On Sperner families in which no 3 sets have an empty intersection

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1. Introduction

Let $\mathscr{G}(r, k)$ denote the set of all Sperner families $\mathscr{F}$ (i.e. $X \not\subset Y$ for all different $X, Y \in \mathscr{F}$) on $R=[1, r]$ (the interval of the first r natural numbers with $r \geqq 3$) satisfying $\bigcup_{i=1}^{k} X_i \subset R$ for all $X_i \in \mathscr{F}$ $(i=1, \dots, k)$ where $\subset$ is used in the strong sense. Furthermore we use the following notations:

$$\mathscr{G}^1(r, k) = \{\mathscr{F}: \mathscr{F} \in \mathscr{G}(r, k), \bigcup_{X \in \mathscr{F}} X = R\},$$

$$\mathscr{G}^0(r, k) = \{\mathscr{F}: \mathscr{F} \in \mathscr{G}(r, k), \bigcup_{X \in \mathscr{F}} X \subset R\},$$

$$n(r, k) = \max_{\mathscr{F} \in \mathscr{G}} |\mathscr{F}|, \; n^1(r, k) = \max_{\mathscr{F} \in \mathscr{G}^1} |\mathscr{F}| \quad \text{and} \quad n^0(r, k) = \max_{\mathscr{F} \in \mathscr{G}^0} |\mathscr{F}|.$$

We notice that $\mathscr{G}^1(r, k) = \emptyset$ holds for $k \geqq r$.

$n(r, 2)$ was determined by E. C. MILNER [6] (for the dual case) and later by A. BRACE and D. E. DAYKIN [1], and $n(r, k)$ with $k \geqq 4$ was determined by the author [3].

For $n(r, 3)$ the following two configurations are known:

$$n(r, 3) = \binom{r-1}{\left[\frac{r-1}{2}\right]} + 1 \tag{1}$$

and

$$n(r, 3) = \binom{r-1}{\left[\frac{r-1}{2}\right]}. \tag{2}$$

P. FRANKL [2] proved (1) for large enough even r (e.g. for $r > 1000$) and (2) for large enough odd r (e.g. for $r > 300$). The author [3] showed (1) for $r=7$ and even

$r>400$, and (2) for all odd r with the exception of the following 12 values: 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37 and 43.

In the present paper we prove

(1) for $r=4, 6, 114$ and even $r\geqq 120$ and

(2) for $r=11, 17, 23, 29, 35, 43$.

We observe that exchanging all $X\in\mathcal{F}$ by $R\setminus X$ we get analogous results for Sperner families in which no 3 sets have an empty intersection.

We shall sharpen Theorem 5 of [3] in the case $k=3$. There we divided a maximal family $\mathcal{F}\in\mathcal{G}(r, 3)$ to two families $\mathcal{F}_0$ and $\mathcal{F}_1$, and showed

$$|\mathcal{F}_0| \leqq \binom{r-1}{\left[\frac{r-2}{2}\right]} \quad \text{and} \quad |\mathcal{F}_1| \leqq \binom{r-1}{\left[\frac{r-1}{3}\right]-1}.$$

In fact $|\mathcal{F}_1|$ depends on $|\mathcal{F}_0|$. For $k=3$ and even r, $|\mathcal{F}_0|=\binom{r-1}{\left[\frac{r-2}{2}\right]}$ implies $|\mathcal{F}_1|=1$.

In section 2 we shall present our main results and give a new type estimation of families of sets, which will be used in section 3 to prove a theorem analogous to Theorem 5 [3]. Finally, in section 4 we shall prove our main result.

2. Main results

Throughout this paper let $a=\left[\frac{r-2}{2}\right]$ and $b=\left[\frac{r-1}{3}\right]$.

Theorem 1. 1° $n(r, 3)=\binom{r-1}{\left[\frac{r-1}{2}\right]}+1$ *for* $r=4, 6, 114$ *and even* $r\geqq 120$,

2° $n(r, 3)=\binom{r-1}{\left[\frac{r-1}{2}\right]}$ *for* $r=11, 17, 23, 29, 35, 43$.

Let $r\geqq 4$. Then $n(r, 3)$, $n^1(r, 3)$ and $n^0(r, 3)$ exist and it holds $n(r, 3)= =\max(n^1(r, 3), n^0(r, 3))$.

For $\mathcal{F}\in\mathcal{G}^0(r, 3)$ there is an element $v\in R$ such that $\mathcal{F}$ is a Sperner family on $R\setminus\{v\}$, and it follows by SPERNER's theorem [7]:

Lemma 1. $n^0(r, 3)=\binom{r-1}{\left[\frac{r-1}{2}\right]}$.

We shall use the following lemma shown in more general form in [3] (Lemma 2).

Lemma 2. Let $\mathcal{F}\in\mathcal{G}^1(r,3)$ such that $|\mathcal{F}|=n^1(r,3)$ and $\max_{X\in\mathcal{F}}|X|$ is minimal. Then $|X|\leqq a$ holds for all $X\in\mathcal{F}$.

Lemma 3. Let $s\leqq\frac{r}{2}$ be an integer and let $\mathcal{F}_s$ denote an arbitrary family of different s-element subsets of R. Finally, let $\mathcal{F}_{2s}^*$ denote the largest family of $(2s)$-element subsets of R such that for every $X\in\mathcal{F}_{2s}^*$ there is at least one pair (Y, Z) of subsets of $\mathcal{F}_s$ satisfying $Y\cup Z=X$. Then

$$|\mathcal{F}_{2s}^*| \geqq \frac{\binom{r-s}{s}}{\binom{2s-1}{s}}|\mathcal{F}_s| - \binom{r}{2s}.$$

Proof. Let us consider the following families:

$$\overline{\mathcal{F}_s} = \{X: X\subset R,\ |X| = s,\ X\notin\mathcal{F}_s\},$$

$$\overline{\mathcal{F}_{2s}^*} = \{X: X\subset R,\ |X| = 2s,\ X\notin\mathcal{F}_{2s}^*\}.$$

Then for any $X\in\overline{\mathcal{F}_{2s}^*}$ there is no pair (Y, Z) of sets of $\mathcal{F}_s$ with $Y\cup Z=X$. For every such $X\in\overline{\mathcal{F}_{2s}^*}$ there exist exactly $\frac{1}{2}\binom{2s}{s}=\binom{2s-1}{s}$ unordered pairs (Y, Z) with $|Y|=|Z|=s$ and $Y\cup Z=X$. All these sets are mutually disjoint, i.e., at least $\binom{2s-1}{s}$ s-element subsets belong to $\overline{\mathcal{F}_s}$ for every $X\in\overline{\mathcal{F}_{2s}^*}$.

On the other hand for every s-element set Y of R there exist exactly $\binom{r-s}{s}$ disjoint s-element sets Z. Hence

$$|\overline{\mathcal{F}_{2s}^*}|\binom{2s-1}{s-1} \leqq |\overline{\mathcal{F}_s}|\binom{r-s}{s}.$$

Using $|\overline{\mathcal{F}_{2s}^*}|=\binom{r}{2s}-|\mathcal{F}_{2s}^*|$ and $|\overline{\mathcal{F}_s}|=\binom{r}{s}-|\mathcal{F}_s|$ we obtain the inequality of Lemma 3. □

3. An upper bound for $n^1(r, 3)$

Let $\mathcal{F}\in\mathcal{G}^1(r,3)$ such that $|\mathcal{F}|=n^1(r,3)$ and $\max_{X\in\mathcal{F}}|X|$ is minimal. By Lemma 2, we have $|X|\leqq a$ for all $X\in\mathcal{F}$. The numbers $p_i=|\{X: X\in\mathcal{F},\ |X|=i\}|$ $(i=0, \dots, r)$ are called parameters of the family $\mathcal{F}$. $\mathcal{S}\mathcal{F}$ denotes the canonical Sperner family (see A. J. W. Hilton [4]).

Now we decompose $\mathcal{F}$ to the subfamilies $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{H}$ defined as follows.

— $\mathcal{D}$ is a subfamily of $\mathcal{F}$ with $\mathcal{S}\mathcal{D}=\{X: X\in\mathcal{S}\mathcal{F},\ r\notin X\}$.

— $\mathcal{E}=\{X: X\in\mathcal{F}\setminus\mathcal{D},\ |X|\leqq r-2a-1\}$.

— $\mathcal{H}=\{X: X\in\mathcal{F}\setminus\mathcal{D},\ |X|\geqq r-2a\}$.

1. It has been proved by A. J. W. Hilton [4] that all $X\in\mathcal{F}$ with $|X|>b$ belong to $\mathcal{D}$. $\mathcal{S}\mathcal{D}$ is a Sperner family on $R\setminus\{r\}$. Using $\binom{r-1}{|X|}\leqq\binom{r-1}{a-1}$ for $|X|\leqq a-1<\frac{r-1}{2}$, by Lubell's inequality [5] we obtain

$$\sum_{X\in\mathcal{S}\mathcal{D}}\frac{1}{\binom{r-1}{|X|}}=\sum_{\substack{X\in\mathcal{S}\mathcal{D}\\|X|=a}}\frac{1}{\binom{r-1}{a}}+\sum_{\substack{X\in\mathcal{S}\mathcal{D}\\|X|\leqq a-1}}\frac{1}{\binom{r-1}{|X|}}\leqq 1,$$

$$\frac{p_a}{\binom{r-1}{a}}+\frac{|\mathcal{S}\mathcal{D}|-p_a}{\binom{r-1}{a-1}}\leqq 1.$$

and

$$|\mathcal{D}|=|\mathcal{S}\mathcal{D}|\leqq\frac{a}{r-a}\binom{r-1}{a}+\frac{r-2a}{r-a}p_a.$$

2. $\mathcal{J}=\{X\colon X\cup\{r\}\in\mathcal{S}(\mathcal{D}\cup\mathcal{E}),\, r\notin X\}$ is a Sperner family of cardinality $|\mathcal{E}|$ on $R\setminus\{r\}$ and $|X|\leqq r-2a-2$ holds for all $X\in\mathcal{J}$.

By Lubell's inequality [5] we obtain

$$\sum_{X\in\mathcal{J}}\frac{1}{\binom{r-1}{|X|}}\leqq 1,\quad \frac{|\mathcal{J}|}{\binom{r-1}{r-2a-2}}\leqq 1\quad\text{and}\quad |\mathcal{E}|=|\mathcal{J}|\leqq\binom{r-1}{r-2a-2}.$$

3. Let $\mathcal{F}_{2a}^{**}=\{X\colon R\setminus X\in\mathcal{F}_{2a}^{*}\}$. Then $\mathcal{D}\cup\mathcal{H}\cup\mathcal{F}_{2a}^{**}$ is a Sperner family. We notice that $|X|\geqq r-2a$ holds for all $X\in\mathcal{D}\cup\mathcal{H}$ [1] and $|X|=r-2a$ holds for all $X\in\mathcal{F}_{2a}^{**}$. Clearly, $\mathcal{D}\cup\mathcal{H}$ and $\mathcal{F}_{2a}^{**}$ are Sperner families themselves. We have only to show that there is no pair (Y,Z) with $Y\in\mathcal{F}_{2a}^{**}$ and $Z\in\mathcal{D}\cup\mathcal{H}$ satisfying $Y\subseteq Z$. Let us assume the contrary. Then there are two sets $Y_1,Y_2\in\mathcal{D}$ with $Y_1\cup Y_2=R\setminus Y$. Hence for the sets $Y_1,Y_2,Z\in\mathcal{F}$ it follows $Y_1\cup Y_2\cup Z=(R\setminus Y)\cup Z\supseteq(R\setminus Y)\cup Y=R$, which is impossible for $\mathcal{F}\in\mathcal{G}(r,3)$.

$\mathcal{J}'=\{X\colon X\cup\{r\}\in\mathcal{S}(\mathcal{D}\cup\mathcal{H}\cup\mathcal{F}_{2a}^{**}),\, r\notin X\}$ is a Sperner family on $R\setminus\{r\}$. If q_i, q_i' and q_i'' are the parameters of the families $\mathcal{J}'$, $\mathcal{H}$ and $\mathcal{F}_{2a}^{**}$, respectively, then $q_i=q_{i+1}'+q_{i+1}''$ holds. By Lubell's inequality [5], using $\binom{r-1}{|X|}\leqq\binom{r-1}{b}$ for $|X|\leqq b<\frac{r-1}{2}$, we get

$$\sum_{X\in\mathcal{J}'}\frac{1}{\binom{r-1}{|X|}}\leqq 1,\quad \sum_{X\in\mathcal{H}}\frac{1}{\binom{r-1}{|X|-1}}+\sum_{X\in\mathcal{F}_{2a}^{**}}\frac{1}{\binom{r-1}{r-2a-1}}\leqq 1$$

and

$$\frac{|H|}{\binom{r-1}{b-1}}+\frac{|\mathcal{F}_{2a}^{**}|}{\binom{r-1}{r-2a-1}}\leqq 1.$$

By Lemma 3 using $|\mathcal{F}|=n^1(r,3)$ and the estimations for $\mathcal{D},\mathcal{E}$ and $\mathcal{H}$ we obtain

[1] as $\min\limits_{X\in\mathcal{D}\cup\mathcal{H}}|X|\leqq r-2a-1$ would imply $\mathcal{H}=\emptyset$ and, together with 1. and 2., the estimation given in Theorem 2.

Theorem 2.

$$n^1(r,3) \leqq$$

$$\leqq \max_{p_a}\left(\frac{a}{r-a}\binom{r-1}{a}+\frac{r-2a}{r-a}p_a+\binom{r-1}{r-2a-2}+\binom{r-1}{b-1}\frac{2(r-a)}{r-2a}\left(1-\frac{p_a}{\binom{r-1}{a}}\right)\right).$$

4. Proof of Theorem 1

Clearly, $n(r,3)=\max\left(n^1(r,3),\binom{r-1}{\left[\frac{r-1}{2}\right]}\right)$ holds by Lemma 1.

1°. Let r be even. Then all a-element subsets of $R\setminus\{r\}$ and the set $\{r\}$ form a family $\mathcal{F}\in\mathcal{G}(r,3)$ having the cardinality $\binom{r-1}{a}+1$. So we have only to show that the right side of the inequality of Theorem 2 has the value $\binom{r-1}{a}+1$, too.

For $r=4$ it is easy to see that $n^1(4,3)=4$ holds.

Now let $r=6, 114$ or $r\geqq 120$.

The function $f(p_a)$, of which we consider the maximum in Theorem 2, is a linear function in p_a. We have to take the maximum over the interval $\left[0,\binom{r-1}{a}\right]$, as an immediate consequence of A. J. W. Hilton's result [4] which we used in the definition of $\mathcal{D}$. We have $f\left(\binom{r-1}{a}\right)=\binom{r-1}{a}+1$. We have only to show that the factor of p_a in $f(p_a)$ is positive (or equal to o), i.e., using $r-2a=2$,

$$\frac{2}{r-a}-(r-a)\frac{\binom{r-1}{b-1}}{\binom{r-1}{a}}>0. \tag{3}$$

(3) is equivalent to

$$M(r)=\frac{2(r-b)(r-b-1)\dots(r-a+1)}{(r-a)a(a-1)\dots b}>1. \tag{4}$$

$$M(6)=\frac{5}{4} \quad\text{and}\quad M(124)=\frac{35047435882784}{34511088479301}>1.$$

Furthermore,

$$\frac{M(6t+10)}{M(6t+4)}=\frac{2^{10}}{3^6}\,\frac{t+\frac{7}{4}}{t+2}\,\frac{t+\frac{3}{2}}{t+2}\,\frac{t+\frac{5}{4}}{t+\frac{5}{3}}\left(\frac{t+1}{t+\frac{4}{3}}\right)^2\frac{t+\frac{1}{2}}{t+\frac{2}{3}}=g(t)$$

is monotonically increasing, because $\dfrac{t+x}{t+y}$ is monotonically increasing for fixed x and y with $x<y$.

For $t \geqq 20$ we obtain $g(t) \geqq g(20) = \frac{127766373}{99866624} > 1$. By induction it follows that $M(6t+4) > 1$ for $t \geqq 20$.

Moreover we have

$$\frac{M(6t+2)}{M(6t+4)} = \frac{9}{8} \frac{t+1}{t+\frac{3}{4}} \frac{t+1}{t+\frac{2}{3}} \frac{t+\frac{1}{3}}{t} > \frac{9}{8} > 1$$

and

$$\frac{M(6t)}{M(6t+4)} = \frac{3^4}{2^6} \frac{t+1}{t+\frac{3}{4}} \frac{t+1}{t+\frac{1}{2}} \frac{t+\frac{2}{3}}{t-\frac{1}{2}} > \frac{81}{64} > 1,$$

which proves $M(2t) > 1$ for $t \geqq 60$.

Finally we complete our proof by $\frac{M(114)}{M(124)} = \frac{59025914157}{53793208352} > 1$.

2°. In [3] the author proved the following estimation for $|\mathcal{E} \cup \mathcal{H}|$: $|\mathcal{E} \cup \mathcal{H}| \leqq \binom{r-1}{b-1}$. Using our estimation for $|\mathcal{D}|$ we obtain $|\mathcal{F}| \leqq \binom{r-1}{a} \frac{a}{r-a} + \frac{r-2a}{r-a} p_a + \binom{r-1}{b-1}$. Both, this estimation and the bound given in Theorem 2 are valid for each $|\mathcal{F}|$. It suffices to show that for every p_a one of our upper bounds is less than $\binom{r-1}{a+1}$, because in this case r is odd, i.e. $\left[\frac{r-1}{2}\right] = a+1$. We distinguish the following cases.

1. $p_a < \frac{a+3}{3} \left\{ \binom{r-1}{a+1} - \binom{r-1}{a-1} - \binom{r-1}{b-1} \right\}$. Then $|\mathcal{F}| < \binom{r-1}{a+1}$ follows from our last estimation.

2. $p_a \geqq \frac{a+3}{3} \left\{ \binom{r-1}{a+1} - \binom{r-1}{a-1} - \binom{r-1}{b-1} \right\} = \frac{2}{3} \frac{2a+3}{a+1} \binom{r-1}{a} - \frac{a+3}{3} \binom{r-1}{b-1}$.

Then we use the estimation of Theorem 2. First we prove that the factor of p_a in $f(p_a)$ is negative, i.e.

$$\frac{r-2a}{r-a} - \frac{2(r-a)}{r-2a} \frac{\binom{r-1}{b-1}}{\binom{r-1}{a}} < 0. \tag{5}$$

(5) is equivalent to

$$N(r) = \frac{9(r-b)(r-b-1)\dots(a+4)}{2(a+3)a(a-1)\dots b} < 1.$$

We have that

$$\frac{N(6t+5)}{N(6t-1)} = \frac{2^{10}}{3^6} \frac{t+\frac{3}{4}}{t+\frac{4}{3}} \frac{t+\frac{1}{2}}{t+\frac{4}{3}} \frac{t+\frac{1}{4}}{t+\frac{2}{3}} \frac{t-\frac{1}{2}}{t-\frac{1}{3}} = g'(t)$$

is monotonically increasing by our remark above.

For $2 \leqq t \leqq 5$ we obtain $g'(t) \leqq g'(5) = \frac{6072}{6137} < 1$. From $N(11) = \frac{3}{7}$, $N(6t-1) < 1$ follows by induction for $2 \leqq t \leqq 6$. Finally, we get $N(43) = \frac{10179}{59432} < 1$. $f(p_a)$ takes the maximum in the described interval at $p_a = \frac{a+3}{3}\left\{\binom{r-1}{a+1} - \binom{r-1}{a-1} - \binom{r-1}{b-1}\right\}$, consequently. We will complete our proof by showing the following inequality.

$$\left\{\frac{3}{2}\,\frac{2a+3}{a+1}\binom{r-1}{a} - \frac{a+3}{3}\binom{r-1}{b-1}\right\}\left\{\frac{3}{a+3} - \frac{2}{3}(a+3)\frac{\binom{r-1}{b-1}}{\binom{r-1}{a}}\right\} + \frac{a}{a+3}\binom{r-1}{a} +$$

$$+(r-1) + \frac{2}{3}(a+3)\binom{r-1}{b-1} < \binom{r-1}{a+1}.$$

This inequality is equivalent to

$$w(r) = \binom{r-1}{b-1}\left\{1 + \frac{2(a+3)^2}{9(a+1)}\left(1 - (a+1)\frac{\binom{r-1}{b-1}}{\binom{r-1}{a}}\right)\right\} - (r-1) > 0.$$

$$w(11) = 112 > 0.$$

Furthermore we prove the inequality $w'(r) = \frac{(a+3)(a+1)}{(2a+13)}\frac{\binom{r-1}{b-1}}{\binom{r-1}{a}} \leqq \frac{1}{2}$ for $r = 17, 23, 29, 35, 43$ by referring to the following table:

r	17	23	29	35	43
$w'(r)$	$\frac{140}{297}$	$\frac{1}{2}$	$\frac{154}{323}$	$\frac{442}{1035}$	$\frac{9044}{19981}$

Using this estimation of $w'(r)$ we get first

$$w(r) \geqq \binom{r-1}{b-1}\left(1 + \frac{2(a+3)^2}{9(a+1)}\left(1 - \frac{2a+13}{2(a+3)}\right)\right) - (r-1)$$

$$\geqq \binom{r-1}{b-1}\frac{2(a-6)}{9(a+1)} - (r-1),$$

then $w(17) \geqq \frac{311}{9} > 0$. $r \geqq 17$ implies $\frac{a-6}{a+1} \geqq \frac{1}{8}$ and for $2 \leqq i \leqq b-1$ we have $\frac{r-b-1+i}{i} > 3$. Hence for $r \in \{23, 29, 35, 43\}$:

$$\begin{aligned} w(r) &\geqq (r-1)\frac{2(a-6)}{9(a+1)}\prod_{i=2}^{b-1}\frac{r-b-1+i}{i}-(r-1) \\ &\geqq (r-1)\frac{2}{9}\frac{1}{8}3^{b-2}-(r-1) \\ &\geqq (r-1)\frac{1}{4}3^3-(r-1) \\ &> 0 \quad \text{follows.} \ \square \end{aligned}$$

5. Concluding remark

The author conjectures that (1) holds for the remaining even r and (2) holds for the remaining odd r, i.e. 13, 19, 25, 31 and 37.

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