The solvability of the equivalence problem for deterministic frontier-to-root tree transducers

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1. Introduction

In this paper we deal with effective solvability of the equivalence of frontier-to-root tree transducers. T. V. Griffits has shown in [2] that the equivalence problem is unsolvable for λ-free nondeterministic generalized sequential machines which are special frontier-to-root tree transducers, so the equivalence of the nondeterministic frontier-to-root transducers is unsolvable, too. Then in a natural way one can raise the question whether the equivalence of deterministic frontier-to-root tree transducers is solvable. We show the answer is in the affirmative. The proof is based on the proof of the solvability of equivalence problem for λ-free deterministic generalized sequential machines given by F. Gécseg (unpublished result). M. Steinby has called the author's attention to the fact that this result can be employed for minimalization of deterministic frontier-to-root tree transducers. In section 4 we give an algorithm for the minimalization.

A systematic summary of further results concerning frontier-to-root and root-to-frontier tree transducers can be found in [1], where they are called bottom-up and top-down tree transducers, respectively.

2. Notions and notations

Let $X=\{x_1, \dots, x_n, \dots\}$, $Y=\{y_1, \dots, y_m, \dots\}$ and $Z=\{z_1, \dots, z_k, \dots\}$ be countable sets of variables kept fix in this paper. Denote by X_n the subset $\{x_1, \dots, x_n\}$ of X. Consider a nonvoid set F and a mapping ν of F into the set of all nonnegative integers. The pair (F, ν) is called a type. Then the set $T_F(X)$ of *polynomial symbols* over X of type F is defined in the following way:

(a) for each x $(x\in X)$, $x\in T_F(X)$,

(b) if $f\in F$, $\nu(f)=k(\geqq 0)$, and $p_1, \dots, p_k\in T_F(X)$ then $f(p_1, \dots, p_k)\in T_F(X)$,

(c) the polynomial symbols over X of type F are those and only those which we get from (a) and (b) in finite number of steps.

Now we define the *depth* $d(p)$ of $p\in T_F(X)$ as follows:

(a) if $p=x$ $(x\in X)$ then $d(p)=0$,

(b) if $p=f$ ($f\in F$) and $\nu(f)=0$ then $d(p)=0$,

(c) if $p=f(p_1, \ldots, p_k)$ $(\nu(f)=k>0)$ then $d(p)=\max(d(p_i)|i=1, \ldots, k)+1$.

In the literature elements of $T_F(X)$ are called trees, or, in a more detailed form, F-trees.

Next we define the *frontier* $\mathrm{fr}(p)$ of a tree $p\in T_F(X)$ in the following way:

(a) if $p=x$ $(x\in X)$ then $\mathrm{fr}(p)=x$,

(b) if $p=f(p_1, \ldots, p_k)$ $(\nu(f)=k)$ then $\mathrm{fr}(p)=\mathrm{fr}(p_1)\ldots\mathrm{fr}(p_k)$.

We notice that if $p=f$ and $\nu(f)=0$, then $\mathrm{fr}(p)=\lambda$, where λ denotes the empty word over X.

We can define the set $\mathrm{sub}(p)$ of *subtrees* of $p\in T_F(X)$ as follows:

(a) if $p=x$ $(x\in X)$ then $\mathrm{sub}(p)=\{x\}$,

(b) if $p=f(p_1, \ldots, p_k)$ $(\nu(f)=k)$ then
$\mathrm{sub}(p)=\bigcup(\mathrm{sub}(p_i)|i=1, \ldots, k)\cup\{p\}$.

Let $\overline{\mathrm{sub}}(p)=\mathrm{sub}(p)\setminus\{p\}$ be the set of proper subtrees of a tree $p\in T_F(X)$.

Next we define the concept of a substitution. Let $p\in T_F(X_n)$ be an arbitrary tree and $T_1, \ldots, T_n\subseteq T_F(X_n)$. Then $p[T_1\to x_1, \ldots, T_n\to x_n]$ is the set of trees obtained by replacing every occurrence of $x_1, \ldots, x_n$ by a tree in $T_1, \ldots, T_n$, respectively. Formally,

(a) if $p=x_i$ $(x_i\in X_n)$ then $p[T_1\to x_1, \ldots, T_n\to x_n]=T_i$,

(b) if $p=f(p_1, \ldots, p_k)$ $(\nu(f)=k)$ then $p[T_1\to x_1, \ldots, T_n\to x_n]=$
$=\{f(\bar{p}_1, \ldots, \bar{p}_k)|\bar{p}_i\in p_i[T_1\to x_1, \ldots, T_n\to x_n],\ i=1, \ldots, k\}$.

Let $T_1, T_2\subseteq T_F(X_n)$ be arbitrary subsets and $x_i\in X_n$. Then the *x_i-product* $T_1\cdot x_i T_2$ of T_1 by T_2 is the set of trees which can be obtained by replacing every occurrence of x_i in some tree from T_2 by a tree in T_1.

Let $T_1^{0,x_i}=\{x_i\}$ and for every $k>0$

$$T_1^{k,x_i}=T_1^{k-1,x_i}\cdot x_i T_1.$$

Obviously,
$T_1\cdot x_i T_2=\{p[\{x_1\}\to x_1, \ldots, \{x_{i-1}\}\to x_{i-1}, T_1\to x_i, \{x_{i+1}\}\to x_{i+1}, \ldots, \{x_n\}\to x_n]|p\in T_2\}$.

Let us note that a singleton will also be denoted by its element.

Let (F, ν) and (G, μ) be fixed finite types. Moreover, let A be a finite set of states.

A *frontier-to-root rewriting (*FR*) rule* is determined by a triple of the following two forms:

(a) (x, a, q), where $x\in X$, $a\in A$ and $q\in T_G(Y)$,

(b) $(f((a_1, z_1), \ldots, (a_k, z_k)), a, q)$, where $f\in F$, $\nu(f)=k$,
$(a_i, z_i)\in A\times\{z_i\}$ $(i=1, \ldots, k)$, $a\in A$ and $q\in T_G(Y\cup Z_k)$.

In the sequel we write the FR rules in the form $x\to aq$ and $f(a_1z_1, \ldots, a_kz_k)\to aq$, respectively.

A *root-to-frontier rewriting* (RF) *rule* is given by a triple of the following forms:

(a) (a, x, q) where $a\in A$, $x\in X$ and $q\in T_G(Y)$,

(b) $(a, f(z_1, \ldots, z_k), q)$ where $a\in A$, $f\in F$, $\nu(f)=k$ and $q\in T_G(Y\cup A\times Z_k)$.

Further on we write the RF rules in the form $ax\to q$ and $af(z_1, \ldots, z_k)\to q$, respectively.

By a *frontier-to-root tree* (FRT) *transducer* we mean a system $\mathfrak{A}=(F, A, G, A', \Sigma)$, where A' is a subset of A called the set of final states and Σ is a finite set of FR rules. Since Σ is finite thus there is a number n such that the set of symbols x, for which there exists a rule in Σ with left hand side x, is a subset of X_n. Similarly, there exists a number m such that right hand sides of rules from Σ get into $A \times T_G(Y_m \cup Z)$. Then we can restrict ourselves to X_n and Y_m.

For each $a \in A$ and $p \in T_F(X_n)$, the set of all *a-translations of p*, denoted by $\mathfrak{A}_a(p)$, is defined as follows:

(a) if $p = x_i$ $(1 \leqq i \leqq n)$, then $\mathfrak{A}_a(p) = \{q | x_i \to aq \in \Sigma\}$,

(b) if $p = f(p_1, \ldots, p_k)$ $(v(f) = k)$ then

$\mathfrak{A}_a(p) = \{q | f(a_1 z_1, \ldots, a_k z_k) \to a\bar{q} \in \Sigma,\ q \in \bar{q}\,[\mathfrak{A}_{a_1}(p_1) \to z_1, \ldots, \mathfrak{A}_{a_k}(p_k) \to z_k]\}$.

An FRT transducer $\mathfrak{A}$ is *deterministic* (DFRT transducer) if

(a) for all $x_i \in X_n$, there is at most one rule with left hand side x_i,

(b) for all $f \in F$ and $a_1, \ldots, a_k \in A$, there is at most one rule with left hand side $f(a_1 z_1, \ldots, a_k z_k)$.

By a *root-to-frontier tree* (RFT) *transducer* we mean a system $\mathfrak{A} = (F, A, G, A', \Sigma)$, where A' $(\subseteq A)$ is the set of initial states and Σ is a finite set of RF rules. Similarly, in this case we can be restricted to X_n and Y_m for some n and m.

For each $a \in A$ and $p \in T_F(X_n)$, the set of all *a-translations of p*, denoted by $\mathfrak{A}_a(p)$, is defined as follows:

(a) if $p = x_i$ $(1 \leqq i \leqq n)$ then $\mathfrak{A}_a(p) = \{q | ax_i \to q \in \Sigma\}$,

(b) if $p = f(p_1, \ldots, p_k)$ $(v(f) = k)$ then

$\mathfrak{A}_a(p) = \{q | af(z_1, \ldots, z_k) \to \bar{q}(\ldots, \bar{a}z_i, \ldots) \in \Sigma,\ q \in \bar{q}[\ldots, \mathfrak{A}_{\bar{a}}(p_i) \to \bar{a}z_i, \ldots]\}$.

An RFT transducer $\mathfrak{A}$ is *deterministic* (DRFT transducer) if

(a) for all $x_i \in X_n$ and $a \in A$, there is at most one rule with left hand side ax_i,

(b) for all $f \in F$ $(v(f) = k)$ and $a \in A$, there is at most one rule with left hand side $af(z_1, \ldots, z_k)$,

(c) A' is a singleton.

Let $\mathfrak{A} = (F, A, G, A', \Sigma)$ be a FRT (RFT) transducer and $p \in T_F(X_n)$. The *translations of p induced by* $\mathfrak{A}$, denoted by $\mathfrak{A}(p)$, is the set $\cup(\mathfrak{A}_a(p) | a \in A')$.

We define the *transformation induced by* $\mathfrak{A}$ to be the relation $\{(p, q) | p \in T_F(X_n), q \in \mathfrak{A}(p)\}$ from $T_F(X_n)$ into $T_G(Y_m)$.

If $\mathfrak{A}$ is a deterministic FRT (RFT) transducer, then for each $p \in T_F(X_n)$ at most one element is in $\mathfrak{A}(p)$. Therefore, the transformation induced by $\mathfrak{A}$ is a (partial) mapping from $T_F(X_n)$ into $T_G(Y_m)$, and it is denoted by $\mathfrak{A}$, too. This mapping is called the *mapping induced by* $\mathfrak{A}$.

Let $\mathfrak{A} = (F, A, G, A'\ \Sigma_A)$ and $\mathfrak{B} = (F, B, G, B', \Sigma_B)$ be FRT (RFT) transducers. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent* if and only if $\mathfrak{A}$ and $\mathfrak{B}$ induce the same transformation. The FRT (RFT) transducer $\mathfrak{A}$ is *minimal* if and only if for all FRT (RFT) transducer $\mathfrak{C} = (F, C, G, C', \Sigma_C)$ equivalent to $\mathfrak{A}$, $|A| \leqq |C|$ holds.

We say that $\mathfrak{A}$ is a *minimal transducer belonging to* $\mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent and $\mathfrak{A}$ is minimal.

3. The equivalence of deterministic frontier-to-root tree transducers

Let $\mathfrak{A} = (F, A, G, A', \Sigma_A)$ and $\mathfrak{B} = (F, B, G, B', \Sigma_B)$ be deterministic frontier-to-root tree transducers such that the mappings induced by $\mathfrak{A}$ and $\mathfrak{B}$ are from $T_F(X_n)$ into $T_G(Y_m)$. Let us construct, for the states $a \in A$ and $b \in B$, two DFRT transducers

$$\mathfrak{A}^a = (F, A, B, A', \Sigma_A \cup \{\# \to a\#\})$$

and

$$\mathfrak{B}^b = (F, B, G, B', \Sigma_B \cup \{\# \to b\#\}).$$

Then $\mathfrak{A}^a$ and $\mathfrak{B}^b$ induce mappings from $T_F(X_n \cup \{\#\})$ into $T_G(Y_m \cup \{\#\})$.

We define the *#-depth* $\bar{d}(p)$ of a tree $p \in T_F(X_n)$ in the following way:

(a) if $p = x_i$ $(1 \leqq i \leqq n)$ then $\bar{d}(p)$ is undefined,

(b) if $p = \#$ then $\bar{d}(p) = 0$,

(c) if $p = f(p_1, \ldots, p_k)$ $(\nu(f) = k)$ and $\bar{d}(p_i)$ $(i = 1, \ldots, k)$ are undefined then $\bar{d}(p)$ is undefined,

(d) if $p = f(p_1, \ldots, p_k)$ $(\nu(f) = k)$ and one of $\bar{d}(p_i)$ $(1 \leqq i \leqq k)$ is defined, then $\bar{d}(p) = \max(\bar{d}(p_i) \mid \bar{d}(p_i)$ is defined, $1 \leqq i \leqq k) + 1$.

Let T be the set of all trees $p \in T_F(X_n)$ for which both $\mathfrak{A}(p)$ and $\mathfrak{B}(p)$ are defined.

Take a tree $p \in T$ and an arbitrary subtree $\bar{p} \in \text{sub}(p)$. Let $\bar{\bar{p}} \in T_F(X_n \cup \{\#\})$ be the tree obtained by replacing a fix occurrence of $\bar{p}$ by $\#$. Obviously, $\bar{\bar{p}}$ contains exactly one symbol $\#$ on its frontier and $p = \bar{p} \cdot \bar{\bar{p}}$, where $\bar{p} \cdot \bar{\bar{p}}$ denotes the $\#$-product of $\bar{p}$ by $\bar{\bar{p}}$. Since $p \in T$, there exist exactly one state of A and B denoted respectively by $A_{\bar{p}}$ and $B_{\bar{p}}$, such that both $\mathfrak{A}_{A_{\bar{p}}}(\bar{p})$ and $\mathfrak{B}_{B_{\bar{p}}}(\bar{p})$ are defined.

The following two lemmas hold under these notations.

Lemma 1. For each $p \in T$ and $\bar{p} \in \text{sub}(p)$,

$$\mathfrak{A}(p) = \mathfrak{A}_{A_{\bar{p}}}(\bar{p}) \cdot \mathfrak{A}^{A_{\bar{p}}}(\bar{\bar{p}})$$

and

$$\mathfrak{B}(p) = \mathfrak{B}_{B_{\bar{p}}}(\bar{p}) \cdot \mathfrak{B}^{B_{\bar{p}}}(\bar{\bar{p}})$$

hold.

Proof is obvious.

Next let $|A| = M$ and $|B| = N$.

Lemma 2. Let $p \in T$ be an arbitrary tree and $\bar{p} \in \text{sub}(p)$. Then there exists a tree $t \in T_F(X_n \cup \{\#\})$ containing exactly one symbol $\#$ on its frontier such that $\bar{d}(t) < MN$, $d(t) < 2MN - 1$ and $\bar{p} \cdot t \in T$.

Proof. First we give a tree $\bar{t}$, for which $\bar{d}(\bar{t}) < MN$. Construct a sequence $t_1, \ldots, t_s, \ldots$ of trees as follows: Set $t_0 = \bar{\bar{p}}$. Then consider the sequence $q_0, \ldots, q_l$ of maximal length, for which $q_0 = t_s$, $q_l = \#$ and $q_i \in \overline{\text{sub}}(q_{i-1})$ $(i = 1, \ldots, l)$. If $l < MN$ then $\bar{d}(t_s) < MN$, and in this case let $\bar{t} = t_s$. Otherwise, we can find two indices j and k such that $0 \leqq j < k \leqq l$ and $A_{q_j} = A_{q_k}$, $B_{q_j} = B_{q_k}$. Then let t_{s+1} be the tree obtained from t_s by replacing the subtree q_j in t_s by q_k. It is clear that $\bar{d}(t_{s+1}) < \bar{d}(t_s)$. Thus, continuing this process in a finite number of steps we arrive at the desired tree $\bar{t}$. If $d(\bar{t}) < 2MN - 1$ then let $t = \bar{t}$. In the opposite case there exists a sequence $q_0, \ldots, q_l$ of subtrees of $\bar{t}$ with $l \geqq MN$, $\# \notin \text{sub}(q_0)$, $q_l \in X_n$ and $q_i \in \overline{\text{sub}}(q_{i-1})$ $(i = 1, \ldots, l)$. We construct a tree $\bar{\bar{t}}$ from $\bar{t}$ by means of the sequence $q_0, \ldots, q_l$ in the same way as $\bar{t}$ has been constructed from $\bar{\bar{p}}$. The tree $\bar{\bar{t}}$ contains less occurrences of symbols from F than $\bar{t}$ does. It follows that the procedure can be continued till the depth of the resulting tree is not less than $2MN - 1$. The constructed tree satisfies the conclusions of Lemma 2.

Notice that if the frontier of $\mathfrak{A}^{A_{\bar{p}}}(\bar{p})$ contains the symbol #, then it occurs in the frontier of $\mathfrak{A}^{A_{\bar{p}}}(t)$. Similar statement is valid for $\mathfrak{B}^{B_{\bar{p}}}(\bar{p})$ and $\mathfrak{B}^{B_{\bar{p}}}(t)$.

Lemma 3. Let $p\in T$ and $d(p)\geqq 4MN$. Then there exist trees $p_1, p_2, p_3, p_4, p_5, p_6\in \in T_F(X_n\cup\{\#\})$ such that p_2, p_3, p_4, p_5, p_6 contain exactly one symbol # in their frontiers. Moreover, $p=p_1\cdot p_2\cdot p_3\cdot p_4\cdot p_5\cdot p_6$, $d(p_i)\geqq 1$ $(i=2, 3, 4, 5)$ and $d(p_1\cdot p_2\cdot p_3\cdot p_4\cdot p_5)\leqq 4MN$. Finally, the following equations hold:

$$A_{p_1} = A_{(p_1\cdot p_2)} = A_{(p_1\cdot p_2\cdot p_3)} = A_{(p_1\cdot p_2\cdot p_3\cdot p_4)} = A_{(p_1\cdot p_2\cdot p_3\cdot p_4\cdot p_5)} = a,$$

$$B_{p_1} = B_{(p_1\cdot p_2)} = B_{(p_1\cdot p_2\cdot p_3)} = B_{(p_1\cdot p_2\cdot p_3\cdot p_4)} = B_{(p_1\cdot p_2\cdot p_3\cdot p_4\cdot p_5)} = b,$$

$$\mathfrak{A}(p) = \mathfrak{A}_a(p_1)\cdot\mathfrak{A}_a^a(p_2)\cdot\mathfrak{A}_a^a(p_3)\cdot\mathfrak{A}_a^a(p_4)\cdot\mathfrak{A}_a^a(p_5)\cdot\mathfrak{A}^a(p_6),$$

$$\mathfrak{B}(p) = \mathfrak{B}_b(p_1)\cdot\mathfrak{B}_b^b(p_2)\cdot\mathfrak{B}_b^b(p_3)\cdot\mathfrak{B}_b^b(p_4)\cdot\mathfrak{B}_b^b(p_5)\cdot\mathfrak{B}^b(p_6).$$

Proof. Let $\bar{p}$ be an arbitrary subtree of p with depth $4MN$. Then there exists a sequence $q_0, \dots, q_{4MN}$ of trees with $q_0=\bar{p}$ and $q_i\in\overline{\text{sub}}\,(q_{i-1})$ $(i=1, \dots, 4MN)$. Consider the pairs of states (A_{q_i}, B_{q_i}) $(i=0, \dots, 4MN)$. Obviously, there exist indices j_1, j_2, j_3, j_4, j_5 $(4MN\geqq j_1>j_2>j_3>j_4>j_5\geqq 0)$ having the same pairs of states.

Let $p_1=q_{j_1}$. Construct the tree p_k by replacing the subtree $q_{j_{k-1}}$ in the tree q_{j_k} by the symbol # $(k=2, 3, 4, 5)$. Finally, let p_6 be the tree obtained from p by replacing its subtree q_{j_5} by #. From the construction and Lemma 1, it is clear that the trees $p_1, p_2, p_3, p_4, p_5, p_6$ constructed in this way satisfy the conditions of Lemma 3.

Let $L=\max\big(d(\mathfrak{A}(p)),\ d(\mathfrak{B}(p))\big|p\in T,\ d(p)\leqq 6MN\big)$ and $K=4(L+2)MN$.

Lemma 4. Take a tree $p\in T$. Moreover, let $p_1, p_2, p_3, p_4, p_5, p_6\in T_F(X_n\cup\{\#\})$ be trees and $a\in A$ and $b\in B$ states satisfying the conditions of Lemma 3. If $\mathfrak{A}(p)\neq\mathfrak{B}(p)$ and $\bar{d}(\mathfrak{A}^a(p_4\cdot p_5\cdot p_6))$ is undefined, then there is a tree $\bar{p}\in T$, for which $d(\bar{p})<K$ and $\mathfrak{A}(\bar{p})\neq\mathfrak{B}(\bar{p})$.

Proof. Let S be the set of trees with minimal depth satisfying the conditions of Lemma 4. Let $p(\in S)$ be a tree which has minimal number of occurrences of symbols from F among all trees in S. Assume that $d(p)\geqq K$.

The #-depth of the tree $\mathfrak{B}^b(p_3\cdot p_4\cdot p_5\cdot p_6)$ is defined and $\bar{d}(\mathfrak{B}_b^b(p_3))>0$, for otherwise

$$\mathfrak{A}(p_1\cdot p_3\cdot p_4\cdot p_5\cdot p_6) = \mathfrak{A}(p) \neq \mathfrak{B}(p) = \mathfrak{B}(p_1\cdot p_3\cdot p_4\cdot p_5\cdot p_6)$$

or

$$\mathfrak{A}(p_1\cdot p_2\cdot p_4\cdot p_5\cdot p_6) = \mathfrak{A}(p) \neq \mathfrak{B}(p) = \mathfrak{B}(p_1\cdot p_2\cdot p_4\cdot p_5\cdot p_6)$$

holds, which contradicts the minimality of p. Next we define a tree t, for which

$$d(t)<3MN-1 \text{ and } \bar{d}(t)<2MN-1.$$

First we consider the sequence $q_0, \dots, q_l$ of subtrees with maximal length for which $q_0=p_4\cdot p_5\cdot p_6$, $q_l=\#$ and $q_i\in\overline{\text{sub}}\,(q_{i-1})$ $(i=1, \dots, l)$. Then for each q_i there is exactly one state $a_i\in A$ such that $\mathfrak{A}_{a_i}^a(q_i)$ is defined. Let i be the maximal index, for which $\bar{d}(\mathfrak{A}_{a_i}^a(q_i))$ is undefined. Since $\mathfrak{A}_{a_0}^a(q_0)=\mathfrak{A}^a(p_4\cdot p_5\cdot p_6)$, $\bar{d}(\mathfrak{A}^a(p_4\cdot p_5\cdot p_6))$ is undefined and $\mathfrak{A}_{a_l}^a(q_l)=\#$ thus $0\leqq i\leqq l-1$ holds. Now we consider the tree t_2 given by Lemma 2 for the tree p and the subtree $p_1\cdot p_2\cdot p_3\cdot q_i$. Let $q_i=f(r_1, \dots, r_k)$

$(\nu(f)=k)$. Then there exists an index j $(1 \leqq j \leqq k)$ such that $r_j = q_{i+1}$. Let us construct the tree $\bar{r}_j$ from r_j in exactly that way as the tree $\bar{t}$ has been constructed from the tree $\bar{p}$ in the proof of Lemma 2.

Furthermore, let t_1 be the tree arising from the tree $f(r_1, \ldots, r_{j-1}, \bar{r}_j, r_{j+1}, \ldots, r_k)$ in the same way as the tree t has been obtained from the tree $\bar{t}$ in Lemma 2. Let $t = t_1 \cdot t_2$.

Consider the tree $q = p_1 \cdot p_3^{L+1} \cdot t$, where $p_3^{L+1} = \{p_3\}^{L+1, \#}$. It is clear that $q \in T$, and

$$\mathfrak{A}(q) = \mathfrak{A}^a(t)$$

and

$$\mathfrak{B}(q) = \mathfrak{B}_b(p_1) \cdot (\mathfrak{B}_b^b(p_3))^{L+1} \cdot \mathfrak{B}^b(t)$$

hold by Lemma 1. Since $d(\mathfrak{A}(q)) \leqq L$ and $d(\mathfrak{B}(q)) > L$ thus $\mathfrak{A}(q) \neq \mathfrak{B}(q)$. But $d(q) < K$, which contradicts the minimality of p.

Lemma 5. Let $p \in T$ be a tree for which $\mathfrak{A}(p) \neq \mathfrak{B}(p)$. Assume that there exist trees $p_1', p_2', p_3', p_4', p_5', p_6' \in T_F(X_n \cup \{\#\})$ and states $a \in A$ and $b \in B$ satisfying the conditions of Lemma 3. If $\bar{d}(\mathfrak{A}^a(p_4' \cdot p_5' \cdot p_6'))$ is defined, then there exists a tree $\bar{p} \in T$ such that $d(\bar{p}) < K$ and $\mathfrak{A}(\bar{p}) \neq \mathfrak{B}(\bar{p})$.

Proof. Let S be the set of trees with minimal depth satisfying the conditions of Lemma 5. Let $p (\in S)$ be a tree which has minimal number of occurrences of symbols from F among all trees in S. Assume that $d(p) \geqq K$.

Let t be the tree given by Lemma 2 to the tree p and the subtree $p_1' \cdot p_2' \cdot p_3' \cdot p_4' \cdot p_5'$. We introduce the following notations:

$$p_1 = p_1' \cdot p_2' \cdot p_3', \quad p_2 = p_4', \quad p_3 = p_5', \quad p_4 = p_6'$$

$$\mathfrak{A}_a(p_1) = q_1, \quad \mathfrak{B}_b(p_1) = r_1,$$

$$\mathfrak{A}_a^a(p_2) = q_2, \quad \mathfrak{B}_b^b(p_2) = r_2,$$

$$\mathfrak{A}_a^a(p_3) = q_3, \quad \mathfrak{B}_b^b(p_3) = r_3,$$

$$\mathfrak{A}^a(p_4) = q_4, \quad \mathfrak{B}^b(p_4) = r_4,$$

$$\mathfrak{A}^a(t) = \bar{q}_4, \quad \mathfrak{B}^b(t) = \bar{r}_4.$$

First let us illustrate the idea of the proof in a special case. Assume that $\nu(f)=1$ and $\mu(g)=1$ for all $f \in F$ and $g \in G$. Then the DFRT transducers $\mathfrak{A}$ and $\mathfrak{B}$ may be considered as deterministic generalized sequential machines.

In Figure 1 we indicate the trees p, $\mathfrak{A}(p)$, $\mathfrak{B}(p)$. Now let us consider the trees $=p_1 \cdot p_2^l \cdot t$ and $\mathfrak{A}(t_l)$, $\mathfrak{B}(t_l)$ $(l=1, \ldots, L+1)$ (see, Figure 2).

Since $\mathfrak{A}(t_l) = \mathfrak{B}(t_l)$ $(l=1, \ldots, L+1)$, thus Figure 2 shows that the same tree is constructed in two different ways. As it appears from Figure 2, and it can be readily verified, too, $q_2 = \bar{r}_2 \cdot \bar{q}_2$ and $r_2 = \bar{q}_2 \cdot \bar{r}_2$. The idea behind the proof of Lemma 5 is similar, but more involved.

The $\#$-depth of $\mathfrak{B}^b(p_4' \cdot p_5' \cdot p_6')$ is defined, for otherwise, by Lemma 4, there exists a tree $\bar{p} \in T$, for which $d(\bar{p}) < K$ and $\mathfrak{A}(\bar{p}) \neq \mathfrak{B}(\bar{p})$ hold contradicting the minimality of p. Since both $\bar{d}(\mathfrak{A}^a(p_2 \cdot p_3 \cdot p_4))$ and $\bar{d}(\mathfrak{B}^b(p_2 \cdot p_3 \cdot p_4))$ are defined thus all the trees q_2, q_3, q_4 and r_2, r_3, r_4 contain the symbol $\#$ in their frontiers. Moreover, by the note following Lemma 2, the frontiers of the trees $\bar{q}_4$ and $\bar{r}_4$ contain it, too.

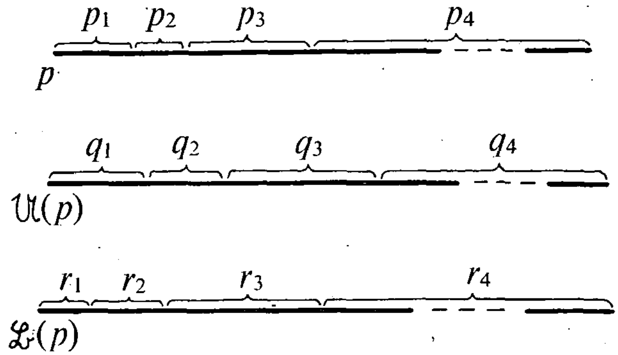

Fig. 1

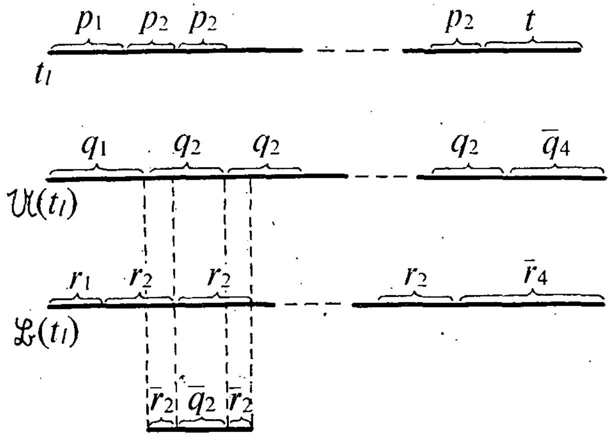

Fig. 2

Assume that $\bar{d}(q_2)=\bar{d}(r_2)=0$. Then

$$\mathfrak{A}(p) = q_1 \cdot q_3 \cdot q_4 = \mathfrak{A}(p_1 \cdot p_3 \cdot p_4)$$

and

$$\mathfrak{B}(p) = r_1 \cdot r_3 \cdot r_4 = \mathfrak{B}(p_1 \cdot p_3 \cdot p_4).$$

i.e. $\mathfrak{A}(p_1 \cdot p_3 \cdot p_4) \neq \mathfrak{B}(p_1 \cdot p_3 \cdot p_4)$, which is a contradiction.

In the same way we obtain that if $\bar{d}(q_3)=\bar{d}(r_3)=0$, then $\mathfrak{A}(p_1 \cdot p_2 \cdot p_4) \neq \mathfrak{B}(p_1 \cdot p_2 \cdot p_4)$, which is impossible.

Now we consider the trees

$$t_l = p_1 \cdot p_2^l \cdot t \quad \text{and} \quad s_l = p_1 \cdot p_3^l \cdot t \quad (l = 0, \ldots, L+1).$$

By Lemma 1, it follows that

$$\mathfrak{A}(t_l) = q_1 \cdot q_2^l \cdot \bar{q}_4, \quad \mathfrak{B}(t_l) = r_1 \cdot r_2^l \cdot \bar{r}_4,$$

$$\mathfrak{A}(s_l) = q_1 \cdot q_3^l \cdot \bar{q}_4, \quad \mathfrak{B}(s_l) = r_1 \cdot r_3^l \cdot \bar{r}_4 \quad (l = 0, \ldots, L+1).$$

Since $d(t_l)$, $d(s_l)<K$ thus $\mathfrak{A}(t_l)=\mathfrak{B}(t_l)$ and $\mathfrak{A}(s_l)=\mathfrak{B}(s_l)$ $(l=0, \ldots, L+1)$. If exactly one of $\bar{d}(q_2)$ and $\bar{d}(r_2)$ is equal to zero, say $\bar{d}(q_2)=0$ and $\bar{d}(r_2)>0$, then $d(\mathfrak{A}(t_{L+1}))<d(\mathfrak{B}(t_{L+1}))$, consequently, $\mathfrak{A}(t_{L+1})\neq\mathfrak{B}(t_{L+1})$, which contradicts the minimality of p. It means that the following equalities are true:

$$d(\mathfrak{A}(t_l)) = \bar{d}(\bar{q}_4)+(l-1)\bar{d}(q_2)+d(q_1\cdot q_2)$$

and

$$d(\mathfrak{B}(t_l)) = \bar{d}(\bar{r}_4)+(l-1)\bar{d}(r_2)+d(r_1\cdot r_2) \quad (l = L, L+1).$$

This implies that $\bar{d}(q_2)=\bar{d}(r_2)>0$. Similarly, we get that $\bar{d}(q_3)=\bar{d}(r_3)>0$.

The tree $\mathfrak{A}(t_{L+1})$ is obtained from the tree $\bar{q}_4$ by replacing all occurrences of the subtree # by the tree $q_1\cdot q_2^{L+1}$, while $\mathfrak{B}(t_{L+1})$ is given by replacing all occurrences of # in $\bar{r}_4$ by the tree $r_1\cdot r_2^{L+1}$.

We have that $d(\bar{q}_4)\leq L$, $d(\bar{r}_4)\leq L$ and $d(q_1\cdot q_2^{L+1})>L$, $d(r_1\cdot r_2^{L+1})>L$. Thus the equality $\mathfrak{A}(t_{L+1})=\mathfrak{B}(t_{L+1})$ implies that $r_1\cdot r_2^{L+1}\in\operatorname{sub}(q_1\cdot q_2^{L+1})$ or $q_1\cdot q_2^{L+1}\in\operatorname{sub}(r_1\cdot r_2^{L+1})$.

Assume that $r_1\cdot r_2^{L+1}\in\operatorname{sub}(q_1\cdot q_2^{L+1})$. Let j be the minimal number, for which $r_1\cdot r_2^{L+1}\in\operatorname{sub}(q_1\cdot q_2^{j})$. Since $r_1\cdot r_2^{L+1}\in\operatorname{sub}(q_1\cdot q_2^{L+1})$ and $d(r_1\cdot r_2^{L+1})>d(q_1\cdot q_2)$ thus $2\leq j\leq L+1$.

Let $\bar{q}_2$ be the tree obtained from the tree $q_1\cdot q_2^j$ by replacing all occurrences of $r_1\cdot r_2^{L+1}$ by the symbol #. Therefore, $r_1\cdot r_2^{L+1}\cdot\bar{q}_2=q_1\cdot q_2^j$ and $r_1\cdot r_2^{L+1}\notin\operatorname{sub}(\bar{q}_2)$. Since j is minimal, it follows that $r_1\cdot r_2^{L+1}\notin\operatorname{sub}(q_1\cdot q_2^{j-1})$. On the other hand $r_1\cdot r_2^{L+1}\cdot\bar{q}_2=q_1\cdot q_2^{j-1}\cdot q_2$ and $r_1\cdot r_2^{L+1}\notin\operatorname{sub}(q_2)$. Therefore, $q_1\cdot q_2^{j-1}\in\operatorname{sub}(r_1\cdot r_2^{L+1})$.

Let $\bar{r}_2$ be the tree given from $r_1\cdot r_2^{L+1}$ by replacing all occurrences of $q_1\cdot q_2^{j-1}$ by the symbol #. Thus $q_1\cdot q_2^{j-1}\cdot\bar{r}_2=r_1\cdot r_2^{L+1}$ and $q_1\cdot q_2^{j-1}\notin\operatorname{sub}(\bar{r}_2)$. It means that $q_1\cdot q_2^{j-1}\cdot\bar{r}_2\cdot\bar{q}_2=q_1\cdot q_2^{j-1}\cdot q_2$.

Next we show that $q_1\cdot q_2^{j-1}\notin\operatorname{sub}(\bar{r}_2\cdot\bar{q}_2)$ holds, too. Indeed, if $q_1\cdot q_2^{j-1}\in\operatorname{sub}(\bar{r}_2\cdot\bar{q}_2)$, then $q_1\cdot q_2^{j-1}\in\operatorname{sub}(\bar{q}_2)$ because of $q_1\cdot q_2^{j-1}\notin\operatorname{sub}(\bar{r}_2)$ and $\bar{r}_2\notin\operatorname{sub}(q_1\cdot q_2^{j-1})$. Thus, in $q_1\cdot q_2^{j-1}\cdot q_2$ there exists a subtree $q_1\cdot q_2^{j-1}$, which is not a subtree of $r_1\cdot r_2^{L+1}$. But this is impossible since in this case one can show that $r_1\cdot r_2^{L+1}\in\operatorname{sub}(q_1\cdot q_2^{j-1})$. Therefore, one have

$$\bar{r}_2\cdot\bar{q}_2 = q_2.$$

Since $\mathfrak{A}(t_{L+1})=\mathfrak{B}(t_{L+1})$ thus

$$r_1\cdot r_2^{L+1}\cdot\bar{r}_4 = q_1\cdot q_2^{L+1}\cdot\bar{q}_4 = q_1\cdot q_2^j\cdot q_2^{L+1-j}\cdot\bar{q}_4 = r_1\cdot r_2^{L+1}\cdot\bar{q}_2\cdot q_2^{L+1-j}\cdot\bar{q}_4.$$

Furthermore, $r_1\cdot r_2^{L+1}$ is not a subtree of any of the trees $\bar{q}_4, q_2, \bar{q}_2, \bar{r}_4$. Thus the preceding equality implies

$$\bar{r}_4 = \bar{q}_2\cdot q_2^{L+1-j}\cdot\bar{q}_4.$$

We have $\mathfrak{A}(t_0)=\mathfrak{B}(t_0)$. Thus $q_1\cdot\bar{q}_4=r_1\cdot\bar{r}_4=r_1\cdot\bar{q}_2\cdot q_2^{L+1-j}\cdot\bar{q}_4$. Therefore,

$$q_1 = r_1\cdot\bar{q}_2\cdot q_2^{L+1-j}.$$

Using the equality $\mathfrak{A}(t_1)=\mathfrak{B}(t_1)$ we get

$$q_1\cdot q_2\cdot\bar{q}_4 = r_1\cdot\bar{q}_2\cdot(\bar{r}_2\cdot\bar{q}_2)^{L+1-j}\cdot(\bar{r}_2\cdot\bar{q}_2)\cdot\bar{q}_4,$$

$$r_1\cdot r_2\cdot\bar{r}_4 = r_1\cdot r_2\cdot\bar{q}_2\cdot(\bar{r}_2\cdot\bar{q}_2)^{L+1-j}\cdot\bar{q}_4.$$

This implies that $r_1 \cdot \bar{q}_2 \cdot \bar{r}_2 = r_1 \cdot r_2$. Furthermore, from the equalities $\mathfrak{A}(t_l) = \mathfrak{B}(t_l)$ $(l=0, \ldots, L+1)$, by induction, we obtain $r_1 \cdot (\bar{q}_2 \cdot \bar{r}_2)^{L+1} = r_1 \cdot (\bar{q}_2 \cdot \bar{r}_2)^L \cdot r_2$. Since $2 \leqq d(\mathfrak{A}(p_1 \cdot p_2 \cdot p_3 \cdot t)) \leqq L$, thus $d(r_1 \cdot (\bar{q}_2 \cdot \bar{r}_2)^L) > d(r_1 \cdot \bar{q}_2 \cdot \bar{r}_2) = d(r_1 \cdot r_2) \geqq d(r_2)$. Therefore, $r_1 \cdot (\bar{q}_2 \cdot \bar{r}_2)^L \notin \text{sub}\,(r_2)$, implying

$$\bar{q}_2 \cdot \bar{r}_2 = r_2.$$

Now consider the trees $s_l = p_1 \cdot p_3^l \cdot t$ $(l=0, \ldots, L+1)$. Then $r_1 \cdot r_3^{L+1} \in \text{sub}\,(q_1 \cdot q_3^{L+1})$ because of $\bar{r}_4 = \bar{q}_2 \cdot q_2^{L+1-j} \cdot \bar{q}_4$. In the above way we get that there are trees $\bar{q}_3, \bar{r}_3$ and a number i $(2 \leqq i \leqq L+1)$ such that

$$q_1 = r_1 \cdot \bar{q}_3 \cdot q_3^{L+1-i},$$

$$q_3 = \bar{r}_3 \cdot \bar{q}_3,$$

$$r_3 = \bar{q}_3 \cdot \bar{r}_3.$$

Since p is minimal thus

$$\mathfrak{A}(p_1 \cdot p_4) = \mathfrak{B}(p_1 \cdot p_4) \quad \text{and} \quad \mathfrak{A}(p_1 \cdot p_2 \cdot p_4) = \mathfrak{B}(p_1 \cdot p_2 \cdot p_4),$$

i.e.,

$$q_1 \cdot q_4 = r_1 \cdot r_4 \qquad \text{and} \qquad q_1 \cdot q_2 \cdot q_4 = r_1 \cdot r_2 \cdot r_4.$$

The first equality implies that $r_1 \cdot r_4 = r_1 \cdot \bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_4$. Consequently, r_4 can differ from $\bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_4$ in the tree r_1 only, i.e. whenever $\#$ is a subtree in one of them then the corresponding subtree in the other one should be r_1 or $\#$. By the above second equality we get

$$r_1 \cdot \bar{q}_2 \cdot \bar{r}_2 \cdot r_4 = r_1 \cdot \bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot (\bar{r}_2 \cdot \bar{q}_2) \cdot q_4.$$

Thus r_4 and $\bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_4$ can differ only in $r_1 \cdot r_2$. Thus, by $r_1 \cdot r_2 \neq r_1$, we have

$$r_4 = \bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_4.$$

Similarly, using the trees $p_1 \cdot p_4$ and $p_1 \cdot p_3 \cdot p_4$, we obtain

$$r_4 = \bar{q}_3 \cdot (\bar{r}_3 \cdot \bar{q}_3)^{L+1-i} \cdot q_4.$$

Therefore, $\bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_4 = \bar{q}_3 \cdot (\bar{r}_3 \cdot \bar{q}_3)^{L+1-i} \cdot q_4$ implying

$$\bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} = \bar{q}_3 \cdot (\bar{r}_3 \cdot \bar{q}_3)^{L+1-i}.$$

Finally, using the above equalities, we get

$$q_1 \cdot q_2 \cdot q_3 \cdot q_4 = r_1 \cdot \bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot (\bar{r}_2 \cdot \bar{q}_2) \cdot q_3 \cdot q_4 =$$
$$= r_1 \cdot (\bar{q}_2 \cdot \bar{r}_2) \cdot \bar{q}_2 \cdot (\bar{r}_2 \cdot \bar{q}_2)^{L+1-j} \cdot q_3 \cdot q_4 = r_1 \cdot r_2 \cdot \bar{q}_3 \cdot (\bar{r}_3 \cdot \bar{q}_3)^{L+1-i} \cdot (\bar{r}_3 \cdot \bar{q}_3) \cdot q_4 =$$
$$= r_1 \cdot r_2 \cdot (\bar{q}_3 \cdot \bar{r}_3) \cdot \bar{q}_3 \cdot (\bar{r}_3 \cdot \bar{q}_3)^{L+1-i} \cdot q_4 = r_1 \cdot r_2 \cdot r_3 \cdot r_4,$$

i.e., $\mathfrak{A}(p) = \mathfrak{B}(p)$ contradicting our assumption.

Similarly, we arrive at a contradiction by assuming

$$q_1 \cdot q_2^{L+1} \in \text{sub}\,(r_1 \cdot r_2^{L+1}).$$

This means that the depth of p is smaller than K ending the proof of this lemma.

Theorem 6. The equivalence problem of deterministic frontier-to-root tree transducers is effectively solvable.

Proof. Consider two arbitrary DFRT transducers $\mathfrak{A}=(F, A, G, A', \Sigma_A)$ and $\mathfrak{B}=(F, B, G, B', \Sigma_B)$. The set of all trees p, for which $\mathfrak{A}(p)$ and $\mathfrak{B}(p)$ are defined, is a regular set of trees, which can be given effectively (see, Corollary 3.12. in [1]). Thus, the problem whether or not the domains of mappings induced by $\mathfrak{A}$ and $\mathfrak{B}$ are equal is solvable. If they are not equal, then the transducers are not equivalent. In the opposite case, by Lemmas 4 and 5 it is sufficient to check whether their translations coincide on a finite number of trees. This ends the proof of Theorem 6.

Finally, we present a result concerning the equivalence problem in a special class of deterministic root-to-frontier tree transducers.

Let $\mathfrak{M}$ be the set of deterministic root-to-frontier tree (DRFT) transducers $\mathfrak{A}=(F, A, G, A', \Sigma)$ with the following property: if $af(z_1, \dots, z_k)\to q$ is in Σ $(v(f)=k, k>0)$, then there are states $a_1, \dots, a_k\in A$ such that $q\in T_G(Y\cup\{(a_i, z_i)|i=1, \dots, k\})$. For such DRFT transducers one can prove Lemmas 1—5. Thus we have

Theorem 7. The equivalence problem of DRFT transducers in $\mathfrak{M}$ is effectively solvable.

4. Minimalization of DFRT transducers

Take a DFRT transducer $\mathfrak{A}=(F, A, G, A', \Sigma_A)$ such that the mapping induced by $\mathfrak{A}$ is from $T_F(X_n)$ into $T_G(Y_m)$. Moreover, let p be an arbitrary tree, for which $\mathfrak{A}(p)$ is defined, i.e., $p\in\mathfrak{A}^{-1}(T_G(Y_m))$. In this case for any $\bar{p}\in\text{sub}(p)$ of the form $\bar{p}=f(p_1, \dots, p_k)$ or $\bar{p}=x_i$, there is exactly one rule in Σ_A, denoted by $\sigma(\bar{p})$ such that if $\sigma(\bar{p})=f(a_1z_1, \dots, a_kz_k)\to A_{\bar{p}}q$ then

$$\mathfrak{A}_{A_{\bar{p}}}(\bar{p}) = q[\mathfrak{A}_{a_1}(p_1)\to z_1, \dots, \mathfrak{A}_{a_k}(p_k)\to z_k],$$

and

$$\mathfrak{A}_{A_{\bar{p}}}(\bar{p}) = q \quad \text{if} \quad \sigma(\bar{p}) = x_i \to A_{\bar{p}}q.$$

Lemma 8. Let $p\in\mathfrak{A}^{-1}(T_G(Y_m))$ and $\bar{p}\in\text{sub}(p)$ be arbitrary. Then there exist a $p'\in\mathfrak{A}^{-1}(T_G(Y_m))$ and a $\bar{p}'\in\text{sub}(p')$, such that $\sigma(\bar{p})=\sigma(\bar{p}')$ and $d(p')<2|A|$.

Proof. Let $\bar{\bar{p}}$ denote the tree obtained by replacing the subtree $\bar{p}$ in p by $\#$. Let $\bar{\bar{p}}'$ be the tree given by Lemma 2 to the tree p and its subtree $\bar{p}$. Assume, that $\bar{p}=f(p_1, \dots, p_k)$. Let us construct the tree $\bar{p}_i$ from p_i $(i=1, \dots, k)$ in exactly that way as the tree $\bar{t}$ has been constructed from the tree $\bar{\bar{p}}$ in the proof of Lemma 2 $(i=1, \dots, k)$. Let $\bar{p}'=f(\bar{p}_1, \dots, \bar{p}_k)$ and $p'=\bar{\bar{p}}'\cdot\bar{p}'$. From the construction it is clear, that the trees p' and $\bar{p}'$ satisfy the conditions of Lemma 8. A similar argument can be used in the case $\bar{p}=x_i$.

Let $L=\max(d(\mathfrak{A}(p))|p\in\mathfrak{A}^{-1}(T_G(Y_m)), d(p)<2|A|)$.

Lemma 9. There exists a minimal DFRT transducer $\mathfrak{B}=(F, B, G, B', \Sigma_B)$ belonging to $\mathfrak{A}$ such that if $x_i\to bq$ or $f(b_1z_1, \dots, b_kz_k)\to bq$ is in Σ_B then $d(q)\leq L$.

Proof. Let $\mathfrak{B}$ be a minimal DFRT transducer belonging to $\mathfrak{A}$. Assume that there exist $p\in\mathfrak{B}^{-1}(T_G(Y_m))$ and $\bar{p}\in\text{sub}(p)$ such that the depth of the right hand side of

$\sigma(\bar{p})$ is greater, than L. We show that $\bar{d}(\mathfrak{B}(\bar{\bar{p}}))$ is undefined, where $\bar{\bar{p}}$ is obtained by replacing $\bar{p}$ in p by $\#$.

Indeed, by Lemma 8, there exist trees p' and $\bar{p}'$, $\bar{\bar{p}}'$, for which $p'=\bar{p}'\cdot\bar{\bar{p}}'$, $\sigma(\bar{p})=$ $=\sigma(\bar{p}')$, $p'\in\mathfrak{B}^{-1}(T_G(Y_m))$ and $d(p')<2|B|\leqq 2|A|$.

By the note following Lemma 2, if $\bar{d}(\mathfrak{B}(\bar{\bar{p}}))$ is defined then so is $d(\mathfrak{B}(\bar{\bar{p}}'))$. But $d(\mathfrak{B}(p'))\geqq\bar{d}(\mathfrak{B}(\bar{\bar{p}}'))+d(\mathfrak{B}(\bar{p}'))$. Furthermore, by our assumption $d(\mathfrak{B}(\bar{p}'))>L$. Thus $d(\mathfrak{B}(p'))>L$ which is a contradiction since $\mathfrak{B}(p')=\mathfrak{A}(p')$ and $d(p')<2|B|\leqq$ $\leqq 2|A|$.

Now for all $\sigma=f(b_1z_1,\dots,b_kz_k)\to bq$ and $\sigma=x_i\to bq$ with $d(q)>L$, let us replace σ in Σ_B by $\bar{\sigma}=f(b_1z_1,\dots,b_kz_k)\to by_1$ and $\bar{\sigma}=x_i\to by_1$, respectively, and denote the resulting set of rules by $\bar{\Sigma}_B$. Then the DFRT transducer $\bar{\mathfrak{B}}=(F,B,G,B',\bar{\Sigma}_B)$ is equivalent to $\mathfrak{B}$, completing the proof of Lemma 9.

Theorem 10. There exists an algorithm for determining to any DFRT transducer $\mathfrak{A}=(F,A,G,A',\Sigma_A)$ a minimal DFRT transducer belonging to $\mathfrak{A}$.

Proof. Let $|A|=M$ and $L=\max\big(d(\mathfrak{A}(p))|p\in\mathfrak{A}^{-1}(T_G(Y)),\ d(p)<2M\big)$. Then for a minimal DFRT transducer belonging to $\mathfrak{A}$, it holds that the number of its states is less than or equal to M. Furthermore, by Lemma 9, we can assume that the depths of right hand sides of rules of a minimal DFRT transducer belonging to $\mathfrak{A}$ are less than or equal to L. But there is only a finite number of DFRT transducers satisfying these two assumptions. This means that it is enough to check only for finitely many DFRT transducers whether they are equivalent to $\mathfrak{A}$.

After determining all such DFRT transducers equivalent to $\mathfrak{A}$, we choose one of them with minimal number of states.

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