A possible new model of neurons and neural processes based on the quantum-mechanical theory of measurement

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Abstract

A new model of neurons and neural processes is proposed which aims at providing a framework for treating the phenomena of statistical nature in the nervous system. Its conceptual base is the quantum-mechanical theory of measurement and some general characteristics of the interactions between systems. The simpler form of the model takes into account two observables, the stimulus intensity and length, measured by individual neurons, with only one threshold for each. By considerations analogous to those of quantum mechanics an uncertainty relation is derived between the possible accuracy of the measured intensity and time length values. The model is extended to the case of many thresholds and to measurements made by neuron populations which, in fact, generally occur in the nervous system.

Introduction

The paradigms of various sciences, particularly that of biology, have always shown characteristic relationships to the current theories of physics. Approaches to the problem of neural processes have in all ages, and today as well, depended on the generally accepted physical model of the world.

In classical considerations the changes of state of the units are always related to some interactions — "stimulation — excitation" — but there is no attention paid for the unique character of this interaction, namely that it means a kind of measurement, too. In our present knowledge, the measuring interaction is in general not negligible to the interaction determining the change of state. The principle of strict individuality of neural objects results, however, in an essentially dynamic treatment in which the neural units — like the organism as a whole — behave as the subject in the interaction with their own environment and remain separable from it during the process. Therefore, the states and their transitions belong to the units themselves and not to the interactions.

In constrast to this dynamic picture the recent use of some methods of statistical mechanics in the theoretical approach to neural systems was a serious step forward (Wiener, 1965; Cowan, 1968; Cowan, 1970, Amari, 1974).

Today the application of techniques developed in statistical mechanics seems unavoidable in the study of the nervous system and these techniques meet quite general acceptance. One can, nevertheless, expect that in the light of further experiences these statistical concepts would not be satisfactory enough, and the group of available theoretical methods should again be enlarged by developing ideas more departed from those of classical physics.

In our opinion three important features of the neural phenomena point to this direction. First, the discrete character of the structural and functional organization of the nervous system on various levels. Second, the probabilistic character of the distribution of activity in space and time. Third, last but not least, the existence of the above mentioned "measuring process" itself.

Generalization of the concept of "measurement"

Before trying to describe any hypothetical "measuring process" in the nervous system it is worth discussing the meaning of some general terms to be used. If we want to generalize the concept of measurement we must be aware that, in the definition and quantitative characterization of any measurable quantity — i.e. any observable —, the task of finding an appropriate device for the measurement can not be rejected. At the first steps of generalization, however, we need not identify these devices immediately with some concrete physical objects, in particular if we start with empirical experiences in the cases when objects and events are not separable from each other.

Let us try first to find a mathematical model that is fairly general to serve as a framework for any possible structure of events. Mathematically, the concept of an event is considered a primitive notion that is not otherwise defined; our ultimate aim is to get to a formalism for treating the systems of events taking place in the nervous system. Thus every object will be defined only by the system of events belonging to it.

Fortunately, there is already existing a general model of physical systems which applies not only to physical but any other systems as well and is adequate for the description of the measuring process (Mackey, 1963). Let us now sum it up briefly.

Suppose the structure of our system is not changing in time and the values of all the observable quantities are real numbers. The distribution of any of these quantities can be determined by measurements, i.e., by processes which select out a subset of the sample set given for the observables. The observables can have several different distributions; the state α of the system determines which one of them would result as the outcome of the measurement. Thus a real-numbered random variable φ belongs to each observable. Mathematically, φ is a Borel function mapping an (Ω, S, P) measurable space into the set R of real numbers:

$$\varphi\colon \Omega \to R \tag{1}$$

where (Ω, S) is the set of events being subsets of the sample set Ω and P is a probability measure defined on (Ω, S).

Now, let φ be assigned to a given observable 0 and let E be a given subset of the σ-algebra B of the Borel sets in the set R of real numbers, i.e., $E \in B$. In simple cases E is an interval. The question, then, arises: what is the probability that the value of φ falls into the set E?

According to the definition of φ, there is a subset $\varphi^{-1}(E)$ in (Ω, S) which is mapped to E by φ (Fig. 1). $\varphi^{-1}(E)$ is, of course, an event. As P is the probability measure defined in the same space (Ω, S, P) the probability we asked is $P[\varphi^{-1}(E)]$.

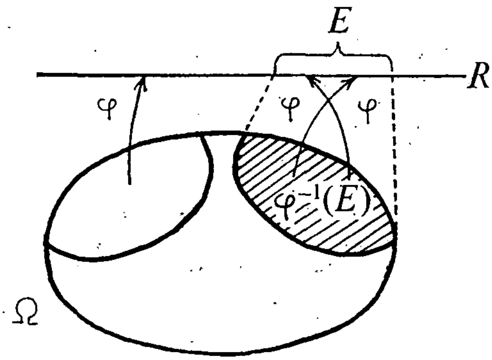

Fig. 1

The distribution of the values of φ given in this way can be called *the distribution induced by P*. We shall denote it by P^{α}_{φ} expressing the dependence of this distribution on the state α of the system and on the observable φ Therefore, every system has a family of distributions P^{α}_{φ} (Fig. 2).

Mathematically, if we consider several different (Ω, S, P) probability spaces the induced distributions P^{α}_{φ} may also be different. Physically, however, we expect that if the state of the system, denoted by α, is the same while the measured observables $\varphi_1, \varphi_2, \ldots, \varphi_n$ are different, the distributions $P^{\alpha}_{\varphi_1}, P^{\alpha}_{\varphi_2}, \ldots, P^{\alpha}_{\varphi_n}$ should be induced by the same probability measure $P = P^{\alpha}$ depending only on α in a fixed (Ω, S) measurable space.

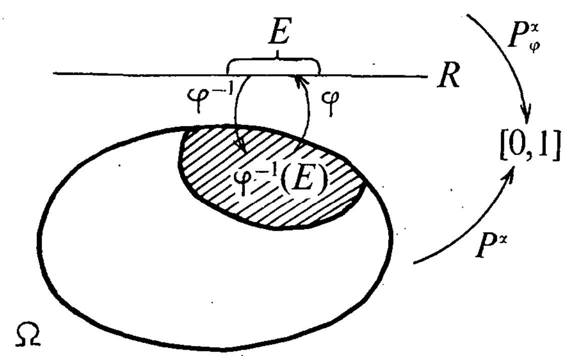

Fig. 2

In other words, as we want to characterize the system by the simultaneous description of the different observables — i.e., physical quantities — we must have an event space common to all of them. This common event space, then, represents the system by representing the states that determine the induced distributions.

As it is well known in probability theory, the most general event space in which a probability measure can be defined — i.e., in which the events are all compatible with each other — is the Boolean σ-algebra. Accordingly, Boolean σ-algebra can be an adequate structure for the common space outlined above *if all combinations* *of simultaneous events are physically possible in the system.* For example, in this case any set of predetermined values $f_1, f_2, \ldots, f_n$ for the observables $\varphi_1, \varphi_2, \ldots, \varphi_n$ can be given as an outcome of a measurement. The structure of the event space in classical physics — involving statistical mechanics and such extensions as e.g. the present models of neural systems — is therefore Boolean σ-algebra but in quantum mechanics where, in fact, it is not possible for certain events to occur simultaneously, another structure must be chosen.

In quantum mechanics the subspace lattice of the Hilbert space H(of infinite dimensions) is used as a common event space. Thus the mapping φ^{-1} operates from the intervals of real numbers — more generally, the subsets of the set of Borel sets of real numbers — to the subspaces of H:

$$\varphi^{-1}: B \to H \tag{2}$$

where $\varphi^{-1}(E) = \{\varphi^{-1}(x) | x \in E\}$, and x is a real number (that is taken by the random variable φ at an element of H) (Fig. 3).

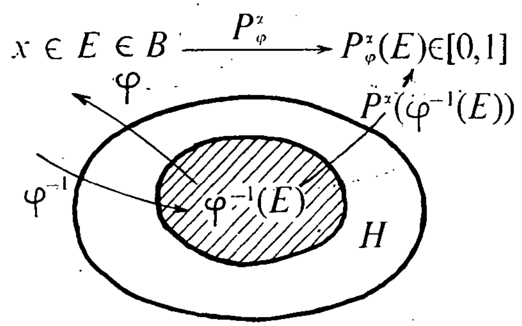

Fig. 3

It is obvious that the mapping

$$\varphi^{-1}: B \to H$$

is a homomorphism.

Now we can construct a one-to-one correspondence I between the subspaces of H and the operators P projecting to these subspaces. Then the mapping

$$I \circ \varphi^{-1}: B \to P \tag{3}$$

operating from the intervals of real numbers to the set of projection operators will be a so-called projection measure. As it is known, each projection measure is equivalent to a self-adjoint operator; the theory of self-adjoint operators (Neumann, 1932; Araki and Yanase, 1960) then provides us with the formalism adequate for deducing all the consequences essential in quantum mechanics.

The operators corresponding to the observables in classical physics are all commutative. Mathematically this follows from the fact that the common event space in this case is a distributive lattice; in quantum mechanics, on the contrary, the subspace lattice of H is not distributive, thus the operators do not always commute. From the point of view of measurement the non-commutativity involves the existence of observables whose values are not measurable simultaneously. Therefore, if we want to decide whether a system can be described by means of a formalism of classical type or not, we examine the physical possibilities for simultaneous determination of any set of values of the various observables.

The aim of this paper is to suggest and outline a model of abstract neural objects in which the common event space is a non-distributive lattice; i.e., the formalism of treatment is analogous to that of quantum mechanics. Apart from the mathematical construction described above one can have another, more general, possibility to approach the problem that in which cases an essentially probabilistic view of a given system is necessary.

Measurement and complexity

It is trivial that any process by which information can be obtained — i.e., any measurement — involves an interaction between at least two systems, say A and B. A and B are connected together in such a way that as a result of the measuring interaction the states $S_1^A, S_2^A, \ldots, S_n^A$ of A will inevitably correspond to some states $S_1^B, S_2^B, \ldots, S_n^B$ of B and vice versa. Thus, if $O^{(A)}$ is an observable belonging to A and $O^{(B)}$ is another belonging to B, any given value of $O^{(A)}$ (determined by the state of A) corresponds to a given value of $O^{(B)}$ (determined by the corresponding state of B). Let now A be the system to be measured and B the measuring one. Assume that both A and B have more or less complex structure. The notion of complexity does not need a strict definition here; it is enough to consider that the more complex a system is, the more complex the changes of its states and the parameters describing these changes will be. In the case of the measuring interaction between A and B there are two basic possibilities.

If A and B have equal complexity or B is more complex than A then the state changes in B can reflect in an adequate way the state changes in A. If, howewer, B is less complex than A, B does not have a large enough number of states for this purpose and in this case the measurement can lead to only a probabilistic description of A via the parameters of the state changes of B.

In practice, there is possible an important compromise. Namely, if though the system A is the more complex one, but it does not take part, as a whole, in the interaction, then the description may be dynamic. The necessary condition for this is that the part of A interacting with the measuring system B should not be more complex than the totality of B. H is possible only in this case — that all observables describing A are measurable simultaneously.

The quantum-mechanical concept of measurement is, therefore, the adequate tool for studying systems exhibiting non-negligible complexity in their interactions. In our opinion the nervous system does have this property. The application of the theory of measurement for this branch of biology is possible because the interactions between the nervous system and its environment, or between the parts of the nervous system itself, can be viewed as measuring processes (Jólesz and Gyöngy, 1975). In addition, any measurement has an aspect regarding to information, as the result of the measurement appears as a given state of the system (or its parameters), and the same is true for the new information obtained by the measurement. If we observe the measuring processes in the nervous system are of statistical nature we accordingly tend to discard the dynamic principles that would uniquely determine all details of the interactions occuring in the system.

A simple model of the neural measuring process

The simple abstract model of the neuron, discussed here, is somewhat similar to the so-called formal neuron, but a few of its properties are essentially different. As the mathematical concepts and procedures are all well known from quantum mechanics, for the sake of conciseness we will confine the treatment to a brief outline.

The basic postulates are as follows:

1. The system is characterized by a wave function Φ; the state of the system is fully determined by Φ. Φ is an element of the Hilbert space H.

2. The properties of the system are described by giving the possible values (the so-called eigenvalues) of the observables and by associating with each of them one or more state-functions in the Hilbert space, termed eigenfunctions. In addition, each wave function Φ can be expanded as a linear combination of the eigenfunctions of any observable. Thus, the observables $O_1, O_2, \ldots, O_n, \ldots$ are characterized by the appropriate sequences of real numbers:

$$O_1\colon k_1^{(1)}, k_2^{(1)}, \ldots, k_r^{(1)}, \ldots$$

$$O_2\colon k_1^{(2)}, k_2^{(2)}, \ldots, k_r^{(2)}, \ldots$$

$$\ldots$$

To each observable a set of probability values is assigned:

$$O_1\colon W(k_1^{(1)}, \Phi),\ W(k_2^{(1)}, \Phi), \ldots, W(k_r^{(1)}, \Phi), \ldots$$

$$O_2\colon W(k_1^{(2)}, \Phi),\ W(k_2^{(2)}, \Phi), \ldots, W(k_r^{(2)}, \Phi), \ldots$$

$$\ldots$$

where the probabilities W depend also on Φ.

The wave functions are elements of the space H and can be demonstrated as vectors in the Euclidean space of infinite dimensions. The observables are operators in the space of functions or matrices in the Euclidean space. Matrices and operators are both linear mappings of vector spaces.

3. The probability of that any given observable O takes on a value from the given interval (k', k'') can be determined in any state. (See Fig. 4.)

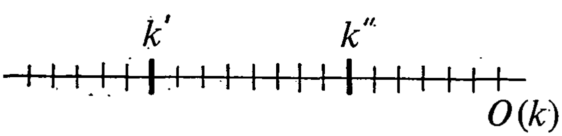

Fig. 4

4. The probability of that the values of two observables O_1 and O_2 falls simultaneously into the intervals (k', k'') and (l', l''), respectively, can be computed for some observables but can not be done so for others. The operators of these latter are not commutable:

$$O_1 O_2 \neq O_2 O_1 \tag{4}$$

or

$$O_1 O_2 - O_2 O_1 = cI \tag{5}$$

where c is an imaginary number and I is the unit operator or matrix.

In the model we assume that from the point of view of the measurement only two observables are relevant: *the stimulation intensity I* and *the time length T during which the stimulation operates*. In other words we assume that the stimulation has the only physical properties of intensity and length. As for both these observables the number of possible values is infinitely large we need some further simplifications.

In accordance with the existence of intensity and length thresholds — i_0 and t_0 — in real neurons suppose that both these observables can take on only two values each (Fig. 5).

To be more clear, let the states be represented by vectors of unit length on the plane. Let φ_1 and φ_2 be the state vectors associated with the two possible values

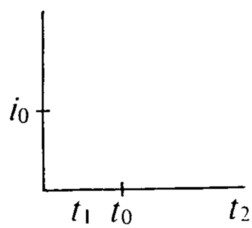

Fig. 5
Where i_1 and t_1 denote values below, while i_2 and t_2 above the threshold

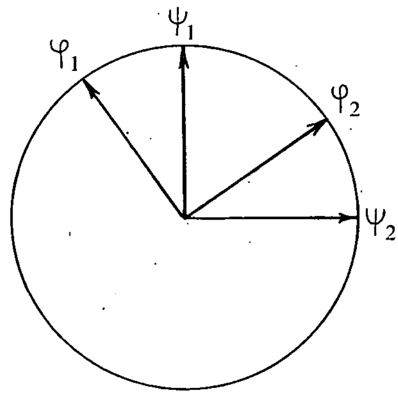

Fig. 6

i_1 and i_2 of the observable I and ψ_1 and ψ_2 the state vectors similarly for the observable T (Fig. 6).

It may be worth noting that all the eigenfunctions are mutually orthogonal, i.e., the scalar product of all pairs of them is equal to zero.

Now we have the symbols necessary for working with the model: the stimulus intensity I and length T as quantities to be measured, the possible values i_1, i_2 and t_1, t_2 of them, being equal either 0 or 1:

$$I\begin{cases} i_1 = 0\,(\varphi_1) \\ i_2 = 1\,(\varphi_2) \end{cases} \qquad T\begin{cases} t_1 = 0\,(\psi_1) \\ t_2 = 1\,(\psi_2) \end{cases}$$

Operators, probabilities, expectations and variances

Let the projection of an arbitrary state Φ to the direction of the eigenstates φ_1 and φ_2 be a_1 and a_2, respectively, and to the direction of ψ_1 and ψ_2 be b_1 and b_2 (Fig. 7).

Thus, Φ can be written as

$$\Phi = a_1\varphi_1 + a_2\varphi_2 = b_1\psi_1 + b_2\psi_2 .$$

The expansion of Φ as a linear combination of the eigenstates of I (or T) is called the I (or T) representation of Φ.

The operators projecting to a given state σ will be denoted by P_σ. Thus applying P_{φ_1} to Φ we get

$$P_{\varphi_1}\Phi = a_1\varphi_1.$$

It is obvious that projecting φ_1 to itself it remains unchanged

$$P_{\varphi_1}\varphi_1 = \varphi_1,$$

and projecting to a direction orthogonal to it the result will be zero

$$P_{\varphi_2}\varphi_1 = 0.$$

In the simple model presented here we consider only two operators A and B representing the observables I and T, respectively. As it was assumed, I and T can take on two values each and with every one of these values a corresponding state is associated. Note that in quantum mechanics this kind of characterization of the states and observables is quite general but the number of states is usually infinite. Therefore, the operators representing I and T can be written as

$$I \to A = i_1 P_{\varphi_1} + i_2 P_{\varphi}$$

and

$$T \to B = t_1 P_{\psi_1} + t_2 P_{\psi_2}$$

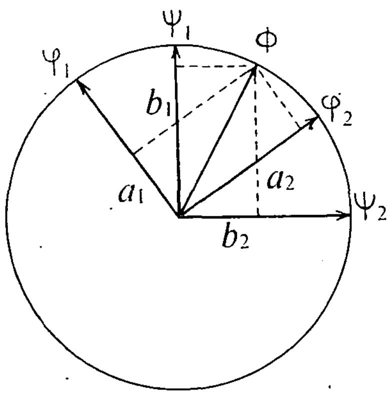

Fig. 7

If we want to compute the probability of obtaining the various values of the observables in a given state Φ we should multiply by the proper projection operators and then take the scalar product of the result with itself:

Probability {The measurement yields the value k}$=(P_\varkappa\Phi,\ P_\varkappa\Phi)$ (6)

where $\varkappa$ is the eigenstate associated with k.

Applying the above procedure to i_1, i_2, t_1 and t_2 the probabilities essential in our model are

$$\begin{aligned} W(i_1, \Phi) &= (P_{\varphi_1}\Phi, P_{\varphi_1}\Phi) = a_1^2, \\ W(i_2, \Phi) &= (P_{\varphi_2}\Phi, P_{\varphi_2}\Phi) = a_2^2, \\ W(t_1, \Phi) &= (P_{\psi_1}\Phi, P_{\psi_1}\Phi) = b_1^2, \\ W(t_2, \Phi) &= (P_{\psi_2}\Phi, P_{\psi_2}\Phi) = b_2^2. \end{aligned} \tag{7}$$

Note that

$$a_1^2 + a_2^2 = 1 \tag{8}$$

and

$$b_1^2 + b_2^2 = 1 \tag{9}$$

as in both cases one of the two possible values will certainly be obtained with probability 1.

As two important special cases, it is apparent that obtaining the value i_2 in the state φ_1 is impossible

$$W(i_2, \varphi_1) = (P_{\varphi_2}\varphi_1, P_{\varphi_2}\varphi_1) = 0, \tag{10}$$

and obtaining the value i_1 in the state φ_1 is certain

$$W(i_1, \varphi_1) = (P_{\varphi_1}\varphi_1, P_{\varphi_1}\varphi_1) = 1, \tag{11}$$

In general, if the state of the system coincides with one of the eigenstates of a given operator, the observable corresponding to that operator has a uniquely determined value by the measurement; and in any other state the probabilities of obtaining any permissible value of that observable can be computed in the way outlined above. If, however, we want to determine the probability of that in an arbitrary state Φ the observable I takes on the value i_1 *and* the observable T takes on the value t_1 we may easily get to a confusing result. In this case, Φ should first be projected to φ_1 belonging to i_1

$$P_{\varphi_1}\Phi = a_1\varphi_1, \tag{12}$$

then the result of this projection should be projected to ψ_1 belonging to t_1

$$P_{\psi_1}(P_{\varphi_1}\Phi) = P_{\psi_1}(a_1\varphi_1) = a_1 P_{\psi_1}\varphi_1. \tag{13}$$

The sought probability according to (6) is

$$(a_1 P_{\psi_1}\varphi_1, a_1 P_{\psi_1}\varphi_1) = a_1^2(P_{\psi_1}\varphi_1, P_{\psi_1}\varphi_1) = a_1^2(\varphi_1\psi_1)^2. \tag{14}$$

But, if we follow the reverse order of this procedure, namely projecting first to ψ_1 and then to φ_1

$$P_{\psi_1}\Phi = b_1\psi_1, \tag{15}$$

$$P_{\varphi_1}(P_{\psi_1}\Phi) = P_{\varphi_1}(b_1\psi_1) = b_1 P_{\varphi_1}\psi_1, \tag{16}$$

$$(b_1 P_{\varphi_1}\psi_1, b_1 P_{\varphi_1}\psi_1) = b_1^2(P_{\varphi_1}\psi_1, P_{\varphi_1}\psi_1) = b_1^2(\psi_1, \varphi_1)^2. \tag{17}$$

It is trivial that the two computed values for the same probability are not equal:

$$a_1^2(\varphi_1, \psi_1)^2 \neq b_1^2(\psi_1, \varphi_1)^2 \tag{18}$$

as because of the result of the scalar product being independent of the order of the factors

$$(\varphi_1, \psi_1) = (\psi_1, \varphi_1) \tag{19}$$

and thus

$$a_1^2 \neq b_1^2 \tag{20}$$

causes the inequality to hold.

Accordingly, it is not possible to make a unique assertion about the probability of obtaining simultaneous values for I and T. The order of applying the operators corresponding to I and T to a given state function Φ is not commutable:

$$AB\Phi \neq BA\Phi \tag{21}$$

or

$$AB\Phi - BA\Phi = C\Phi \tag{22}$$

where $C \neq 0$ is called commutator. C is independent of the state, i.e., of the wave function Φ.

In the model it is of paramount importance to compute the expectation value and the variance of the stimulus intensity and length. The expectations in a given state Φ are

$$\bar{I} = i_1 a_1^2 + i_2 a_2^2 = i_1(P_{\varphi_1}\Phi, P_{\varphi_1}\Phi) + i_2(P_{\varphi_2}\Phi, P_{\varphi_2}\Phi) =$$

$$= i_1(P_{\varphi_1}\Phi, \Phi) + i_2(P_{\varphi_2}\Phi, \Phi) = ([i_1 P_{\varphi_1} + i_2 P_{\varphi_2}]\Phi, \Phi) = (A\Phi, \Phi) \tag{23}$$

and

$$\bar{T} = t_1 b_1^2 + t_2 b_2^2 = (B\Phi, \Phi). \tag{24}$$

The expectation value thus can be given without knowing the analytic form of the wave function and the operator representing the observable to be measured:

$$\bar{A} = (A\Phi, \Phi) \tag{25}$$

$$\bar{B} = (B\Phi, \Phi) \tag{26}$$

In a quite similar way we get the variances:

$$(\Delta I)^2 = ([A - \bar{A}\mathbf{I}]^2 \Phi, \Phi) \tag{27}$$

$$(\Delta T)^2 = ([B - \bar{B}\mathbf{I}]^2 \Phi, \Phi) \tag{28}$$

where $\mathbf{I}$ is the unit operator.

Appearance of the uncertainty relations

In the light of the above considerations a question arises concerning the meaning of the simultaneous measurement of the stimulus intensity and time length. It is obvious that the neuron can be regarded as a physiological device for measuring the intensity and length of various stimuli and not less obviously this measuring process relates to some threshold conditions. In our simple model what consequences can be drawn if, as we have just seen, there is an inherent ambiguity in the process of simultaneous determination of the probabilities for I and T?

If the measurement of one of the observables, e.g. the stimulus intensity, can yield two permitted values i_1 and i_2 according to the existence of a threshold, the states φ_1 and φ_2 associated with the eigenvalues i_1 and i_2 respectively assign the probability 1 to the corresponding i_1 or i_2 values of the observable I. If by the same measurement the neuron does determine the length of the stimulus, too, the measurement yields either the value t_1 or t_2 for the observable T and, consequently, we can be sure that the system was either in the state ψ_1 or ψ_2. Thus, in a simultaneous measurement one of the eigenstates of I would be the same as one of the eigenstates of T. This involves the commutativity of the operators belonging to I and T. In the model suggested here, however, like in quantum mechanics, neither φ_1 or φ_2 is equal to ψ_1 or ψ_2. The contradiction disappears only if we accept that in the measuring process of the neuron there are measurable quantities whose statistics can not be correlated with each other. In other words, as there are no common eigenstates of the intensity and length of the stimulus, these two observables can not be measured simultaneously however obey them, separately, quite well-defined probabilistic laws.

The variances of the observables relevant in our model have a relationship to each other similar to that between the variances of canonical conjugate variables described by the Heisenberg uncertainty relation. This relation expresses, in fact, that the two observables have no common eigenstates.

The farer is the state of a neural object from the eigenstate, for a given observable, the more uncertain the value of that observable; the variance is zero only in an eigenstate. If the equation (21)—(22) holds true, the variances of the two quantities in that equation can not be zero simultaneously. Of course, this means that the simultaneous measurement of these quantities can not, even theoretically, be arbitrarily accurate. We can ask only to what extent the $(\Delta A)^2$ and $(\Delta B)^2$ variances can be simultaneously lowered.

To see this let us introduce two auxiliary operators:

$$A' = A - \bar{A}\mathbf{I}$$

and

$$B' = A - \bar{B}\mathbf{I} \tag{29}$$

As it is obvious the commutation relations remain true for A' and B'

$$A'B' - B'A' = C.$$

The variances

$$(\Delta A)^2 = (\Phi, A'^2\Phi) = (A'\Phi, A'\Phi)$$

$$(\Delta B)^2 = (\Phi, B'^2\Phi) = (B'\Phi, B'\Phi). \tag{30}$$

Now let

$$f = A'\Phi$$

$$g = B'\Phi. \tag{31}$$

As f and g are quadratically integrable functions, according to the Schwartz inequality

$$|(f, g)|^2 \leqq \|f\| \cdot \|g\|. \tag{32}$$

Substituting from (30) into (32):

$$(\Delta A)^2(\Delta B)^2 \geqq |(A'\Phi, B'\Phi)|^2 = |(\Phi, A'B'\Phi)|^2 \tag{33}$$

With some trivial transcriptions:

$$\Delta A \Delta B \geqq \frac{1}{2}|\bar{C}| \tag{34}$$

This inequality characterizes the correlation between the uncertainties in the measured value of observables corresponding to non-commutable operators. The choice of the state Φ of the model neuron does not affect the validity of (34).

For the measurements of stimulus intensity and length

$$\Delta I \Delta T \geqq C. \tag{35}$$

This relationship is analogous to the Heisenberg uncertainty relation with the important difference of C being not a universal constant but only one independent of the state. (35), (27), (28) assert that in the proposed model the accurate simultaneous measurement of the stimulus intensity and length has an absolute (theoretical) limitation (Jólesz and Szilágyi, 1974.). This does not mean that the neuron could not measure either the intensity or the length of a given stimulus; it states only that it can not measure them simultaneously with an arbitrary precision. In other words, if the state function Φ is not an eigenvector of a given operator A, by the measurement of the observable corresponding to A we can be given several different results and the farer is the state of the neuron from an eigenstate of A, the more uncertain the value of the given observable.

To sum up the process of measurement by the model neuron, the most important points, in our opinion, are the following. The neuron measures the property I or T of the stimulus, or rather their thresholds; in the process of this measurement its original state Φ turns into φ_1 or φ_2 (associated with the numbers i_1 and i_2 characterizing the value of I being above or below the threshold) in the case of measuring I, and into ψ_1 or ψ_2 (associated with t_1 or t_2) in the case of measuring T:

$$\Phi \to \begin{matrix}\varphi_1 \\ \varphi_2\end{matrix} \qquad \Phi \to \begin{matrix}\psi_1 \\ \psi_2\end{matrix} \tag{36}$$

This state change is called the measuring process. It corresponds to the projection process of the state vector Φ onto the direction of one of the eigenvectors. In the measuring process the object being in the state Φ turns into one of the eigenstates φ_j of the operator A of the measured variable (I or T). The process itself does not require any description; what is relevant are only the probabilities of the occurrence of the various possible φ_j final states.

Denoting the projection operator projecting onto the direction of φ_j by P_{φ_j} we obtained that the probability of the

$$\Phi \to P_{\varphi_j}\Phi = (\Phi, \varphi_j)\varphi_j \tag{37}$$

transition is $|(\Phi, \varphi_j)|^2$ as a result of the measuring process.

As the state vectors are normalized to unity the multiplicative coefficient of φ_j can be eliminated by normalization. Thus the original state Φ becomes completely vanished from the expression of the final state. The original state takes part only in the expression of the transition probability. The measuring process has a representative only in the set of the projection operators.

Apart from the measuring process — which is in some sense a singular one — another process is existing in the model: the spontaneous change of state of the undisturbed system. This process can be described by a continuous rotation:

$$\Phi \to U(\alpha)\Phi \tag{38}$$

where $U(\alpha)$ a unitary operator with the rotation parameter α.

It is obvious that both processes outlined are well identifyable in the case of our model and of real neurons as well. In contrast to the continuous transition in spontaneous processes, the measuring process represents a discrete change of state.

Finally, it may be worth noting that it is possible to draw conclusions about the state before a measurement. One can do so by the measurement itself, because the probability distribution of the measured spectrum reflects just the distribution of the possible states before the measurement. This latter but characterizes a real state for, in the case of neurons as analogous systems to micro-objects in quantum mechanics, it is the probability distribution of the possible eigenvalues which contains the whole information about any given state independently of the measurement.

An extension of the model

As we have stated before, the neuron can be viewed as a device for measuring the stimulus intensity and length. The neuron can have, however, not only one threshold for the measured observables, but many different ones for each. It involves the need for extending the previous simple model to be able to treat a series of eigenvalues $\{i_1, i_2, \dots, i_n, \dots\}$ and $\{t_1, t_2, \dots, t_n, \dots\}$.

Consider the points k_i on the real line permitted for a given observable R to take on as measured values. Thresholds can be taken into account by leaving out certain points from the set of the possibilities; mathematically this is done by means of a projection operator $E(k)$ which increases in the permitted points and remains constant elsewhere. The whole set of the permitted values will be called the spectrum of R.

It is obvious that operators having discrete series of eigenvalues are adequate for the description of observables whose permitted values constitute also a discrete series. Thus, this kind of operators can be used in a neuron model with a number of discrete thresholds.

For theoretical derivation of the possible values of a given observable — e.g. thresholds of intensity — we need to know the operator corresponding to that observable. It is enough to determine the two observables characterizing the stimulus because all the other observables — e.g. which relates to the speed of the stimulus intensity change in time — and consequently their operators can be deduced from these.

The formal neuron (McCulloch and Pitts, 1943) and the variations of it can also be regarded as devices for measuring the stimulus intensity (Lábos, 1975). In some experiments (Lábos, 1973; Sclabassi, Lábos et al., 1973) neurons have yielded response characteristics the analysis of which by means of model frequency code points towards the idea of the neuron with more than one threshold. A similar system of thresholds can be obtained from the Hodgkin—Huxley model (successive current thresholds, Lieberstein, 1973) as well.

According to Lábos (1975) any neuron having response characteristic with generalized distribution function can be regarded as a measuring device. The response characteristics which refer to more than one threshold generate discrete Lebesgue—Stieltjes measurable spaces. In addition, Lábos stated that neurons have various different sets of thresholds depending on the length of the stimulus: the shorter the stimulus in time, the fewer levels of intensity can be distinguished. As it was mentioned before, the neuron should be represented by not only some thresholds in intensity but in length as well. Therefore, the outlined model should be extended.

In the most general case there must be a solution both for discrete and continuous eigenvalues. Thus, let i be an arbitrary real number (being one of the eigenvalues of the stimulus intensity as an observable) and $E(i)$ an operator with the argument i. $E(i)$ is a generalization of projection operators of the simple model, projecting to different subspaces depending on the value i. Let f and g be two arbitrary elements of the domain of $E(i)$. Taking the inner product

$$(f, E(i)g) = (E(i)f, g) \tag{39}$$

if the following Lebesgue—Stieltjes integral exists:

$$\int_{-\infty}^{\infty} id\,(f, E(i)g) \tag{40}$$

where

$$E(-\infty) = 0, \quad E(+\infty) = \mathbf{I} \tag{41}$$

$\mathbf{I}$ being the unit operator, then an operator A can be constructed:

$$(f, Ag) = \int id(f, E(i)g) \tag{42}$$

$$A = \int idE(i) \tag{43}$$

$E(i)$ is called the spectral decomposition of the operator A. In the case of discrete spectrum $E(i)$ depends on i in the following way

$$E(i) = \sum_{i_n \leqq 1} P_n. \tag{44}$$

The intervals where $(f, E(i)g)$ is constant may be excluded from the domain of integration. These values do not belong to the spectrum of A; to this spectrum do belong only the values i whose corresponding product $(f, E(i)g)$ is changing. Where the change is continuous, so is the spectrum, while the points where there is an abrupt change in $(f, E(i)g)$ constitute the point spectrum of A. In this way the existence of thresholds may be taken into consideration. There can be no projection operator attributed to the isolated points in the domain of any continuous spectrum. On the contrary, to the interval (i', i'') the following projection operator belongs

$$E = E(i'') - E(i'). \tag{45}$$

The commutation relations for the operators are closely related to those for their spectral decompositions. Let

$$A = \int idE(i) \tag{46}$$

and

$$B = \int tdF(t). \tag{47}$$

If

$$E(i)F(t) = F(t)E(i) \tag{48}$$

then

$$AB = BA. \tag{49}$$

It would hold true only, however, if both A and B were functions of the same operator but, according to our assumptions, now this is not the case.

In our opinion the extended neuron model is characterizable by the set of thresholds of stimulus intensity and length. The operators associated with these observables are in one-to-one correspondence with some subspaces of H and hence the equality $EF = FE$ would be valid only if EF and FE projected onto the same subspace. As but the system of eigenvectors and so the subspaces in question are not common the operators do not commute.

According to the outlined model, neural objects are represented by the Hilbert space; events are represented by its subspaces or the projection operators being in one-to-one correspondence with the subspaces.

The relevance of measurement should be stressed in particular. The outcome of the measurement is affected by chance; the probability of any given transition from a state before to another after the measurement depends on the beginning state and the measured observable. The beginning state Φ of the object is an element of the Hilbert space representing the object in question. The resulting state φ_n is always an eigenstate of the operator of the measured observable. The measured value of the observable is the i_n eigenvalue (in the case of intensity measurement) belonging to the eigenstate φ_n. The distribution function determined by the transition probabilities, i.e., the probability of that the value i_n of I is not greater than a given value i is as follows

$$W(i_n \leqq i\,|\Phi) = (\Phi, E(i)\,\Phi) \tag{50}$$

or, with some trivial transcriptions

$$W(i_n \leqq i|\Phi) = (f, E(i)\,\Phi) = (\Phi, E^2(i)\,\Phi) = (E(i)\,\Phi, E(i)\,\Phi) = \|E(i)\,\Phi\|^2. \tag{51}$$

Knowing the probability distribution, the expected value is easily computed

$$\bar{A} = \int id(\Phi, E(i)\Phi). \tag{52}$$

The scatter

$$\overline{(A-\bar{A}\mathbf{I})^2} = (\Phi, (A-\bar{A}\mathbf{I})^2\,\Phi) = |(A-\bar{A}\mathbf{I})\,\Phi, (A-\bar{A}\mathbf{I})\,\Phi| = \|(A-\bar{A}\mathbf{I})\,\Phi\|^2. \tag{53}$$

As regards to the measuring process, the following are important. In the case of the measuring interaction the object being in the state Φ turns into another state φ_n. φ_n is one of the eigenvectors of the operator of the measured observable, i.e., I, where I can be expanded by the projection operators P_n:

$$\begin{gathered} A = \sum i_n P_n = \int id(E(i)) \\ P_n\varphi_n = \varphi_n, \quad A\varphi_n = i_n\varphi_n \\ E(i) = \sum_{i_n \leqq i} P_n \end{gathered} \tag{54}$$

The distribution function of the transition probabilities by transcribing (51)

$$W(i_n \leqq i|\Phi) = \|E(i)\Phi\|^2 = \sum_{i_n \leqq i} |(\varphi_n, \Phi)|^2 \tag{55}$$

where $|(\varphi_n, \Phi)|^2$ is the probability of the transition $\Phi \to \varphi_n$.

The measurement yields a correct value for I if φ_n uniquely determines i_n and vice versa. The eigenvalue i_n is really the correct value, because in the state φ_n the deviation for the operator I is zero:

$$\begin{aligned} (\varphi_n, [A-\bar{A}\mathbf{I}]^2\varphi_n) &= ([A-\bar{A}\mathbf{I}]\,\varphi_n, [A-\bar{A}\mathbf{I}]\,\varphi_n) = \\ &= \|(A-\bar{A}\mathbf{I})\,\varphi_n\|^2 = \|A\varphi_n - \bar{A}\varphi_n\|^2 = 0 \end{aligned} \tag{56}$$

Another case occurs if the observable to be measured has a continuous spectrum (i.e. set of eigenvalues). Let the spectral decomposition of I be $E(i)$. A measurement with given correctness means that the outcome i falls into a given interval (i', i''). The probability of this is

$$W(i' < i \leqq i''\,|\Phi) = (\Phi, E(i)\,\Phi) \tag{57}$$

where Φ is the state before the measurement. The length of the interval (i', i'') may be arbitrarily small the value of $W(i'<i\leqq i''|\Phi)$ remains finite; it follows then that the probability of the case of absolute correct measurement is zero:

$$W(i' < i \leqq i'' \mid \Phi) = 0. \tag{58}$$

Consequently, in the continuous range of the spectrum (according to the model suggested) there exist measurements only with non-zero uncertainty.

Probabilistic interpretation of the neural measuring process

Theoretical interpretations regarding to the operation of the nervous system — in spite of that they contradict to each other in some respects — have the common feature of accepting (at least at present) the probabilistic nature of the neural processes. In this respect the opinions are diverging in whether this nature is the same as that of other disciplines in physics (classical physics, thermodynamics, quantum mechanics) or is inherently different.

In our model the spontaneous and the measuring process may be in close connection with the probabilistic interpretation. The measuring process influences the state of the system, so obtaining information is connected to the state change. When we state that the new information, i.e., the result of the measurement, is reflected in the new state of the system, we lay stress on the statistical meaning of the state. Namely, while during the spontaneous process the state transition $\Phi\to\Phi'$ is not statistic (the system turns from $U=P_\Phi$ into $U=P_{\Phi'}$ in a continuous way), the measuring process causes the state Φ to transform into one of the eigenstates $\varphi_1, \varphi_2, \ldots$, this transformation being only stochastically determined: the probabilities $|(\Phi, \varphi_1)|$, $|(\Phi, \varphi_2)|, \ldots$ of the states $\varphi_1, \varphi_2, \ldots$ are uniquely determined and not so is the final state itself. During the measuring process the states turn into mixed form

$$U = P_\Phi \to U' = \sum_{n=1}^{\infty} |(\Phi, \varphi_n)|^2 P_{\varphi_n}. \tag{59}$$

In the language of information theory the measurement is a kind of mixing processes hence it is necessarily irreversible.

The basic difference between the spontaneous and measuring processes is that while in the time interval between two measurements the variation of the state vector is determined and continuous, the variation owing to the measurement is sudden and discontinuous. This latter can be described only by probability laws.

After the measurement the state of the system is a compound consisting of the eigenstates of the operator of the measured observable. All these statements are of importance if one considers a measurement taken by the neuron: from the point of view of neural networks, the outcome of the measurement means that a group of neurons is not in a homogeneous state but its members have different states with different probabilities. Any combination of these states can be a measurement outcome if the measuring device is the given group. After the measurement the group can yield only probabilistic relationships.

In the field of theoretical neurobiology relatively large area is occupied by statistical mechanics (Wiener, 1958; Cowan, 1968; 1970; Amari, 1974). Regarding to the origin of the probabilistic laws applied, the analysis may show different levels of deepness (Griffith, 1971). In the application of the theory of random processes to macroscopic neural networks one can disregard even the existence of the network structure. It causes, then, the treatment to confine itself to statistical fluctuations. However, there are a number of theories of considerable efficiency by the utilization of probabilistic concepts of neural processes.

In addition to the use of the methods of statistical mechanics some examples of the use of quantum statistics can also be found (Winograd and Cowan, 1963; Cowan, 1965; Agin, 1963; Michalov, 1967, 1968). In our opinion this way is very promising. By means of the formalism of quantum mechanics essential features of the neural measuring process may become known. Moreover, we can extend the borders of the probabilistic interpretation by taking into account that only the statistics of the observables are really "observable". The connection between the state functions and the observables has an inherent statistical nature because, in the model suggested here, in the case of a system with k degrees of freedom the states are characterized by a function $\Phi(g_1, g_2, ..., g_k)$ which is an element of the Hilbert space; therefore, with even a full giving of Φ one can make only statistical assertions about the system. (Obviously it is irrelevant that the probability of the truth of these assertions lies sometimes near 0 or 1).

Characterization of the state of a neural micro-object by a state function yields the possibility of making probabilistic statements, but the validity of the statements can be checked only on some groups of micro-objects, i.e., on neural populations. This means also that this formalism expects immediately the measurement to be made by neuron populations or, equivalently, it expects the recording of the statistics of the measured observables. In classical statistical mechanics the question concerning the probability of finding a given neuron from the population in a given state can always be asked and answered as well. On the contrary, the probabilistic expressions in the formalism of quantum theory give possibility only of answering the question about the probabilities of a given value to fall into the interval (i', i'') or the interval (t', t'') separately. There is no probability measure common to both intervals because there is no common state in which both probabilities can be measured.

Assuming that populations of neurons are generally not in pure state (i.e., all neurons are not in the same state) we should consider mixtures. If a measurement is made on a system in mixed state it forces the system to turn into an eigenstate. The sudden change of the state function at the moment of the measurement can be described by means only of probabilistic relationships. In the theory of measurement (Neumann, 1932) it is generally assumed that measuring an observable on a single object R is not all that possible and the measurement even should be made on a system of very many objects. With the attitude of measurement theory the large dimensionality of the nervous system and also the concept of redundancy can gain a new interpretation. Namely, according to this attitude, it is advisable to make measurements on large statistical groups consisting of a number of micro-objects $R_1, R_2, ..., R_N$, where N is a large number. On such a group the distribution of the values of the measured observable is determined. The advantage of this procedure is that though the measurement disturbs the object on which it was made, the disturbance of the population as a whole may be arbitrarily small if N is large enough. Furthermore, though two observables having non-commuting operators can not be measured simultaneously to any degree of accuracy, in the population their probability distributions can be determined with an arbitrarily small error. It is enough to measure a part of the whole system, if the number of elements M of that part is large, i.e. $M \gg 1$, but it is much smaller than $N (M \ll N)$. In this case the measurement affects only the M/N part of the total system. Measuring another observable on another part, made up by K elements, of the system, the two measurements do not interfere if $(K+M) \ll N$ and $(K+M)/N \ll 1$. These requirements can easily be fulfilled if N is large enough; in this case K and M may also be large.

In the nervous system by means of statistically large populations of neurons there is a possibility of objective measurements being independent of occasional disturbations and of that any single neural object is unable to make simultaneous accurate measurements of two non-commuting observables. However, as it will soon be demonstrated, the measurement can not be absolutely accurate even in this case.

Consider, for example, the simpler one of the models described, in which the observable I can take on the values i_1 and i_2 only. Let us measure I on a population $\{R_1, R_2, \ldots, R_n\}$; then we get i_1 as a result at a part $\{R'_1, \ldots, R'_{N1}\}$ of the population and i_2 at another part $\{R''_1, \ldots, R''_{N2}\}$. As a consequence of the measuring process, however, the neuron states will change in both parts of the population and, for this reason, if we measure another observable T on the same population (with possible values t_1 and t_2) it is no more possible to obtain scatterless results because the eigenstates of I can not be eigenstates of T. Accordingly, simultaneous measurements can not produce pure populations.

It is generally stated that probability is a property of certain classes of populations. This permits, however, to apply probability calculus to some individual (e.g. neural) processes, if we know that probability calculus is applicable if the process at hand leads to statistical populations. In the study of neural processes the reason for probability to play role is twofold. On the one hand, in the starting mixed state of the system certain properties — e.g. intensity and length thresholds — have statistical distributions and this causes a statistical distribution of the measured values as well. On the other hand, during the measurement the nervous system passes through a series of interactions which gives rise to statistical distributions of certain parameters. In both cases it is plausible that the individual objects making up the population (i.e. the neurons) obey some dynamic laws different from the statistical ones. Therefore, the concept of the so-called measuring interaction requires an adequate framework to be placed in, in order to be distinguishable from another types of interactions.

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Summary

The purpose of the present paper is to investigate some properties of measuring processes performed by single nerve cells and neural nets. We applied the formalism of quantum mechanics and the quantum-mechanical concept of measurement. We used the subspace lattice of the Hilbert space as a common event space. Two neuron models were analyzed in which we assumed that only

two observables (stimulus intensity and time) are relevant. In these models we considered only two operators representing the observables.

The first model was characterizable by one threshold, the second one by the set of thresholds. In the proposed models the simultaneous measurement of the observables has an absolute limitation and the variances of them have a relationship which is analogous to the Heisenberg uncertainty relation, with the important difference of C being not a universal constant. Statistical properties of the neuronal measuring processes were examined.

The mathematical methods for dealing with neuronal systems that we have described in this paper seem to have many advantages over the methods usually used. There is a strong analog between these methods and the techniques generally used for physical systems. Although we have limited ourselves in this paper to two neuron models, the presented new method is generalizable to neuronpopulations that are composed of elements with n thresholds and in which not only two observables are measured.

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