}$-automata

A *partial p-vector with minimal dependent elements* is a subset of $\mathscr{P}^m$ defined as

$$\tilde{p} = \{p \mid p = (p_1, p_2, \ldots, p_m), p_i \in \tilde{\sigma}_i \subseteqq [0, 1], \sum_i p_i = 1\}.$$

Such a partial p-vector is called a *partial $p\tilde{\sigma}$-vector* and is specified in form

$$\tilde{p} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_m) \tag{14}$$

where $\tilde{\sigma}_i$ $(i=\overline{1, m})$ are defined subsets of [0, 1] and the condition $\sum_i p_i=1$ is omitted as obvious.

A partial p-matrix defined as

$$\tilde{P} = \{P \mid P = (p_{ij})_{m,n}, \; p_{ij} \in \tilde{\sigma}_{ij} \subseteqq [0, 1], \sum_j p_{ij} = 1, \; i = \overline{1, m}, \; j = \overline{1, n}\}$$

may be specified in form

$$\tilde{P} = (\tilde{\sigma}_{ij})_{m,n} \tag{15}$$

where $\tilde{\sigma}_{ij}$ $(i=\overline{1, m}, j=\overline{1, n})$ are defined subsets of [0, 1] and the conditions $\sum_j p_{ij}=1$ $(i=\overline{1, m})$ are omitted as obvious. Such a partial p-matrix is called a *partial $p\tilde{\sigma}$-matrix*. It is clear that each partial pf-matrix with independent rows and minimal dependent columns may be specified in form of a partial $p\tilde{\sigma}$-matrix.

We say [1] that a partial $p\tilde{\sigma}$-vector (14) is *correctly specified* if $\tilde{\sigma}_i \neq \emptyset$ $(\tilde{\sigma}_i \subseteqq [0, 1], \; i=\overline{1, m})$ and for each $p_j \in \tilde{\sigma}_j$ there exists $p_i \in \tilde{\sigma}_i$ $(i \neq j)$ such that $\sum_{s=1}^{m} p_s = 1$ $(j=\overline{1, m})$. A partial $p\tilde{\sigma}$-matrix is *correctly specified* if each of its rows is a correctly specified partial $p\tilde{\sigma}$-vector. For example,

$$\tilde{P} = \begin{pmatrix} \left\{0, \frac{1}{4}\right\} & \left[\frac{1}{4}, \frac{1}{2}\right) & \left(\frac{1}{4}, \frac{3}{4}\right] \\ 0 & \left\{\frac{1}{2}, \frac{3}{4}\right\} & \left\{\frac{1}{4}, \frac{1}{2}\right\} \end{pmatrix}$$

is a correctly specified partial $p\tilde{\sigma}$-matrix.

A *partial $p\tilde{\sigma}$-automaton* is a system

$$\tilde{\mathbf{A}}_{pr} = \langle X, A, Y, \tilde{p}^{(0)}, \tilde{P} \rangle$$

where $\tilde{p}^{(0)}=(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_m)$ is a correctly specified partial $p\tilde{\sigma}$-vector (a partial probabilistic distribution on the state set) and $\tilde{P}=(\tilde{\sigma}_{si,lj})_{nm,km}$ is a correctly specified partial transition-output $p\tilde{\sigma}$-matrix.

9. Partial i-vectors, i-matrices, i-automata

Let us propose the following notations, where $\alpha, \beta \in \{0, 1\}$:

$$\underset{\alpha}{|} = \begin{cases} (& \text{if } \alpha = 0, \\ [& \text{if } \alpha = 1 \end{cases} \qquad \overset{\beta}{|} = \begin{cases}) & \text{if } \beta = 0, \\ ] & \text{if } \beta = 1. \end{cases}$$

A partial $\tilde{\sigma}$-vector (11) is called a *partial vector with interval elements* (a *partial i-vector*) if in (11)

$$\tilde{\sigma}_i = \underset{\alpha_i}{|}\, a_i, b_i \,\overset{\beta_i}{|} \quad (i = \overline{1, m})$$

where $\alpha_i, \beta_i \in \{0, 1\}$, $a_i, b_i \in \mathscr{R}$, $a_i < b_i$ if $\alpha_i \beta_i = 0$, $a_i \leqq b_i$ if $\alpha_i \beta_i = 1$. Thus a partial i-vector is a partial $\tilde{\sigma}$-vector such that each of its elements is an interval (closed or unclosed) .

Accordingly, a *partial i-matrix* is a partial $\tilde{\sigma}$-matrix (12) such that

$$\tilde{\sigma}_{ij} = \underset{\alpha_{ij}}{|}\, a_{ij}, b_{ij} \,\overset{\beta_{ij}}{|} \quad (i = \overline{1, m},\ j = \overline{1, n})$$

where $\alpha_{ij}, \beta_{ij} \in \{0, 1\}$, $a_{ij}, \beta_{ij} \in \mathscr{R}$, $a_{ij} < b_{ij}$ if $\alpha_{ij} \beta_{ij} = 0$, $a_{ij} \leqq b_{ij}$ if $\alpha_{ij} \beta_{ij} = 1$. For example,

$$\tilde{R} = \begin{pmatrix} \left[-\frac{1}{2}, 1\right] & \left(\frac{2}{3}, 1\right] \\ \left(-\frac{1}{2}, \frac{1}{2}\right) & \left[\frac{1}{8}, \infty\right) \end{pmatrix}.$$

A *partial generalized i-automaton* is a system

$$\tilde{\mathbf{A}}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{\tilde{R}} \rangle, \tag{16}$$

$$\tilde{r}^{(0)} = \left(\underset{\alpha_1}{|}\, a_1, b_1 \,\overset{\beta_1}{|}, \underset{\alpha_2}{|}\, a_2, b_2 \,\overset{\beta_2}{|}, \ldots, \underset{\alpha_m}{|}\, a_m, b_m \,\overset{\beta_m}{|}\right),$$

$$\tilde{\tilde{R}} = \left(\underset{\alpha_{si,lj}}{|}\, a_{si,lj}, b_{si,lj} \,\overset{\beta_{si,lj}}{|}\right)_{nm,km}.$$

A partial generalized i-automaton (16) defines a set of completely specified generalized automata such that

$$\mathbf{A}_{\text{gen}} \in \tilde{\mathbf{A}}_{\text{gen}} \Leftrightarrow r_i \in \underset{\alpha_i}{|}\, a_i, b_i \,\overset{\beta_i}{|} \ \&\ r_{si,lj} \in \underset{\alpha_{si,lj}}{|}\, a_{si,lj}, b_{si,lj} \,\overset{\beta_{si,lj}}{|} \text{ for all } s, i, l, j$$

where $\mathbf{A}_{\text{gen}}$ is defined by (13).

10. Partial *pi*-vectors, *pi*-matrices, *pi*-automata

A *partial pi-vector* (a partial probabilistic vector with interval elements) is a partial $p\tilde{\sigma}$-vector (14) such that

$$\tilde{\sigma}_i = \underset{\alpha_i}{|}\; a_i, b_i \overset{\beta_i}{|} \subseteqq [0, 1] \quad (i = \overline{1, m}).$$

A *partial pi-matrix* is a partial $p\tilde{\sigma}$-matrix (15) such that

$$\tilde{\sigma}_{ij} = \underset{\alpha_{ij}}{|}\; a_{ij}, b_{ij} \overset{\beta_{ij}}{|} \subseteqq [0, 1] \quad (i = \overline{1, m},\ j = \overline{1, n}).$$

For example,

$$\tilde{P} = \begin{pmatrix} [0;\ 0{,}3) & [0{,}2;\ 0{,}4] & (0{,}3;\ 0{,}8] \\ (0{,}1;\ 0{,}2] & (0{,}3;\ 0{,}5] & [0{,}3;\ 0{,}6) \\ [0{,}2;\ 0{,}3] & [0{,}5;\ 0{,}6] & 0{,}2 \end{pmatrix}$$

is a correctly specified partial square *pi*-matrix of order 3.

A *partial pi-automaton* is a system

$$\tilde{\mathbf{A}}_{pr} = \langle X, A, Y, \tilde{p}^{(0)}, \tilde{P} \rangle$$

where $\tilde{p}^{(0)}$ is a correctly specified partial m-dimensional *pi*-vector and $\tilde{P}$ is a correctly specified partial *pi*-matrix of size $nm \times km$. In the case of closed intervals the problem of partial *pi*-automata minimization was studied in [1].

11. The conditions of correct specification

Now we are going to find the conditions which must be satisfied for correct specification of a partial *pi*-vector (*pi*-matrix, *pi*-automaton). Such conditions in case of $\alpha_i = \beta_i = 1$ $(i = \overline{1, m})$ were found in [1].

Theorem. *Let p be a partial pi-vector defined as*

$$\tilde{p} = \left(\underset{\alpha_1}{|}\; a_1, b_1 \overset{\beta_1}{|},\ \underset{\alpha_2}{|}\; a_2, b_2 \overset{\beta_2}{|},\ \ldots,\ \underset{\alpha_m}{|}\; a_m, b_m \overset{\beta_m}{|} \right) \tag{17}$$

Vhere

$$\underset{\alpha_i}{|}\; a_i, b_i \overset{\beta_i}{|} \neq \emptyset, \quad \underset{\alpha_i}{|}\; a_i, b_i \overset{\beta_i}{|} \subseteqq [0, 1], \quad i = \overline{1, m}.$$

ahen p is correctly specified if and only if the following conditions hold for $j = \overline{1, m}$:

(a)
$$a_j \geqq 1 - \sum_{i \neq j} b_i \tag{18}$$

and

$$a_j = 1 - \sum_{i \neq j} b_i \ \&\ \exists\, i : i \neq j, \quad \beta_i = 0 \Rightarrow \alpha_j = 0, \tag{19}$$

(b)
$$b_j \leqq 1 - \sum_{i \neq j} a_i \tag{20}$$

and

$$b_j = 1 - \sum_{i \neq j} a_i \;\&\; \exists\, i : i \neq j, \quad \alpha_i = 0 \Rightarrow \beta_j = 0. \tag{21}$$

Proof. For the proof of the necessity let $\tilde{p}$ be a correctly specified partial *pi*-vector. Since $\tilde{p} \neq \emptyset$ thus $\sum_i a_i \leqq 1$, $\sum_i b_i \geqq 1$, and for every j,

$$b_j \geqq 1 - \sum_{i \neq j} b_i, \quad a_j \leqq 1 - \sum_{i \neq j} a_i. \tag{22}$$

Assume now that the condition (18) does not hold for any j and $b_j - a_j > 0$,

$$a_j < 1 - \sum_{i \neq j} b_i. \tag{23}$$

Then we take

$$p_j = \frac{a_j + 1 - \sum_{i \neq j} b_i}{2}. \tag{24}$$

Since (22) and (23) hold thus $a_j < p_j < b_j$ and $p_j \in |a_j, b_j|_{\alpha_j}^{\beta_j}$. Since $\tilde{p}$ is correctly specified thus there must be a probabilistic vector $p = (p_1, p_2, \ldots, p_m) \in \tilde{p}$ such that p_j has a value (24). Then for p,

$$\sum_i p_i = 1 = \frac{a_j + 1 - \sum_{i \neq j} b_i}{2} + \sum_{i \neq j} p_i \leqq \frac{a_j + 1 - \sum_{i \neq j} b_i}{2} + \sum_{i \neq j} b_i$$

holds. This implies that

$$a_j \geqq 1 - \sum_{i \neq j} b_i$$

which contradicts our assumption (23). Therefore in the case $b_j - a_j > 0$ the condition (18) holds.

In exeptional case when $\tilde{\sigma}_j = [a_j, a_j] = a_j$, every probabilistic vector $p \in \tilde{p}$ has $p_j = a_j$ and, therefore,

$$\sum_i p_i = 1 = a_j + \sum_{i \neq j} p_i \leqq a_j + \sum_{i \neq j} b_i,$$

i.e., the condition (18) also holds.

Assume now that $a_j = 1 - \sum_{i \neq j} b_i$ (i.e., $a_j + \sum_{i \neq j} b_i = 1$) and there is an $s \neq j$ such that $\beta_s = 0$ but $\alpha_j = 1$. Since $\tilde{p}$ is correctly specified thus in this case there is a probabilistic vector $p \in \tilde{p}$ such that $p_j = a_j$, $p_s < b_s$. Then for the vector p,

$$\sum_i p_i = 1 = a_j + \sum_{i \neq j} p_i < a_j + \sum_{i \neq j} b_i$$

holds. But this contradicts our assumption. Therefore $\alpha_j = 0$ and the condition (19) holds. This ends the proof of the necessity of the conditions (a).

The necessity of condition (b) can be shown similarly.

Conversely, assume that conditions (a) and (b) hold for $\tilde{p}$. We prove that $\tilde{p}$ is correctly specified. Let us take any j and any $p_j \in |a_j, b_j|_{\alpha_j}^{\beta_j}$. It follows from (18) and (20) that

$$1 - p_j \geqq 1 - b_j \geqq \sum_{i \neq j} a_i, \quad 1 - p_j \leqq 1 - a_j \leqq \sum_{i \neq j} b_i. \tag{25}$$

We take for $i \neq j$ the following elements of a vector $p=(p_1, p_2, \ldots, p_m)$

$$p_i = a_i + \frac{1-p_j-\sum_{i\neq j} a_i}{\sum_{i\neq j}(b_i-a_i)}(b_i-a_i) \quad (i \neq j), \tag{26}$$

where $\sum_{i\neq j}(b_i-a_i)>0$ (if $\sum_{i\neq j}(b_i-a_i)=0$ then $a_i=b_i$ $(i=\overline{1,m})$ and $\tilde{p}$ is a completely specified probabilistic vector). Then for the vector p,

$$\sum_i p_i = p_j + \sum_{i\neq j}\left(a_i + \frac{1-p_j-\sum_{i\neq j} a_i}{\sum_{i\neq j}(b_i-a_i)}(b_i-a_i)\right) = 1,$$

i.e., p is a probabilistic vector. Now we shall prove that $p\in\tilde{p}$.

From (25) and (26) we have that $p_i \geqq a_i$ $(i\neq j)$ and for any $i\neq j$,

$$p_i = a_i \Leftrightarrow b_i = a_i \vee 1-p_j = \sum_{i\neq j} a_i.$$

If $b_i=a_i$ then $\tilde{\sigma}_i=[a_i, a_i]=a_i$. If $1-p_j=\sum_{i\neq j} a_i$ then in accordance with (25), $p_j=b_j$ $\beta_j=1$, $1-b_j=\sum_{i\neq j} a_i$ and it follows from (21) that $\alpha_i=1$ $(i\neq j)$. Thus if $p_i=a_i$ for any $i\neq j$ then $\alpha_i=1$ and $p_i\in\tilde{\sigma}_i=[a_i, b_i\overset{\beta_i}{|}$. Moreover, it follows from (25) that

$$1-p_j-\sum_{i\neq j} a_i \leqq \sum_{i\neq j}(b_i-a_i).$$

Therefore, $p_i \leqq b_i$ $(i\neq j)$, and for any $i\neq j$,

$$p_i = b_i \Leftrightarrow b_i = a_i \quad \vee \quad 1-p_j-\sum_{i\neq j} a_i = \sum_{i\neq j}(b_i-a_i).$$

If $b_i=a_i$ then $\tilde{\sigma}_i=[a_i, a_i]=a_i$. If $1-p_j-\sum_{i\neq j} a_i=\sum_{i\neq j}(b_i-a_i)$ then $1-p_j=\sum_{i\neq j} b_i$ and, in accordance with (25), $p_j=a_j$, $\alpha_j=1$. In this case the condition (19) implies $\beta_i=1$ $(i\neq j)$. Thus if $p_i=b_i$ for any $i\neq j$ then $\beta_i=1$ and $p_i\in\tilde{\sigma}_i=\underset{\alpha_i}{|}a_i, b_i]$. Finally, if for any $i\neq j$, $a_i<p_i<b_i$ then $p_i\in\underset{\alpha_i}{|}a_i, b_i\overset{\beta_i}{|}$. Thus, we proved that the constracted vector p is probabilistic and $p\in\tilde{p}$. Therefore, $\tilde{p}$ is correctly specified. This completes the proof of the Theorem.

12. Partial *b*-vectors, *b*-matrices, *b*-automata

A partial f-vector

$$\tilde{r} = (f_1(\{\xi_\nu\}), f_2(\{\xi_\nu\}), \ldots, f_m(\{\xi_\nu\})) \quad (\xi_\nu\in\{0, 1\}, \quad \nu=\overline{1,q}), \tag{27}$$

where $f_i(\{\xi_\nu\})$ $(i=\overline{1,m})$ are boolean (logical) functions, is called a *partial boolean vector* (a *partial b-vector*). A partial f-matrix

$$\tilde{R} = (f_{ij}(\{\xi_\nu\}))_{m,n} \quad (\xi_\nu\in\{0, 1\}, \ \nu=\overline{1,q}) \tag{28}$$

where $f_{ij}(\{\xi_\nu\})$ $(i=\overline{1,m},\ j=\overline{1,n})$ are boolean functions, is a *partial b-matrix*. For b-vectors and b-matrices the domain of every parameter is $\{0, 1\}$, therefore it may be omitted. For example,

$$\tilde{R} = \begin{pmatrix} \xi_1 \vee \xi_2 & \xi_1\bar{\xi}_2 & \bar{\xi}_1\xi_2 \\ \xi_2\xi_3 & \bar{\xi}_2 \vee \xi_3 & \xi_1\bar{\xi}_2 \vee \xi_3 \end{pmatrix}.$$

If a partial b-vector (b-matrix) is a partial $\tilde{\sigma}$-vector ($\tilde{\sigma}$-matrix) then its elements may be 0, 1 or $\{0, 1\}$. In this case it is convenient to replace $\{0, 1\}$ by "—". For example,

$$\tilde{R} = \begin{pmatrix} 0 & - & 1 \\ - & 0 & - \\ 1 & 1 & - \end{pmatrix}.$$

A *partial generalized b-automaton* is a system

$$\tilde{\mathbf{A}}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle,$$

$$\tilde{r}^{(0)} = (f_1, f_2, \ldots, f_m), \quad \tilde{R} = (f_{ij})_{m,n}$$

where $f_i = f_i(\{\xi_\nu\}), f_{ij} = f_{ij}(\{\xi_\nu\})$ are boolean functions of the parameters $\xi_1, \xi_2, \ldots, \xi_q$ $(\xi_\nu \in \{0, 1\},\ \nu = \overline{1,q})$.

13. Partial d-vectors, d-matrices, d-automata

If a partial b-vector (27) is also a partial p-vector then $\tilde{p} \subseteq \mathscr{D}^m$ and

$$f_i f_j \equiv 0 \quad (i \neq j), \quad \bigvee_i f_i \equiv 1. \tag{29}$$

Such a partial vector is called a *partial d-vector*. Thus if a partial b-matrix (28) is also a partial p-matrix then it is of form

$$\tilde{D} = (f_{ij})_{m,n}, \quad f_{ij} f_{il} \equiv 0 \quad (j \neq l), \quad \bigvee_j f_{ij} \equiv 1 \quad (i = \overline{1,m}) \tag{30}$$

and $\tilde{D} \subseteq \mathscr{D}^{m,n}$. Such a partial matrix is called a *partial d-matrix*. It is useful to notice that any subset of $\mathscr{D}^m (\mathscr{D}^{m,n})$ may be specified as a partial d-vector (d-matrix). For example,

$$\tilde{D} = \begin{pmatrix} \xi_1 \vee \xi_2 & \bar{\xi}_1\bar{\xi}_2 & 0 \\ \bar{\xi}_1 & \xi_1\bar{\xi}_2 & \xi_1\xi_2 \\ \bar{\xi}_2\xi_3 & \xi_2 \vee \bar{\xi}_3 & 0 \end{pmatrix}$$

is a partial square d-matrix of order 3.

If a partial d-matrix is a partial $p\tilde{\sigma}$-matrix then $\{0, 1\}$ may also be replaced by "—", but it is necessary to keep in mind the conditions (30).

A *partial deterministic automaton* (a *partial d-automaton*) is a system

$$\tilde{\mathbf{A}}_{\text{det}} = \langle X, A, Y, \tilde{d}^{(0)}, \tilde{D} \rangle$$

where $\tilde{d}^{(0)} = (f_1, f_2, \ldots, f_m)$ is a partial d-vector and $\tilde{D} = (f_{si,lj})_{nm,km}$ is a partial d-matrix.

14. Automata programming

Above the most general definitions of incompletely specified finite automata were proposed and some special classes of such automata were introduced. For these automata all classical problems of the automata theory may be formulated. Some of such problems were investigated, for example in [1]—[4] for certain partial p-automata, partial pi-automata and partial d-automata. But a partial automaton is a more interesting object for investigation than a completely specified automaton and there are many special important problems in its theory. One class of such problems which we shall call "the problems of automata programming" may be formulated in the following way.

Let $\tilde{\mathbf{A}}_{\text{gen}}^{(1)}, \tilde{\mathbf{A}}_{\text{gen}}^{(2)}, \dots, \tilde{\mathbf{A}}_{\text{gen}}^{(q)}$ be partial automata (for example generalized) and Ψ be a mapping

$$\Psi: \tilde{\mathbf{A}}_{\text{gen}}^{(1)} \times \tilde{\mathbf{A}}_{\text{gen}}^{(2)} \times \dots \times \tilde{\mathbf{A}}_{\text{gen}}^{(q)} \to \mathscr{R}.$$

It is necessary to find partial automata $\tilde{\mathbf{A}}_{\text{gen}}^{(1)\prime}, \tilde{\mathbf{A}}_{\text{gen}}^{(2)\prime}, \dots, \tilde{\mathbf{A}}_{\text{gen}}^{(q)\prime}$ such that

$$\tilde{\mathbf{A}}_{\text{gen}}^{(i)\prime} \subseteq \tilde{\mathbf{A}}_{\text{gen}}^{(i)} \quad (i = \overline{1, q})$$

and

$$\mathbf{A}_{\text{gen}}^{i} \in \tilde{\mathbf{A}}_{\text{gen}}^{(i)\prime} \quad (i = \overline{1, q}) \Leftrightarrow \Psi(\mathbf{A}_{\text{gen}}^{(1)}, \mathbf{A}_{\text{gen}}^{(2)}, \dots, \mathbf{A}_{\text{gen}}^{(q)}) = \Psi_{\max}$$

where

$$\Psi_{\max} = \max_{\substack{\mathbf{A}_{\text{gen}}^{(i)} \in \tilde{\mathbf{A}}_{\text{gen}}^{(i)} \\ i = \overline{1, q}}} \Psi(\mathbf{A}_{\text{gen}}^{(1)}, \mathbf{A}_{\text{gen}}^{(2)}, \dots, \mathbf{A}_{\text{gen}}^{(q)}).$$

Such problems, for example, are very important for optimization of automata or some systems and processes which may be described in terms of automata. One such problem concerned with automata reliability was solved in [1].

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