Strongly connected digraphs in which each edge is contained in exactly two cycles

By B. ZELINKA

In [1] A. ÁDÁM suggests a problem to characterize strongly connected digraphs without cut vertices with the property that each edge of such a graph is contained at most in two cycles. (See Problem 2, p. 189 in [1].) In this note we do not solve this problem in general, but we consider a particular case when each edge is contained exactly in two cycles. We consider finite digraphs without loops and without pairs of equally oriented edges joining the same pair of vertices.

We start by a definition.

DEFINITION. Let $A_1, A_2, \ldots, A_n$ for $n \geqq 2$ be pairwise disjoint cycles. On each A_i for $i=1, \ldots, n$ choose two distinct vertices a_i, b_i. Then identify b_i with a_{i+1} for all $i=1, \ldots, n-1$ and b_n with a_1. The class of all digraphs obtained in this way will be denoted by $\mathscr{A}$ (Fig. 1).

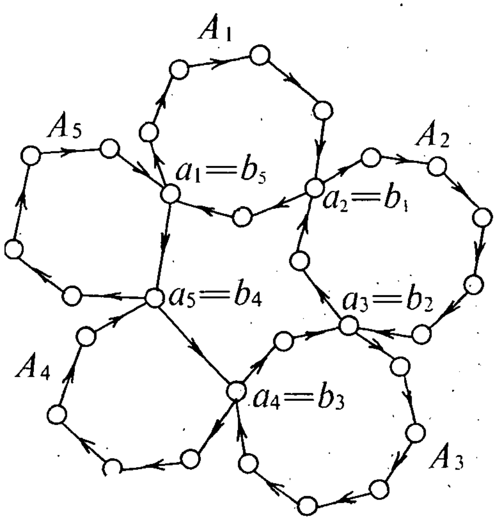

Fig. 1

Further, by a diagonal path of a cycle C we shall mean a directed path whose initial and terminal vertices are in C, while its edges and inner vertices (if any) are not.

THEOREM. *Let G be a strongly connected finite digraph without cut vertices. Then the following two assertions are equivalent:*

(i) $G \in \mathscr{A}$.

(ii) *Each edge of G is contained in exactly two cycles of G.*

PROOF: (i)$\Rightarrow$(ii). Let $G \in \mathscr{A}$. Let e be an edge of G. The edge e is contained in some cycle A_i for $1 \leqq i \leqq n$. The cycle A_i is the union of two directed paths $P_1^{(i)}, P_2^{(i)}$, where $P_1^{(i)}$ is the path from a_i into b_i in A_i and $P_2^{(i)}$ is the path from b_i into a_i in A_i; these two paths are edge-disjoint. If e belongs to $P_1^{(i)}$ then, evidently, each cycle containing e contains the whole $P_1^{(i)}$, therefore, it must contain also a directed path from b_i into a_i in G. There are exactly two such paths; one of them is $P_2^{(i)}$, the other is the union of all $P_1^{(j)}$ for $1 \leqq j \leqq n, j \neq i$, where $P_1^{(j)}$ is defined analogously as $P_1^{(i)}$. Therefore, there are exactly two cycles in G which contain e. For the case when e is in $P_2^{(i)}$ the proof is analogous, obtained from this proof by interchanging subscripts 1 and 2.

(ii)$\Rightarrow$(i). Let G satisfy (ii). Let C_0 be a cycle in G. Let $\overrightarrow{u_1u_2}$ be an edge of C_0. As $\overrightarrow{u_1u_2}$ must be contained in two cycles, there exists a cycle C_1 containing $\overrightarrow{u_1u_2}$ and distinct from C_0. Evidently, there exists the longest directed path P_1 which contains $\overrightarrow{u_1u_2}$ and is contained in both C_0 and C_1. Let this path go from a vertex u_3 into a vertex u_4. Let P_1' be the path in C_1 from u_4 into u_3. Suppose that P_1' contains a vertex u' of C_0 distinct from u_3 and u_4; let u_1' be the first vertex of P_1' with this property. Then there exists a cycle which is the union of P_1, the subpath of P_1' from u_4 into u_1' and the path in C_0 from u_1' into u_3. This cycle is evidently distinct from both C_0 and C_1 and contains $\overrightarrow{u_1u_2}$, which is a contradiction. Thus P_1' is a diagonal path of C_0. Let u_5 be the terminal vertex of the edge of C_0 whose initial vertex is u_4. There exists a cycle C_2 distinct from C_0 and C_1 which contains the edge $\overrightarrow{u_4u_5}$. Let P_2 be the longest path which contains $\overrightarrow{u_4u_5}$ and is contained in both C_0 and C_2, let it go from a vertex u_6 into a vertex u_7. Let P_2' be the path in C_2 from u_7 into u_6; it is a diagonal path of C_0. Suppose that P_1' and P_2' have a common inner vertex; and let v be the first inner vertex of P_2' belonging to P_1'. If $u_7 \neq u_3$, then any edge belonging to the intersection of the paths in C_0 from u_6 into u_4 and from u_3 into u_7 belongs to three cycles, namely C_0, C_1 and the cycle which is the union of the path from u_3 into u_7 in C_1, the subpath of P_2' from u_7 into v and the subpath of P_1' from v into u_3, which is a contradiction. An analogous contradiction will be obtained for $u_6 \neq u_4$. Therefore P_1' and P_2' can have a common inner vertex only if $u_7 = u_3$ and $u_6 = u_4$; this case will be denoted by (∗), the opposite case by (∗ ∗).

Consider the case (∗). Each edge of the path in C_0 from u_3 into u_4 is contained in C_0 and C_1, each edge of the path in C_0 from u_4 into u_3 is contained in C_0 and C_2. Let v_1 be the first vertex of P_1' distinct from u_4 and belonging to P_2'. The subpath of P_1' from u_4 into v_1 and the subpath of P_2' from v_1 into u_4 form a cycle D_1. Each edge of D_1 is contained in two cycles only, therefore, an inner vertex neither of the subpath of P_1' from v_1 into u_3, nor of the subpath of P_2' from u_3 into v_1 can belong to D_1. If $v_1 \neq u_3$, we repeat this consideration with the subpath of P_1' from v_1 into u_3 instead of P_1' and with the subpath of P_2' from u_3 into v_1 instead of P_2', and analogously as we have obtained v_1 and D_1 we obtain v_2 and D_2. Thus we proceed further, until we obtain $v_k = u_3$ for some k (this will be performed after a finite number of steps). The cycles $C_0, D_1, \ldots, D_k$ correspond to the cycles $A_1, A_2, \ldots, A_n$ from the definition of $\mathscr{A}$. The graph G evidently cannot contain further vertices or edges, because then (ii) would be violated. Therefore $G \in \mathscr{A}$ (Fig. 2).

Now consider the case (∗ ∗). Suppose that $u_6 \neq u_4$. As C_2 must contain $\overrightarrow{u_4u_5}$, the vertex u_4 lies on the path in C_0 from u_6 into u_7. As $u_6 \neq u_4$, also the edge of C_0 whose terminal vertex is u_4 is contained in this path and in the cycle C_2. Then this edge is contained in three cycles C_0, C_1, C_2, which is a contradiction. Therefore, $u_6 = u_4$. If u_7 is an inner vertex of P_1, then an arbitrary edge of the path in C_0 from u_3 into u_7 is contained in C_0, C_1 and the cycle which is the union of P_2', P_1' and the path in C_0 from u_3 into u_7, which is a contradiction. Therefore, u_7 lies on the path in C_0 from u_4 into u_3. We see that C_1 and C_2 have only one common vertex u_4. Thus we may proceed further and we obtain further cycles $C_3, \ldots, C_k$. The cycles $C_1, C_2, \ldots, \ldots, C_k$ then correspond to the cycles $A_1, A_2, \ldots, A_n$ from the definition of $\mathscr{A}$. As G cannot contain further vertices and edges, we have $G \in \mathscr{A}$ (Fig. 3).

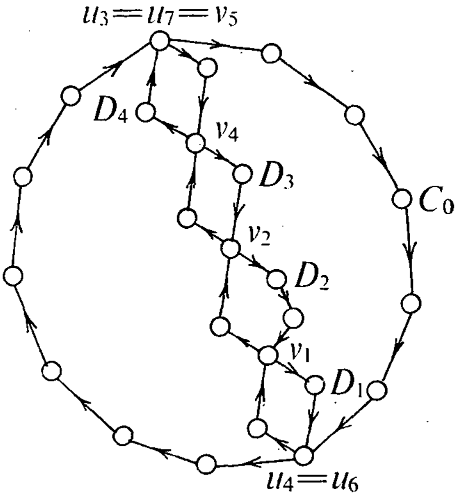

Fig. 2

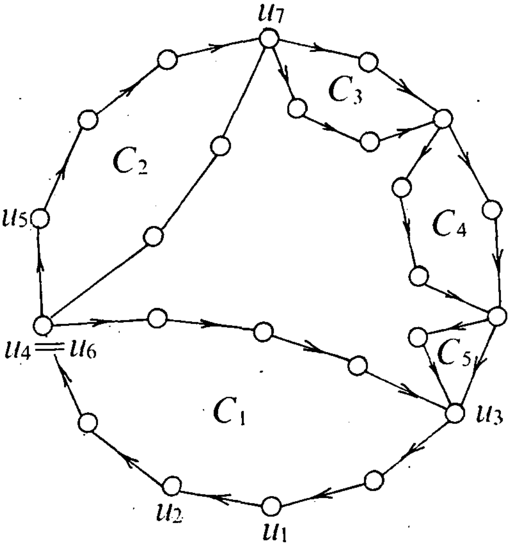

Fig. 3

Сильно связные орграфы, в которых каждая дуга принадлежит точно двум циклам

В статье характеризован класс всех конечных сильно связных ориентированных графов, в которых каждая дуга принадлежит точно двум циклам. Это является частичным решением одной проблемы предложенной A. Ádám-ом.

DEPARTMENT OF MATHEMATICS,
INSTITUTE OF MECHANICAL
AND TEXTILE TECHNOLOGY,
KOMENSKÉHO 2.
460 01 LIBEREC, CZECHOSLOVAKIA

Reference

[1] Ádám, A., On some open problems of applied automaton theory and graph theory (suggested by the mathematical modelling of certain neuronal networks), *Acta Cybernetica*, v. 3, 1977, pp. 187—214.

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